Non trivial zeros of Riemann’s Zeta function

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Abstract. In the following research work, we attempt to show, how and why Riemann’s Hypothesis is not false. Avoiding jargon and sophisticated mathematical terminology(whenever it’s possible) using basic concepts in complex Analysis, an indirect proof and classic results in numerical positive Series, likely convergence tests, Cauchy’s Theorem, Cardinality and Cantorian set Theory.

“Without doubt it would be desirable to have a rigorous proof of this proposition; however I have left this research aside for the time being after some quick unsuccessful attempts, because it appears to be unnecessary for the immediate goal of my study...” Bernhard Riemann (1859).

1. Notation and preliminary Statements.

1.1 Starting with, some of the main lines along which progress has been made, include methods or highly specialized mathematics background. Omitting that features, we have revised several well-known articles, dealing with Riemann’s Hypothesis, we believe that computational evidence [1][12], suggests that Riemann’s Conjecture probably is true (apart of Algorithm aspect); although, like everyone know; Probabilities means strictly: Tendency to a percentage, from zero to unity, and unfortunately the conjecture would be false. Numerous and interesting results [8], about Riemann’s Hypothesis are known, [11][19].

To gain insight, we assume all known Properties that characterize $\mathbb{R}$ like an ordered Field Archimedean and Complete [10]. Also We shall denote non-trivial zeros by the symbol $-T_z$. [4]

Whereas $\varphi$, shall be represent the infinitude of Prime Numbers and $Card.(\mathbb{R}) = \aleph$. 

2.1 First part, partitioning the Series $\zeta(z)$. 

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How to attack the guess. In our approach the function \( \zeta(z) \) must be partitioned, and next we proceed to study it separately. Indeed, through the current literature\(^9\), Riemann’s Zeta function (a particular case of Dirichlet-L Series) can be represented in the following way,

\( \zeta(z) = \sum_{n=1}^{\infty} n^{-z} \); Where \( z \in \mathbb{C} \) and \( n \) belongs to \( \mathbb{N} \), likely we can write the formula,

\[
\sum_{n=1}^{\infty} n^{-z} = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{x+iy}} ,
\]

immediately also we have the following expression,

\[
\zeta(z) = \sum_{n=1}^{\infty} n^{-z} = 0 \iff \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^z = \sum_{n=1}^{\infty} \frac{1}{n^{x+iy}} = 0 .
\]

However expanding the summands,

\[
\zeta(z) = \left( \frac{1}{1} \right)^z + \left( \frac{1}{2} \right)^z + \left( \frac{1}{3} \right)^z + \left( \frac{1}{4} \right)^z + \left( \frac{1}{5} \right)^z + \left( \frac{1}{6} \right)^z + \left( \frac{1}{7} \right)^z + \left( \frac{1}{8} \right)^z + \left( \frac{1}{9} \right)^z + \left( \frac{1}{10} \right)^z + \ldots
\]

\[
= \left\{ \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z} + \frac{1}{10^z} + \ldots \right\} + \left\{ \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \ldots \right\}
\]

we can get,

\[
\zeta(z) = \sum_{n=1}^{\infty} n^{-z} = \zeta_1(z) + \zeta_2(z),
\]

where evidently for short we may take respectively,

\[
\left\{ \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z} + \frac{1}{10^z} + \ldots \right\} = \zeta_1(z) \quad \left\{ \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \ldots \right\} = \zeta_2(z)
\]

We observe that, all terms inside both infinite series; are simply Egyptian Fractions, but raised to complex powers. However, grouping together all terms of the same parity, also we have,

\[
\left\{ \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z} + \frac{1}{10^z} + \ldots \right\} = \zeta_1(z) \quad \left\{ \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \ldots \right\} = \zeta_2(z)
\]

, but employing more brief and compact forms (for brevity) so we can see,

\[
\zeta_1(z) = \sum_{n=1}^{\infty} \left( \frac{1}{2n} \right)^z \quad \zeta_2(z) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z} .
\]

Although, according to the required Hypothesis; tacitly quantity \( y \) belongs to the set \( \mathbb{R} \).
Next we can write, $I_m(z) = y$ satisfy this restrictive and linear inequality below,

\[(9) \quad -\infty < I_m(z) < \infty; \] whereas the point $p_o\left(\frac{1}{2}, 0\right)$ belongs to the range $(-\infty, \infty)$.

