A FORMULA FOR SOLVING GENERAL QUINTICS; A FOUNDATION FOR SOLVING GENERAL POLYNOMIALS OF HIGHER DEGREES

ABSTRACT

I explore possible methods of solving general quintics and higher degree polynomials. In this paper I will attempt to show that each quintic has an auxiliary cubic equation. I will therefore attempt to bring a method of deriving a general quintic and its possible auxiliary cubic equation forms. I propose the same method to be used to generate higher degree polynomials.

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METHOD

Consider the simple cubic equation:

\[ x^3 = x + 2 \] \hspace{1cm} 1

Squaring both sides of the above equation we obtain the equation:

\[ x^6 = (x + 2)^2 = x^2 + 4x + 4 \] \hspace{1cm} 2

Multiplying both sides of equation 1 by \( x^3 \) we obtain the equation:

\[ x^6 = (x + 2)x^3 = x^4 + 2x^3 \] \hspace{1cm} 3

Subtracting equation 2 from 1 and rearranging we obtain the equation:

\[ x^4 + 2x^3 - x^2 - 4x - 4 = 0 \] \hspace{1cm} 4

Rearranging equation 1:

\[ x^3 - x - 2 = 0 \] \hspace{1cm} 5

The cubic equation 5 is an auxiliary equation of the quartic equation 4 since:

\[ x^4 + 2x^3 - x^2 - 4x - 4 = (x^3 - x - 2)(x - 2) = 0 \] \hspace{1cm} 6

This analysis projects the idea that each polynomial equation possesses auxiliary lower degree polynomial. I will use this concept to come up with a method that can be used to solve any general quintic.

Consider the cubic equation:

\[ x^3 = (x + a)^2 + bx + c = x^2 + x(b + 2a) + a^2 + c \] \hspace{1cm} 7
Again consider the cubic equation:

\[ x^3 = x^2 + dx + e \]  

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Multiplying equation 8 by 7 we obtain the equation:

\[ x^6 = x^4 + x^3(b + 2a + d) + x^2(a^2 + c + e + bd + 2ad) + x(dc + be + 2ae + a^2d) + ce \]  

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Multiplying both sides of equation 7 by \( x^3 \) we obtain the equation:

\[ x^6 = x^5 + x^4(b + 2a) + x^3(a^2 + c) \]  

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Multiplying both sides of equation 8 by \( x^3 \) we obtain the equation:

\[ x^6 = x^5 + x^4d + x^3e \]  

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Taking equation 10 subtracting equation 9 and rearranging:

\[ x^5 + x^4(b + 2a - 1) + x^3(a^2 + c - b - 2a - d) - x^2(a^2 + c + e + bd + 2ad) - x(dc + be + 2ae + a^2d) - ce = 0 \]  

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I will call equation 13 an extended quintic equation.

Rewriting the cubic equations 7 and 8:

\[ x^3 - x^2 - x(b + 2a) - a^2 - c = 0 \]  

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\[ x^3 - x^2 - dx - e = 0 \]  

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The product of equation 7 and 8 is given by:

\[ \left[ x^3 - \{(x + a)^2 + bx + c\} \left[ x^3 - x^2 - dx - e \right] \right] = 0 \]  

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Equation 10 can be rewritten as:

\[ (x^3 + (x + a)^2 + bx + c)x^3 - x^6 = ((x + a)^2 + bx + c)x^3 = 0 \]  

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Adding equation 16 to 17 and simplifying we get the equation:

\[ x^3(-x^2 - dx - e) - \{(x + a)^2 + bx + c\}(-x^2 - dx - e) = 0 \]

Or \[ [x^3 + dx + e][x^3 - \{(x + a)^2 + bx + c\}] = 0 \]  

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The expansion of equation 18 is given by

\[ x^5 + x^4(d - 1) + x^3(b - 2a - d + e) + x^2(c - a^2 + d(b + 2a) - e) + x[d(c - a^2) + e(b - 2a) + e(c - a^2)] = 0 \]  

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Equation 19 is a second extended general quintic whose factorized form is equation 18.
The second extended general quintic can be used to obtain a general quintic formula since

\[ a_4 = d - 1 \] \[ a_3 = b - 2a - d + e \] \[ a_2 = c - a^2 + d(b + 2a) - e \] \[ a_1 = d(c - a^2) + e(b - 2a) \] \[ a_0 = e(c - a^2) \]

By equation 19:

\[ d = a_4 + 1 \]

Let \( p = c - a^2 \)

Let also \( q = (b - 2a) \)

Then \( a_3 = p - (a_4 + 1) + e \)

\[ a_2 = p + (a_4 + 1)q - e \]

\[ a_1 = p(a_4 + 1) + eq \]

Equation 23 can be written as:

\[ a_0 = -ep \]

Substituting 30 into 26 and simplifying:

\[ p^2 - p(a_4 + a_3 + 1) + a_0 = 0 \]

\[ p = \frac{(a_4 + a_3 + 1) \pm \sqrt{(a_4 + a_3 + 1)^2 - 4a_0}}{2} \]

By equation 30:

\[ e = \frac{-a_0}{p} \]

By equation 28:

\[ q = \frac{a_2 + e - p}{a_4 + 1} \]

To determine \( b \) and \( c \), choose \( a \) and then calculate \( b \) and \( c \) using the equations 25 and 26.

**Example 1**

Solve \( x^5 - x + 1 = 0 \)
Solution

\[ a_4 = 0 \Rightarrow d = 1 \]
\[ p = \frac{1 \pm \sqrt{1 - 27}}{2} = \frac{1 \pm i\sqrt{3}}{2} \]
Take \( p = \frac{1 + i\sqrt{3}}{2} \)
\[ e = \frac{2}{1 + i\sqrt{3}} \]
\[ q = e - p = \frac{3 - i\sqrt{3}}{1 + i\sqrt{3}} \]
Take \( a = 1; c = p + 1 = \frac{3 + i\sqrt{3}}{2} \)
\[ b = q + 2 = \frac{5 + i\sqrt{3}}{1 + i\sqrt{3}} \]

The auxiliary equations of the quintic are:
\[ x^2 + x + \frac{2}{1 + i\sqrt{3}} = 0 \]
\[ x^3 - x^2 + \frac{3 - i\sqrt{3}}{1 + i\sqrt{3}} x + \frac{1 + i\sqrt{3}}{2} = 0 \]

The above two quadratic and cubic equations can be solved by their respective formulae.

The roots of the auxiliary quadratic equation are:
\[ x_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{-7 + i\sqrt{3}}}{2(1 + i\sqrt{3})} \]

One of the roots of the auxiliary cubic equation is:
\[ x_3 = \frac{\sqrt[3]{\frac{4}{27} + \frac{16}{27} + \frac{1}{27 (3(1 + i\sqrt{3}))}}}{3} - \frac{\sqrt[3]{\frac{4}{27} + \frac{16}{27} + \frac{1}{27 (3(1 + i\sqrt{3}))}}}{3} + \frac{1}{3} \]

CONCLUSION

A general formula for solving quintics is achievable. Irreducible quintics can be solved by the formula method proposed. Given \( a_0, a_1, a_2, a_3 \) and \( a_4 \) one can determine \( a, b, c, d, e \) and hence solve the quintic equation.
REFERENCES


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