Geometrical Theorems on Perimeter Relationship of a Circle and Regular Polygons

Gabriel Dawit$^1*$ and Elilta Tesfu$^2$

1. Collage of health sciences, school of pharmacy Asmara-Eritrea

2. Collage of health sciences, Department of clinical laboratory science Asmara-Eritrea

Abstract

Perimeter of a circle and inscribed and circumscribed regular polygons has a mathematical relationship. The objectives are to show how a perimeter of a circle and of inscribed and circumscribed regular polygons is interrelating, and to show the relationship of radius length with side length of a regular of the same perimeter. All these relationships are derived from the previous formulas for perimeter of regular polygons and of a circle. There are three theorems provided in this research, \textit{GEBRIEL'S} and \textit{ELILTA'S} theorem describe the relationship between circumference of a circle and its inscribed and circumscribed regular polygons respectively, each relation has its own constant given in the tables. \textit{JAR} theorem describes a relation between a radius length of a circle and side length of regular polygons of the same circumference. The first two theorems help easily to calculate the needed length of perimeter to inscribed or circumscribed any regular polygon to a circle. \textit{JAR} theorem helps to calculate circumference of a circle and regular polygons in easy way. All these theorems have further advantage over the previous formulas for calculating perimeter of regular polygons, because no need to use any trigonometry ratios (sine, cosine and tan). These theorems are helpful for high school and above level students and also for those which are interested in engineering and designing of different household equipment.

\textbf{Key words}: Perimeter, circle, Inscribed regular polygons, circumscribed regular polygons, radius and side length

Introduction

Geometry is a study of the properties and measurements of figures composed of points and lines. Historically, interest in geometry has been stimulated by its application to the nature.$^5$ It is a very old science and grew out with the needs of people. The early Egyptians and Babylonians (4000-3000 B.C) were able to develop a collection of practical rules for measuring simple geometric
figures and for determining their properties. Applications of these principles were found in the building of the Pyramids and the great Sphinx.¹

Today geometry plays a great role in different part of the world in simplifying the work of engineers in designing and building. Our equipment’s in our houses are all designed and made with geometrical rules.

Geometry as part of mathematics deals with the main fundamental unit of figure, without dimension a point up to those figures of three dimensions with a surface area and with volume capacity.

Point is the fundamental unit in geometry without dimension, as a points combines they produce one dimensional figure which is a line, then one dimensional (lines) produce a plane surface of two dimensional finally three dimensional like containers is formed as a final product of geometry.

In geometry perimeter or circumference is a length of a polygon which bound a specific area surface in a plane figure. Regular polygons are closed figure with same side length and angle. A circle is a set of points with equal distance from the center; also we can define a circle as the final generation of a regular polygon of infinite number of sides.³

Therefore if a circle is considered as the last generation of regular polygons, all the regular polygons are interrelated with a circle. Depending on this idea, this research has done for the following objectives. In this research for the first two theorems there is a great modification in calculating the circumference of a circle that circumscribed about regular polygons and perimeter of regular polygons that inscribe inside a circle. Finally in this research paper there is a theorem which describes the relationship between radius and side length of regular polygons with the same circumference. There are numerous formulas to calculate the circumference of a circle that inscribed and circumscribed regular polygon.

**Methodology**

The pre-existed formulas are used to generate the relationship of the perimeter of a circle and inscribed and circumscribed regular polygons. For both **GEBRIEL’S** and **ELILTA’S** theorem the difference between one unit perimeter of a circle and of each regular is calculated and taken as constant (\(K_{pi}\) and \(K_{pe}\)) respectively, then the difference between any perimeter of a circle and regular polygon is the product of the constant and the radius of a circle. For **JAR** theorem the
same procedure had made as the above mentioned theorems to find the relationship between the radius and side length of a regular polygon of the same perimeter and $K_e$ is the constant used.