What role it plays? Of course, $p_o$ is exactly the **mean point** of the critical band $(0, 1)$.

But it’s the controversial point of the Conjecture. Likely recall that, the Complex roots (Non-trivial zeros) of $\zeta(z)$, all they are placed symmetrically regarding to the real axis and the vertical line $\Re z x = \frac{1}{2}$.

3.1 At this point of the proof, we return to our goal in these notes, having constructed previously, all necessary tools in order to obtain our demonstration by contradiction.

Since $z = x + iy$, coherently with equation (6), working only with **non-Trivial Zeros** of the Zeta function, let be $z$ one of them, therefore by hypothesis, we can write,

\[(10) \quad \zeta(z) = 0 \text{ if and only if } \zeta(z) = \sum_{n=1}^{\infty} n^{-x-iy}; \quad -\infty < y < \infty \quad \text{and } R_s(z) = \frac{1}{2}.\]

As the reader may easily discern, both functions $\zeta_1(z), \zeta_2(z)$, satisfies the arithmetical conditions required at (8); for all elements contained in $T_z$, this enable us to write:

\[(11) \quad 0 + 0 = \sum_{1}^{\infty} \left(\frac{1}{2n}\right)^{\frac{1}{2}} + \sum_{1}^{\infty} \left(\frac{1}{2n-1}\right)^{\frac{1}{2}}.\]

We shall show this equation at Proposition 5.1.1

4. Setting key Requirements

4.1 Like anyone know, the remarkable hypothesis about The Riemann’s Conjecture is:

\[(12) \quad \zeta(z) = 0, \text{ if } I_m(z) = y \in \mathbb{R}, \text{ and simultaneously must be } R_s(z) = x = \frac{1}{2}.\]
Strategically following to Dr B. Riemann (1826-1866) also we adopt that: all non-trivial zeros of \( \zeta(z) \), by hypothesis always lie on the vertical line, \( x - \frac{1}{2} = 0 \).

4.2 By our purpose, we believe that is suffices to assume: there exists a \( z_o \in -z_y \), with \( R_e(z_o) \neq \frac{1}{2} \), being \(-\infty < y < \infty\).

It’s understood the restriction that, all those zeros are placed inside the critical band \((0,1)\). On the other hand, in basic Complex Analysis \([17]\) has been proved that, if \( z = \alpha + i\beta \) is a root of the function \( w = f(z) \), it’s a sufficient condition to say that the number, \( z_0 = \alpha - i\beta \) (It’s conjugate) also satisfy the equation, \( w = f(z_0) = 0 \) as well.

4.3 We here continue analysing the hypothesis, using equations \((10)\) and \((12)\) we can say that, in particular \( \zeta_1(z) \), should be equal to zero, by coherence with \((11)\), thus we have

\[
(14) \quad \zeta_1(z) = \sum_{n=1}^{\infty} \frac{1}{(2n)^z} = 0
\]

