Abstract. This article is concerned with basic Number Theory. Our purpose is to show that, there are infinitely many Mersenne Prime Numbers, for this, we shall use, some known Theorems or results in Arithmetic and Cantorian set Theory.

Key Words. Prime, Mersenne’s Primes, Eclid’s Theorem, Cantor’s Theorem, Schröder-Bernstein’s Theorem, Pomerance’sTheorem.

1. Notation and Preliminary Lemmas.

1.1 Starting with, we need to establish some basic notation. The symbol $M_p$, formally shall represent Mersenne’s Primes numbers, that is $M_p = \{ x \in \mathbb{N}, x = 2^p - 1 \}$, where $p$ and $x$ are both primes. While the symbol $P_r$ stands for the totality of prime numbers $[6][18]$. 

Lemma 1. Euclid’s Theorem. $Card.(P_r) = \aleph_0,[3]$. 

Lemma 2. Schröder-Bernstein’s Theorem.

If $f : S \rightarrow T$ and $g : T \rightarrow S$, are both injective functions, then there exists a Bijective function $h : S \rightarrow T,[16]$. 

Lemma 3. Cantor’s Theorem.

If $\psi : S \rightarrow T$, is an Bijective function, thus $Card.(S) = Card.(T).[2]$. 


Mersenne Primes Cardinality
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2.1 For the sake of completeness, also we must define the following auxiliary set:

\[ D = \{ p \in P_r : 3 \leq p < p + 1 \} \]

Notice that, \( p + 1 \) is even and this set is finite by construction, due to the fact that, for some prime \( p \) given, the number of Primes from three to \( p + 1 \) is finite \([14]\). Furthermore, \( D \) can be represented by this equivalent form:

\[ D = \{ p_1, p_2, ..., p_{\lambda - 1}, p_\lambda \} \]

consequently it contains \( \lambda \) primes and we can write the relation,

\[
(1) \quad \text{Card.}(D) = \lambda \in \mathbb{N}.
\]

It should be observed that, this set plays a very important role in our approach.

To solve our guess, also we need define a generating function \( \phi : P_r \to \mathbb{N} \) as follows,

\[
\phi(p) = \begin{cases} 
2^n - 1, & \text{if and only if } 2^n - 1 \text{ is prime} \\
2^n & \text{Otherwise}
\end{cases}
\]

As we shall see, images of this function \( I_\phi \) does contain (in particular) two infinite subset:

Mersenne Primes \( M_p \) and \( H_0 = \{ y \in \mathbb{N} : y = 2^n \}, n \) belongs to \( \mathbb{N} \), our aim in this article is to prove the following statement; which is immediate consequence of our preceding definitions.

Proposition 1.2.1 Sequence \( H_0 \) and \( M_p \) are both infinite sets.

Proof.

We first prove that \( H_0 \) is infinite. This arithmetical fact is immediate, because we have infinite possibilities for the choice of the quantity \( n = p \), whenever \( p \in P_r \), by applying Lemma 1 or Euclid’s Theorem. It follows that, \( \text{Card.}(H_0) = \text{Card.}(P_r) = \aleph_0 \), next we must prove that,

\[
(2) \quad \text{Card.}(M_p) = \aleph_0.
\]

In order to show that important Cantorian relation, we need to construct another two “bizarre functions”, by using our basic sets \( P_r, M_p \) and \( D \) in the following way:

\[
\psi_1(p) = \begin{cases} 
2^n - 1, & \text{if and only if } 2^n - 1 \text{ is prime} \\
2^n & \text{if } 2^n - 1 \text{ is not prime}
\end{cases}
\]
Remark 1. The quantity \( p \) is to be the maximum prime number inside \( D \). But a sample it is needed to explain how \( \psi_1(p) \) works: if \( p = 11 \), thus \( \psi_1(11) \not\in M_p \). Since \( D \) belongs to the range of \( \psi_1 \), this means that, we must get the maximum prime of \( D = \{3, 5, 7, 11, 12\} \), accordingly our definition at 2.1 above, obviously our finite algorithm provides the value,

\[
(3) \quad \psi_1(11) = 11
\]

Under that crucial restriction, function \( \psi_1 : P_r \to M_p \cup D \) remains well defined (because \( 11 \in P_r \), but it don’t generates none Mersenne’s prime). Likely, existence and uniqueness of a maximum prime \( P \) belonging to \( D \), is based upon Lemma 4 or Pomerance’s Theorem.