Symbols used: $\theta_2 = \frac{(n-2)90}{n}, \theta_1 = \frac{180}{n}$

### Pre-requisite formulas

<table>
<thead>
<tr>
<th></th>
<th>Inscribed regular polygons.</th>
<th>$P=2nrsin \theta_1$ or $P = 2nrcos \theta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.</td>
<td>Circumscribed regular polygons</td>
<td>$P=2nrtan \theta_1$ or $P = \frac{2nrtan \theta_1}{\tan \theta_2}$</td>
</tr>
<tr>
<td>3.</td>
<td>Circle</td>
<td>$P=2\pi r$</td>
</tr>
</tbody>
</table>
1. **GEBRIEL’S Theorem**

[Perimeter relationship between a circle and its inscribed regular polygon]

The difference between the perimeter of a circle and its inscribed regular polygon is the product of \( K \) and the radius of a circle i.e. \([K \pi r]\)

\[
P_{circle} = P_{inscribed \ reg.\ polygon} + K \pi r \quad \to \quad P_{circle} = P_{inscr} + K \pi r
\]

\[
K \pi = 2[\pi - n \sin \theta_1] \quad K = 2[\pi - n \cos \theta_2]
\]

**Proof**

Pre-requisite the perimeter of the inscribed regular polygon can be calculated as

\[
P = 2n r \sin \theta_1 \quad \text{or} \quad P = 2n r \cos \theta_2 \quad \text{and} \quad P_{circle} = 2\pi r
\]

\[
P_{circle} = P_{inscr} + K \pi r
\]
Application

1. How much perimeter length do you need to draw a regular pentagon that is inscribed in a circle with 248m circumference?

Solution: \( P_{\text{circle}} = 248 \text{m} \), \( n=5 \)

\[
K_{\text{pi}} = 0.405332784 \quad r = 39.47042589 \text{m}
\]

\[
\begin{array}{l}
P_{\text{inscr}} = P_{\text{circle}} - K_{\text{pi}} \times r \\
P_{\text{inscr}} = 248 \text{m} - 0.405332784 \times 39.47042589 \text{m} \\
P_{\text{inscr}} = 248 \text{m} - 15.99865762 \text{m} \\
P_{\text{inscr}} = 232.0013424 \text{m}
\end{array}
\]

\[
\begin{array}{l}
P = 2nr \sin \theta_1 \\
P = 10rsin36 \\
P = 232.0013424 \text{m}
\end{array}
\]

\[
\begin{array}{l}
P = 2nr \cos \theta_2 \\
P = 10rcos54 \\
P = 232.0013424 \text{m}
\end{array}
\]

2. Find the circumference of a circle that inscribes a regular 20-gon which has 230m perimeter.

Given \( n=20 \) \( P_{\text{inscr}}=230 \text{m} \)

\[
K_{\text{pi}} = 0.025806705 \quad r = \frac{230 \text{m}}{40 \times \sin 9} \\
r = 36.75660602 \text{m}
\]

\[
P_{\text{circle}} = P_{\text{inscr}} + K_{\text{pi}} \times r
\]

\[
P_{\text{circle}} = 230 \text{m} + 0.025806705 \times r
\]

\[
P_{\text{circle}} = 230.9485669 \text{m}
\]
2. ELILTA’S Theorem

[Perimeter relationship between a circle and its circumscribed regular polygon]

The difference between the perimeter of regular polygon and its inscribed circle is the product of $K_{pc}$ and the radius of a circle.

i.e. $[K_{pc} \times r]$ Therefore the relation is $P_{\text{reg.polygon}} = P_{\text{inscribed circle}} + [K_{pc} \times r]$

$$P_{\text{regular poly}} = P_{\text{circle}} + [K_{pc} \times r]$$

The constants are calculated as the following:

$$K_{pc} = 2[n\tan \theta_1 - \pi]$$

$$K_{pc} = 2\left[\frac{n}{\tan \theta_2} - \pi\right]$$

Proof

Pre-requisite the perimeter of the circumscribed regular polygon can be calculated $P=2nrtan \theta_1$ or $P = \frac{2nr}{\tan \theta_2}$ and $P_{\text{circle}} = 2\pi r$