Although by using \((8)\), but whenever \( y \) belongs to \( \mathbb{R} \), we may choose for convenience the real number \( y = 0 \). It is a reasonable assumption, because it satisfy all requirements adopted at 4.1. Further, never we need divide by \( y \) in the rest of these pages. Substituting \( z \) and \( y \) in \( \zeta_1(z) \), immediately our equation \((15)\) may be written as follows,

\[
(15) \quad \zeta_1(z) = \sum_{n=1}^{\infty} \frac{1}{(2n)^z} = \sum_{n=1}^{\infty} \frac{1}{(2n)^y} = \sum_{n=1}^{\infty} \frac{1}{(2n)^{z+y}} = \sum_{n=1}^{\infty} \frac{1}{(2n)^y (2n)^y} \]

obtaining:
\[
\sum_{i=1}^{\infty} \frac{1}{(2n)^2} = \sum_{i=1}^{\infty} \frac{1}{(2n)^4} = \sum_{i=1}^{\infty} \frac{1}{(2n)^6}.1 = \sum_{i=1}^{\infty} \frac{1}{(2n)^2}.\text{ However, in accordance with } (12),
\]
since the value of \( x = \frac{1}{2} \), it is clear that finally in this direct way we have obtained:

\[
(16) \quad \zeta'(z) = \sum_{i=1}^{\infty} \frac{1}{(2n)^{1/2}} = 0.
\]

Which we have deduced algebraically from

4.4 Continuing, now we proceed to examine that infinite series

\[
\zeta(z) = \sum_{i=1}^{\infty} \frac{1}{(2n)^{1/2}}.
\]

A moment’s reflection shows that, being \( n \) a natural number, and (for the sake of completeness) necessarily we can see,

\[
(17) \quad \forall n \geq 1: n \geq n^{1/2} \Rightarrow 2n \geq 2\sqrt{n} \geq \sqrt{2n}. \text{ By employing the transitivity we may write,}
\]

\[
(18) \quad 2n \geq \sqrt{2n} \text{ or } \frac{1}{2n} \leq \frac{1}{\sqrt{2n}} \Rightarrow \sum_{i=1}^{\infty} \frac{1}{2n} \leq \sum_{i=1}^{\infty} \frac{1}{\sqrt{2n}}. \text{ Like the reader can see, this last series has } a_n = \frac{1}{\sqrt{2n}}, \text{ and it is obvious that } a_n \text{ vanishes to zero, when } n \to \infty.
\]

Furthermore, it remains decreasing and positive, whenever \( n \) satisfy \( 1 \leq n < \infty \).

But these conditions are only necessary not sufficient, so it would be divergent. However, we have together all the required hypothesis to apply, a well-known result in the theory of Numerical Series. Consequently now we shall employ a basic Cauchy’s Theorem. [6].

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4.5 The series \( \sum_{i=1}^{\infty} \frac{1}{\sqrt{2n}} \), converges if the integral \( \int_{1}^{\infty} \frac{dn}{\sqrt{2n}} = \alpha \); where \( \alpha \in \mathbb{R} \).

Continuing our task, we must calculate the value of that unknown quantity \( \alpha \):

Let us \( \alpha = \int_{1}^{\infty} \frac{dn}{\sqrt{2n}} = \lim_{\lambda \to \infty} \int_{1}^{\lambda} \frac{dn}{\sqrt{2n}} \), equivalently now we may rewrite,

\[
\alpha = \lim_{\lambda \to \infty} \frac{1}{\sqrt{2}} \int_{1}^{\lambda} n^{\frac{1}{2}} dn = \frac{2}{\sqrt{2}} \lim_{\lambda \to \infty} \left[ \sqrt{n} \right]_{1}^{\lambda} = \frac{2\sqrt{\lambda}}{\lambda} \lim_{\lambda \to \infty} \left[ \sqrt{\lambda} - \sqrt{1} \right]
\]

or

\[
\alpha = \sqrt{2} \left[ \sqrt{\infty} - \sqrt{1} \right] = \sqrt{2} \left[ \infty - 1 \right] = \sqrt{2} \left[ \infty \right] = \infty \Rightarrow \alpha \in \mathbb{R}.
\]

(21) So the value of \( \alpha = \sum_{i=1}^{\infty} \frac{1}{(2n)^{\frac{1}{2}}} \to \infty \), therefore that Series Diverges, contradicting (12) and (15) above. In such case, we shall return on this notable and crucial result later.