3.1 Continuing our task, also we define another generating function \( \psi_2 : M_p \cup D \to P_r \),

\[
\psi_2(x) = \begin{cases} 
\log_2(x+1) & \text{if } x \in M_p \\
11 & \text{if } x \not\in M_p
\end{cases}
\]

Remark 2. By definition, only two cases can occur: either \( x \in M_p \) or \( x \not\in M_p \) (similarly to preceding case) \( p \) is to be the maximum prime number inside \( D \). With this caution, we are ready to prove a basic result, by employing functions \( \psi_1, \psi_2 \) previously constructed; in the next section we shall do it.

2. Injectivity of \( \psi_1 : P_r \to M_p \cup D \) \( \psi_2 : M_p \cup D \to P_r \).

Proposition 1.2.2 Functions \( \psi_1 \) and \( \psi_2 \) defined above, are both injective.

Proof. Formally, it is necessary to distinguish the following two cases:

Firstly, we study \( \psi_1 : P_r \to M_p \cup D \). Let us \( p_1, p_2 \) belonging to \( P_r \). Whenever \( p_1 \) and \( p_2 \) satisfy these conditions, \( \psi_1(p_1) \in M_p \) and \( \psi_1(p_2) \in M_p \) as well.

We can assume by hypothesis that,

\[
(5) \quad \psi_1(p_1) = \psi_2(p_2).
\]

This means that there exist two primes \( p_1, p_2 \), contained in \( P_r \) satisfying the system:
But applying (5) we may write, 
\[ 2^{p_1} - 1 = 2^{p_2} - 1 \Rightarrow 2^{p_1} = 2^{p_2} \]
and so: \( p_1 = p_2 \).

We conclude that (in this case) \( \psi_1 : P_r \to M_p \cup D \) is injective.

We have therefore left to examine, when \( \psi_1(p_i) \in D_1 \) and \( \psi_1(p_j) \in D_2 \); thence there exists two primes \( p_i, p_j \) which satisfy the conditions: \( p_i \prec p_j \) or \( p_j \prec p_i \) now it is easily seen that, in either case we have,

(7) \( p_i \neq p_j \). In accordance with definition of \( D \), numbers \( p_i \) and \( p_j \) satisfy,

(8) \( p_i = \text{maximum prime of } D_1 \) and simultaneously \( p_j = \text{maximum prime of } D_2 \).

In turn from (7) we have immediately,

(9) \( \psi_1(p_i) \neq \psi_1(p_j) \). And \( \psi_1 : P_r \to M_p \cup D \) is injective. As was to be proved.

2.2 Well, at this point of the proof, we now analyze the function \( \psi_2 : M_p \cup D \to P_r \).

**Case 1.** Let \( x_1, x_2 \) two arbitrary elements belonging to \( M_p \), whence there exists two primes \( p_1, p_2 \) which by hypothesis satisfy the Diophantine equations:

(10) \( x_1 = 2^{p_1} - 1 \) and \( x_2 = 2^{p_2} - 1 \), we can suppose that: \( p_1 \neq p_2 \), hence one has,

(11) \( 2^{p_1} - 1 \neq 2^{p_2} - 1 \Rightarrow x_1 \neq x_2 \), and from this, just we deduce the relation below:

(12) \( \log_2 (x_1 + 1) \neq \log_2 (x_2 + 1) \Rightarrow \psi_2(p_1) \neq \psi_2(p_2) \).

We have showed (only in this case) the injectivity of the function \( \psi_2 : M_p \cup D \to P_r \).

**Case 2.** If we get two prime numbers \( p_i \in D_1 \) and \( p_j \in D_2 \), for some \( i, j \) natural numbers. Being \( p_i \) and \( p_j \) the maximum primes contained in \( D_1 \) and \( D_2 \) respectively,
Next, by hypothesis we may assume that $p_i \neq p_j$, obtaining immediately,

\[(13) \quad \text{max. prime of } D_i \neq \text{max. prime of } D_j.\]

It is quickly seen that, from (13) we may write,

\[(14) \quad \psi_2(p_i) \neq \psi_2(p_j).\]  
So, we have showed that $\psi_2 : M_p \cup D \to P_r$ is injective.