$$P_{\text{regu.poly}} = P_{\text{circle}} + K_{pc} \times r$$

<table>
<thead>
<tr>
<th>$P_{\text{regu.poly}}$</th>
<th>$P_{\text{circle}} + K_{pc} \times r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{\text{regu.poly}} = 2\pi r + 2[n \tan \theta_1 - \pi] \times r$</td>
<td>$P_{\text{regu.poly}} = 2\pi r + 2\left[\frac{n}{\tan \theta_2} - \pi\right] \times r$</td>
</tr>
<tr>
<td>$P_{\text{regu.poly}} = 2\pi r - 2\pi r + 2nrtan \theta_1$</td>
<td>$P_{\text{regu.poly}} = 2\pi r - 2\pi r + \frac{2nr}{\tan \theta_2}$</td>
</tr>
<tr>
<td>$P_{\text{regu.poly}} = 2nrtan \theta_1$</td>
<td>$P_{\text{regu.poly}} = \frac{2nr}{\tan \theta_2}$</td>
</tr>
</tbody>
</table>

N.B $P_{\text{regu.poly}}$ is the circumscribed regular polygon.
Application

1. Find the perimeter of a CD room [regular heptagon in shape] which has a CD with 8cm radius inscribe in it.

\( K_{pc} = 0.458859356 \)

\[
\begin{array}{|c|c|}
\hline
P_{regu.poly} &= 2\pi r + K_{pc} \times r \\
P_{regu.poly} &= 50.26548246 + K_{pc} \times r \\
P_{regu.poly} &= 53.93635731m \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
P_{regu.poly} &= 2n \times r \times \theta_1 \\
P_{regu.poly} &= 53.93635731m \\
\hline
\end{array}
\]

\[
\begin{array}{|c|}
\hline
P_{regu.poly} &= \frac{2n \times r}{\tan \theta_2} \\
P = 53.93635731m \\
\hline
\end{array}
\]

2. How much length fence do we need to fence an inscribed circle in a square field?

Solution: \( P_{regular.polygon} = 140m \) \hspace{1cm} \( r = 17.5m \) \hspace{1cm} \( K_{pc} = 1.716814693 \)

\[
P_{regular.poly} = P_{circle} + K_{pc} \times r
\]

\[
\begin{array}{|c|}
\hline
P_{circle} &= P_{regular.poly} + K_{pc} \times r \\
P_{circle} &= 127.1449424m - 1.716814693 \times 17.5m \\
P_{circle} &= 109.9557429m \\
\hline
\end{array}
\]

Therefore we need 109.9557429m long fence.
3. JAR THEOREM

Side length and radius relationship  $\pi$ should be $\frac{22}{7}$

A. For regular polygons of 3,4,5,6 sided, if the perimeter of a regular polygon and the circumference of a circle is the same then the difference between the side length of regular polygon and the radius of a circle is:

$$P \times \left[ \frac{1}{n} - \frac{1.75}{11} \right] \text{ or }$$

$$P \times |k_e| \quad k_e \text{ is } \left[ \frac{1}{n} - \frac{1.75}{11} \right]$$

Where $P$ is the circumference or perimeter, $n$ is the number of sides and $k_e$ is the constant.

N.B this is true when $is \frac{22}{7}$.

Proof

$$P = 2\pi r \quad \text{side length } [s] = \frac{P}{n} \quad \text{and } r = \frac{P}{2\pi}$$

Therefore: $s - r = \frac{P}{n} - \frac{P1.75}{11}$

$$P = 2\pi r \rightarrow 2 \times \frac{22}{7} \times r$$

<table>
<thead>
<tr>
<th>$s - r$</th>
<th>$P$</th>
<th>$s$</th>
<th>$r$</th>
<th>$P$ is $s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s - r$</td>
<td>$\frac{P}{n}$</td>
<td>$\frac{22}{7} \times 1.75 \times r$</td>
<td>$\frac{77}{11} x r$</td>
<td>$\frac{n}{P}$</td>
</tr>
</tbody>
</table>

