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DISPROOF OF ABEL'S IMPOSSIBILITY THEOREM

ABSTRACT

I prove that solution to the general quintic equation is expressible by radicals, that we can state the equation of each root in terms of the coefficients of the general quintic. By the fundamental theorem of algebra, a general quintic should have five roots.

AUTHORS NAME: SAMUEL BONAYA BUYA

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Neil's Abel produced an explicit proof to defend the impossibility of expressing the solution of the general quintic by radicals. He started by assuming the existence of a solution in radicals of a general quintic and then went ahead to disprove the assumption using a carefully selected strategy . The assumption in Abel's impossibility proof is:

1. "Assume that a solution of a general quintic can be expressed in radicals. That is we can state each equation for the root $\boldsymbol{y}$ in terms of **a, b, c, d, e** such that
$y^5 - ay^4 + by^3 - cy^2 + dy - e = 0$."

I show that there exists another framework by which the rejection of this assumption is uncalled for. Polynomial equations with symmetric group $\mathbf{S_4}$ are and below are solvable in radicals because it is there exists an easy connection between the root and the coefficients four or less coefficients. In the case of general quintic there is an additional fifth coefficient apart from the fifth root because of the $\mathbf{S_5}$ symmetric group that puts an additional constraint. To resolve the problem, one the coefficient is treated as a parameter dependent on the roots and with connections to the other coefficients.

Equation of roots of quintics in terms of its coefficients

Consider a general quintic:

$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ ------------------------------ 1.

If **r, s, t, u, v** are the roots of a general quintic, then each root can be expressed in terms of the coefficients of the roots [1] such that:

$a_4 = -(\boldsymbol{r} + \frac{a_3}{\boldsymbol{r}} + \frac{a_2}{\boldsymbol{r}^2} + \frac{a_1}{\boldsymbol{r}^3} + \frac{a_0}{\boldsymbol{r}^4})$ ------------------------- 2a

$a_4 = -(\boldsymbol{s} + \frac{a_3}{\boldsymbol{s}} + \frac{a_2}{\boldsymbol{s}^2} + \frac{a_1}{\boldsymbol{s}^3} + \frac{a_0}{\boldsymbol{s}^4})$ ------------------------- 2b

$a_4 = -(\boldsymbol{t} + \frac{a_3}{\boldsymbol{t}} + \frac{a_2}{\boldsymbol{t}^2} + \frac{a_1}{\boldsymbol{t}^3} + \frac{a_0}{\boldsymbol{t}^4})$ ------------------------- 2c

$a_4 = -(\boldsymbol{u} + \frac{a_3}{\boldsymbol{u}} + \frac{a_2}{\boldsymbol{u}^2} + \frac{a_1}{\boldsymbol{u}^3} + \frac{a_0}{\boldsymbol{u}^4})$ ------------------------- 2d

$a_4 = -(\boldsymbol{v} + \frac{a_3}{\boldsymbol{v}} + \frac{a_2}{\boldsymbol{v}^2} + \frac{a_1}{\boldsymbol{v}^3} + \frac{a_0}{\boldsymbol{v}^4})$ -------------------------- 2e

Each of the above roots takes the form:

$\boldsymbol{r}^{\mathbf{5}} + a_4\boldsymbol{r}^{\mathbf{4}} + a_3\boldsymbol{r}^{\mathbf{3}} + a_2\boldsymbol{r}^{\mathbf{2}} + a_1\boldsymbol{r} + a_0 = 0$

Substituting 2a into

$x^5 - (\boldsymbol{r} + \frac{a_3}{\boldsymbol{r}} + \frac{a_2}{\boldsymbol{r}^2} + \frac{a_1}{\boldsymbol{r}^3} + \frac{a_0}{\boldsymbol{r}^4})x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ ------------- 3

Equation 3 can further be factorized into:

$x^4(x - \boldsymbol{r}) - \frac{a_3x^3}{r}(x - \boldsymbol{r}) - \frac{a_2x^2}{r^2}\left(x^2 - \boldsymbol{r}^{\mathbf{2}}\right) - \frac{a_1x}{r^3}\left(x^3 - \boldsymbol{r}^{\mathbf{3}}\right) - \frac{a_0}{r^4}(x^4 - \boldsymbol{r}^{\mathbf{4}}) = 0$ ----------- 4

Equation 4 can further be simplified to:

$\{x - \boldsymbol{r}\}\{x^4 - \frac{a_3x^3}{\boldsymbol{r}} - \frac{a_2x^2}{r^2}(x + \boldsymbol{r}) - \frac{a_1x}{\boldsymbol{r}^3}\left(x^2 + x\boldsymbol{r} + \boldsymbol{r}^{\mathbf{2}}\right) - \frac{a_0}{\boldsymbol{r}^4}(x^{\mathbf{3}} + x^2\boldsymbol{r} + x\boldsymbol{r}^{\mathbf{2}} + \boldsymbol{r}^{\mathbf{3}}\}$ ------- 5

By equation 2a the parameter a_4 is dependent on r.

By expressions 5 all quintics can be factorized into linear and quartic factors over all rationals.

By equation 5 given **r**, a_3, a_2, a_1 and a_0 the solution of a general quintic equation is always expressible in radicals.

Solution of a general quintic in radicals

2. In the form of equation 3 above the solution of the general quintic is represented using the coefficients a_3, a_2, a_1 and a_0 and a parameter **r** (representing the coefficient a_4). In this transformed form the solution of a general quintic is always expressible by radicals.

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REFERENCE

1. **Buya S. B. (2013). A calculator for polynomial equations of degree five and above. Open science repository mathematics, Online (open access), e23050424. doi: 10.7392/openaccess.23050424**