This completes the proof of the Proposition 1.2.2

3. Applying Schröder--Bernstein’s Theorem.

In this section we shall show, our most important result of this brief research work. Before,

We must point out the following.

3.1 By definition $3 \in M_p$ and $3 \in D$, which shows clearly that $M_p \cap D \neq \emptyset$, although

being $D$ finite; $\text{Card} \,(M_p \cap D)$ is finite or not. Indeed, we can to show that,

$\text{Card} \,(M_p \cap D)$ is finite. Of course, if is not so; assume that $\text{Card} \,(M_p \cap D)$ is infinite, whence both sets $M_p$ and $D$ should be infinite; eventually $M_p$ would be infinite (we don’t knows), [15]; but in particular $D$ necessarily should be infinite, which contradicts Equation (1) at section 2.1 above. Taking in account all these arithmetical features, thus we may write:

\[(14) \quad \text{Card} \,(M_p \cap D) = \alpha \in \mathbb{N}.\]

Finally we can pay attention to the cardinality of Mersenne’s prime numbers. As we saw already, functions $\psi_1 : P_r \to M_p \cup D$ and $\psi_2 : M_p \cup D \to P_r$ are both injective.

Lemma 1, or Schröder-Bernstein’s Theorem guarantees that: there exists a Bijective

Function $\Psi : M_p \cup D \to P_r$, additionally Lemma3 or Cantor’s Theorem implies $M_p \cup D$ and $P_r$ are certainly, Equipollent sets. In other words, they have the same cardinality.

This is another way of saying that, the relation below,

\[(15) \quad \text{Card} \,(M_p \cup D) = \text{Card} \,(P_r) \text{ Holds.}\]

Consequently, from (15), by applying Cantorian set Theory; we may write at once:
Card\( (M_p) + Card(D) - Card(M_p \cap D) = Card(P).

Hence, substituting every known result in (16), we have obtained the equation:

\[
\text{(17) } Card(M_p) + \lambda - \alpha = \emptyset \quad \Rightarrow \quad Card(M_p) = \emptyset + (\alpha - \lambda).
\]

Let \(\alpha - \lambda = \beta\), it is clear that, in any case \(\beta \in \mathbb{Z}\), ending this section we have that, (17) becomes, \(Card(M_p) = \emptyset + \beta = \emptyset\). Summarizing, equation (2) is true.

3.2 In the current literature, it is well known that: there exists a relationship (so important like interesting) between Even Perfect Numbers and Mersenne’s prime. The classic result so-called \textbf{Euclid-Euler’s Theorem} characterize Mersenne primes and Perfect Numbers\cite{8}, which assert that: \(\theta\) is an \textbf{Even Perfect Number}, if and only if, it can be written in the form, \(\theta = 2^{p-1} \cdot (2^p - 1)\), where \(p\) and \(2^p - 1\) are both Primes.

It was not still \textbf{eighteenth century}, that Mathematicians become aware of such a Characterization\cite{4}\cite{9}, due to the works of famous and prolific Swiss Mathematician, \textbf{Leonhard Euler (1707-1783)}\cite{1}. He studied in every branch of Mathematics then known \cite{22}, and his extensive writings are more numerous than those of any other Mathematician\cite{10}.

3.3 As a result of Proposition 1.2.1; we have obtained the following statement,

\textbf{Corollary 3.3.1} There exist infinitely many \textbf{Even Perfect Numbers}.

We must point out a historical fact: Mathematician \textbf{Nicomachus of Gerasa (60-120 AC)}, conjectured this result \cite{7} He claimed: “ \textbf{Perfect Numbers are infinite}”. However, accordingly actual references, \textbf{He does not prove} that important assertion\cite{19}. 

\(\textbf{(16)}\)
4. Concluding Comments.

From several references [5], French monk: Jean Baptist Marin Mersenne (1588-1648), failed taking like primes the cases corresponding to \( p = 11, 23, 83, 131, 179, 191, 239, \) and 251. However, the posterity acknowledge his heroic contribution, by constructing its formula \( 2^p - 1 \), to generate the first known Messene’s prime [12], much earlier of computation age [21].

5. References

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