$\frac{P}{n}$ is $s$
Application:

1. Find the radius of a circle if the circumference of a circle is 280 cm.

   \[ s = \frac{280}{5} \]

   \[ s = 56 \text{ cm} \]

   \[ s - r = P \times \left( \frac{1}{n} - \frac{1.75}{11} \right) \]

   \[ r = s - 280 \left( \frac{1}{5} - \frac{1.75}{11} \right) \]

   \[ r = 56 - 11.454545 \]

   \[ r = 44.54545 \text{ cm} \]

   \[ r = \frac{280}{2\pi} \]

   \[ r = \frac{280 \times 7}{44} \]

   \[ r = 44.54545 \text{ cm} \]

B. If the perimeter of regular polygons of 7-sided and above (heptagon to n-sided polygon) and the circumference of a circle is the same then the difference between the radius \( r \) of a circle and the side \( s \) length of regular polygon is:

\[ P \times \left[ \frac{1.75}{11} - \frac{1}{n} \right] \text{ or} \]

\[ P \times \left| k_e \right| \]

Where \( n = \text{number of sides} \)

\[ P = \text{perimeter [circumference]} = 2\pi r \]

i.e. \[ r - s = \left[ \frac{1.75}{11} - \frac{1}{n} \right] P \]

\[ N.B \ this \ is \ exactly \ true \ for \ \pi = \frac{22}{7} \]

\[ r = \left[ \frac{1.75P}{11} - \frac{P}{n} \right] + s \]

\[ s = \left[ \frac{1.75P}{11} - \frac{P}{n} \right] - r \]
Proof pre-requisite  
\[ r = \frac{P}{2\pi} \quad \pi = \frac{22}{7} \quad s = \frac{P}{n} \]

\[ r - s = P\left[\frac{1.75}{11} - \frac{1}{n}\right] \]

\[ = \left[\frac{P \times 1.75}{11} - \frac{P}{n}\right] \]

\[ = \left[\frac{1.75 \times 2\pi r}{11} - \frac{P}{n}\right] \]

\[ = \left[\frac{1.75 \times 2 \times \frac{22}{7} \times r}{11} - \frac{P}{n}\right] \rightarrow \left[\frac{77r}{77} - \frac{P}{n}\right] \rightarrow \left[r - \frac{P}{n}\right] \]

\[ r - s = r - s \]

Application

1. If the circumference of a paracetamol tablet is 8cm, find the diameter of the tablet.

Solution: P=8cm  \( n=10 \)  \( s=0.8 \) cm

A.  
\[ r = \left[\frac{1.75P}{11} - \frac{P}{n}\right] + s \]

\[ r = \left[\frac{1.75 \times 8}{11} - \frac{8}{10}\right] + 0.8 \]

\[ r = \left[\frac{14}{11} - 0.8\right] + 0.8 \text{cm} \]

\[ r = \frac{14}{11} \]

\[ r = 1.2727273 \]

\[ \text{diameter} = r \times 2 \]

\[ D = 2.5454545 \text{cm} \]

B.  
\[ r = \frac{P}{2\pi} \]

\[ r = \frac{56}{44} \]

\[ r = 1.2727273 \]

\[ D = r \times 2 \]

\[ D = 2.5454545 \text{cm} \]
Result and Discussion

The results from the methodology all have the same application as the pre-existed formulas. The result obtained with the present formula is the same. **GEBRIEL’S theorem** provides infinite ways to calculate the circumference of a circle and other new way for calculating perimeter of inscribed regular polygons. In the same way **ELILTA’S theorem** provides infinite ways to calculate the circumference of a circle and other new way for calculating perimeter of circumscribed regular polygons. These both theorems have the following advantage over the pre-existed formulas:

1. From a single formula we can calculate both circumference of a circle and inscribed and circumscribed regular polygons.

2. There is no requirement of any trigonometry ratios (sine cosine and tan) for calculating the inscribed and circumscribed regular polygons.