5. Other Important Results.

5.1 After their precedent argumentations, it’s natural ask the following question:

What happen if \( y \neq 0 \) inside the function \( \zeta(z) = \sum_{i=1}^{\infty} (2n-1)^{-i,y} \)? \( y = 0 \) No matter if, either \( y > 0 \); in accordance with previous conditions established at equation (13).

Usually, that Series must be studied in the context of Calcul in several Complex variables,

\[
\zeta(z) = f(n,y) = \sum_{i=1}^{\infty} (2n-1)^{-i,y} \quad \text{w} = (2n-1)^{-i,y}
\]

Let we know that both numbers \( n \) and \( y \) occur an infinitely many of times, that is to say:

\[
(22) \quad w = (2n-1)^{-i,y} \Rightarrow w = e^{ln(2n-1)}^{-i,y} \quad \text{or} \quad \lim_{(n,y) \to (-\infty,\infty)} w = e^{\lim_{(n,y) \to (-\infty,\infty)} (-i,y)[ln(2n-1)]}.
\]


Notice that, the Series \( \zeta_2(z) = f(n, y) = \sum_{i} (2n - 1)^{-i} \) therefore converges.

Like the reader can see that, the function \( \zeta_2(z) \) does contain infinitely many odd Numbers; in particular infinitely many primes \( p \), for a certain value of \( n \geq 2 \), all they satisfying the inequality \( 3 \leq p \prec \infty \). Euclid’s Theorem asserts that: \( Card.(\mathcal{P}) = \aleph_0 \).

Obviously, primality of quantity \( 2n-1 \); oscillates impredictably, following the irregular-asymptotic density of Prime numbers. [5].

5.2 Up to this point, we must establish the second part, that is: Why the R-H is not false.

It’s needed to prove the general case, for when \( I_m(z) = y \), is to be an arbitrary real Number, inside the interval, \( I_0 = (-\infty, 0) \cup (0, \infty) \). Notice that, \( y = o \notin I_0 \). By coherence with (21). What we wish to prove here is the important theorem below:

**Proposition 5.1.1** If \( y \neq o \), \( \forall z \in \mathbb{C}: \) The Series \( \zeta_1(z) = \sum_{i} (2n)^{-i} \) converges.

Let’s look at a proof of this basic assertion in Complex variables.

\[
(23) \quad \lim_{(y,n) \to (-\infty,\infty)} w = e^{\lim_{n \to \infty} \lim_{x \to \infty} \left[ L_n(2n-1) \right]} = e^{\lim_{n \to \infty} L_n(\infty)} = \frac{1}{e^{\infty,00}} = \frac{1}{e^{\infty}} = 1 = 0.
\]

Firstly, \( \zeta_1(z) = \sum_{i} \left( \frac{1}{2n} \right)^z = \sum_{i} (2n)^{-i} = \sum_{i} (2n)^{-x-i, y} = \sum_{i} (2n)^{-z} \cdot (2n)^{-i, y}, \) we have,

\[
(24) \quad \zeta_1(z) = \left[ \sum_{i} (2n)^{-x} \right] \cdot \left[ \sum_{i} (2n)^{-i, y} \right] = \beta \cdot \delta \text{, from this equation, tacitly one has,}
\]
\[ (25) \beta = \sum_{i=1}^{\infty} (2n)^{-i} \text{. Setting } w_i = e^{\frac{Ln[2n]}{2^n}} \Rightarrow \lim_{n \to \infty} w_i = e^{\lim_{n \to \infty} \frac{-x \cdot Ln[2n]}{x^2}} \text{, but } x = \frac{1}{2}, \]

\[ (26) \beta = e^{-\frac{1}{2}Ln(2^{\infty})} = e^{-\frac{1}{2}Ln(\infty)} = e^{-\infty} = e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0. \] Next we must find, the value of the series \( \delta = \sum_{i=1}^{\infty} (2n)^{-i} \cdot \). Analogously to precedent case, we may choose:

\[ (27) w_2 = e^{\lim_{n \to \infty} \frac{1}{n^2}Ln(2^{n})} = e^{\lim_{n \to \infty} \frac{1}{n^2}Ln(2^{n})} \text{. In like manner, we have the expression:} \]

\[ (28) w_2 = e^{-\frac{i}{2}Ln(2^{\infty})} = e^{-\frac{i}{2}Ln(\infty)} = e^{-\frac{i}{2} \cdot \infty} = e^{-\frac{i}{2} \cdot \infty} = \frac{1}{e^{i \cdot \infty}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0. \]

It’s clear that, this last equation justly gives us that, \( \delta = \sum_{i=1}^{\infty} (2n)^{-i} = 0. \) as well.

Indeed, we have already proved that the Series, \( \zeta(z) = \sum_{i=1}^{\infty} \left( \frac{1}{2n} \right)^z \) = \( \beta \cdot \delta \). as well.

We may claim that, equation \( (11) \), has been proved entirely but **partitioning** \( \zeta(z) \).

From \( (23) \) above, The Theory of Series in \( \mathbb{C} \) implies that \( \sum_{i=1}^{\infty} (2n-1)^{-z} \) is convergent,

which agree exactly with Hardy-Littlewoods Theorem; see, \[ 2][15]. \]

5.3 Without loss of generality, to each \( n_j \in \mathbb{N} \), for which \( \sum_{i=1}^{\infty} \frac{1}{(2n)^{1/2}} \) diverges, there exist a value \( , \ y_j \in \mathbb{N} \), satisfying these arithmetical and restrictive conditions below:

\[ (29) \ \zeta(z_y) = \zeta \left( \frac{1}{2} + iy_0 \right) \neq 0, \ \zeta(z_j) = \zeta \left( \frac{1}{2} + iy_1 \right) \neq 0, ..., \ \zeta(z_j) = \zeta \left( \frac{1}{2} + iy_j \right) \neq 0. \]
Due to the fact that; each input \( z_j = n_j \in \mathbb{N} \), corresponds to exactly one output: \( \zeta(z_j) \).

From (29) Cardinality of them is infinite. It follows that none their \( z_j \) belong to \( T_z \).

Of course, by construction we know the fact that, \( \forall z_j : \zeta(z_j) \neq 0 \).

5.4 Further, according to well-known references [4], there are infinitely many zeros belonging to \( T_z \), which agree with our partition \( \zeta(z) = \zeta_1(z) + \zeta_2(z) \),

\[ n \in \mathbb{N}, \text{ being } y \in \mathbb{R} \text{ and } x - \frac{1}{2} = 0. \] As required by our hypothesis.

whenever

Remark. It’s natural put the question below: How many points \( z_j \), does contains the sequence (29) above ?

Our answer is exactly: Aleph Null, because in these arithmetical circumstances, we may write the relation:

\[ \text{Card} \{ y_j : \zeta(y_j) \neq 0 \} = \aleph_0. \] Since, \( -\infty < y < \infty \), and \( \text{Card}(-z) = \mathfrak{c} \).

Summarizing, Transfinite arithmetic in Cantorian Set Theory, guarantee that the relation,

\[ \mathfrak{c} = 2^{\aleph_0} > \aleph_0. \] Holds. From (21), \( \sum_{1}^{\infty} \frac{1}{\sqrt{2n}} \) diverges, which contradicts our equation (26) above, so our assumption made at 4.2 above is wrong! meaning

by this: all non-trivial zeros of the function \( \zeta(z) = \zeta_1(z) + \zeta_2(z) \), have \( R(z) = \frac{1}{2}. \)

Plainly for this to happen, then yields to result that; Riemann’s Hypothesis Thus is not false; therefore is true.

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7. References


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