3. We can easily know how much length of perimeter is required to inscribe inside or circumscribed about a circle.

**JAR theorem** describes the relationship between radius of a circle and side length of a regular polygon for both with the same circumference or perimeter. This provides ways for calculating radius of circle and perimeter of a circle and regular polygons.

Conclusion

**ERENA’S, ELILTA’S** and **JAR theorem has the same application as the preexisted formulas for circumference, and radius of a circle and side length of regular polygons. Perimeter of inscribed and circumscribed regular polygons is calculated easily by ERENA’S and ELILTA’S theorems.**

Acknowledgement

We praise the almighty God for his lifelong guidance. Our further appreciation passes to our family and friends for their moral supports.

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* gabrieldawit@gmail.com or elilta2014@yahoo.com
**Table 1.** \( K_{pi} \) value of each inscribed regular polygon.

<table>
<thead>
<tr>
<th>Inscribed Regular polygon of n-sides</th>
<th>( K_{pi} )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>2.054559769</td>
</tr>
<tr>
<td>Square</td>
<td>0.858407346</td>
</tr>
<tr>
<td>Pentagon</td>
<td>0.491112000</td>
</tr>
<tr>
<td>Hexagon</td>
<td>0.322508961</td>
</tr>
<tr>
<td>Heptagon</td>
<td>0.229429678</td>
</tr>
<tr>
<td>Octagon</td>
<td>0.172113455</td>
</tr>
<tr>
<td>Nonagon</td>
<td>0.134139454</td>
</tr>
<tr>
<td>Decagon</td>
<td>0.107604308</td>
</tr>
<tr>
<td>11-gon</td>
<td>0.088298768</td>
</tr>
<tr>
<td>12-gon</td>
<td>0.073797655</td>
</tr>
<tr>
<td>13-gon</td>
<td>0.062619565</td>
</tr>
<tr>
<td>14-gon</td>
<td>0.053815987</td>
</tr>
<tr>
<td>15-gon</td>
<td>0.046755771</td>
</tr>
<tr>
<td>16-gon</td>
<td>0.041005224</td>
</tr>
<tr>
<td>17-gon</td>
<td>0.036258097</td>
</tr>
<tr>
<td>18-gon</td>
<td>0.032293000</td>
</tr>
<tr>
<td>19-gon</td>
<td>0.028946584</td>
</tr>
<tr>
<td>20-gon</td>
<td>0.026096152</td>
</tr>
<tr>
<td>...n-sided</td>
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</tr>
<tr>
<td></td>
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### Table 2. $K_{pc}$ value of each circumscribed regular polygon.

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<td>0.172115345</td>
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<td></td>
</tr>
<tr>
<td>$K_{pc} = 2[\frac{n}{\tan \theta_2} - \pi]$</td>
<td></td>
</tr>
</tbody>
</table>
### Table 3. $K_e$ value of each regular polygon

| Regular polygon of n-sides | $|K_e|$-value |
|----------------------------|--------------|
| Triangle                   | 0.1742       |
| Square                     | 0.0909       |
| Pentagon                   | 0.0409       |
| Hexagon                    | 0.00758      |
| Heptagon                   | 0.01623      |
| Octagon                    | 0.03409      |
| Nonagon                    | 0.04798      |
| Decagon                    | 0.05909      |
| 11-gon                     | 0.06818      |
| 12-gon                     | 0.07576      |
| 13-gon                     | 0.08217      |
| 14-gon                     | 0.08766      |
| 15-gon                     | 0.09242      |
| 16-gon                     | 0.09660      |
| 17-gon                     | 0.10030      |
| 18-gon                     | 0.10354      |
| 19-gon                     | 0.10646      |
| 20-gon                     | 0.10909      |
| ...n-sided                 |              |

$$|K_e| = \left| \frac{1}{n} - \frac{1.75}{11} \right| \text{ or } \left| \frac{1.75}{11} - \frac{1}{n} \right|$$
Reference