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The "Weitzman-Gollier puzzle" is not a paradox but a mistake, and it is most likely moot.

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Abstract

The "Weitzman-Gollier puzzle" refers to the observation that, when interest rates are uncertain, the expected values of discount factors and compound factors for a pair of present and future values yield different certainty-equivalent discount rates, which are either decreasing or increasing as a function of time, and either tend to the lowest or to the highest possible interest rate. There were many attempts to both resolve this apparent paradox and to justify the conclusion that discount rates should be declining in the long term, but none explained why the certainty-equivalents diverge. The cause of the divergence was found to be Weitzman's incorrect formulation of the expected discount factor in his 1998 article. Correcting for this, the puzzle disappears, and Weitzman's model shows that discount rates should be increasing in the long term. However, the entire question is moot, as it depends on annual interest rates displaying nearly perfect autocorrelation.

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1. Introduction

Discounting plays a central role in cost-benefit analysis because it is used to compare monetary values of different time-periods. Numerous controversies surround the concept. One strand of research that has garnered considerable attention is Martin L. Weitzman's (1998) suggestion that for the very long run, discount rates should be declining and will tend to the lowest possible rate.

Gollier and Weitzman (2009) state that "In a pair of articles, Weitzman (1998, 2001) proposed the idea that what should be probability averaged is not future discount rates but future discount factors. In other words, one should not apply the average discount rate $\Sigma_i\, p_i\, r_i$ as if it were a time independent constant. Instead, one should apply the time-dependent average discount factor $A(t) = \Sigma_i\, p_i \exp(-r_i\, t)$," the corresponding certainty-equivalent discount rate of which is

$$R^W(t) = -(1/t)\ln(\Sigma_i\, p_i \exp(-r_i\, t)) \qquad (1)$$

and which tends to the lowest possible rate for large t.

Gollier inverted this reasoning in a series of articles (2004, 2009a, 2009b) deriving a certainty-equivalent rate, not based on discounting net future values but on compounding net present values instead. The corresponding certainty-equivalent discount rate is

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$$R^G(t) = (1/t)\ln(\Sigma_i p_i \exp(r_i t)) \quad (2)$$

which tends to the highest possible rate for large t.

This is the essence of the paradox: certainty-equivalent rates based on the net present value rule (R^W) decrease with time, while those based on the net future value rule (R^G) increase. Numerous papers have tried to provide an explanation for this incongruence.

In "Discounting the Long-Distant Future: A Simple Explanation for the Weitzman-Gollier-Puzzle" (2008) Wolfgang Buchholz and Jan Schumacher state "We show that, while Weitzman's use of the present value approach may indeed seem questionable, its outcome, *i.e.,* a discount rate that is declining over time, is nevertheless reasonable, since it can be justified by assuming a plausible degree of risk aversion."

Gollier tackled this puzzle in "Expected Net Present Value, Expected Net Future Value, and the Ramsey Rule" (2009) He wrote "In this paper, we propose to introduce risk aversion into the picture to solve the puzzle" and concluded that "Weitzman and Gollier's approaches lead to the same term structure of discount rates when consumption paths are optimal."

In "How Should the Distant Future be Discounted When Discount Rates are Uncertain?" (2009) Gollier and Weitzman conclude "When future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value."

In "What's the Rate? Disentangling the Weitzman and the Gollier Effect" (2012) Christian P. Traeger concludes that "We show that increasing and decreasing discount rates mean different things, can coexist, are created by different channels through which risk affects evaluation."

In "Yes, We Should Discount the Far-Distant Future at Its Lowest Possible Rate: A Resolution of the Weitzman–Gollier Puzzle" Mark C. Freeman (2013) "shows that Weitzman (1998) is 'right' and that [...] the far-distant future should be discounted at its lowest possible rate."

Weitzman's recommendation is well on its way to becoming a new orthodoxy among people worried about the "excessive" discounting that results from the application of constant discount rates hundreds of years into the future. The OECD's cost-benefit analysis manual (Pearce *et al*, 2006) endorses this approach. The UK Government has formally adopted it for its public spending decisions. (HM Treasury Green Book, 2003).

Undoubtedly then, this is an important question. It is an academic discussion that could have significant real world consequences, as it is likely to affect the allocation of real resources.

This paper is organized as follows: Section 2 describes the cause of the apparent paradox; Section 3 analyzes the effect of autocorrelation of interest rates through time that is implicit in Weitzman's model, as this assumption is a key element of the apparent paradox; Section 4 concludes.

2. The cause of the puzzle

The objective of discounting is to ascertain the present value of a future sum of money in the presence of a capital market in which money invested in the present will be retuned with interest in the future. It is worth restating the obvious because the Weitzman-Gollier puzzle resides precisely in the breakdown of this relationship. If investing the present value of a future sum at the interest rate used for discounting does not yield the same future value, then something must be wrong.

The relationship between a present value (PV) and the corresponding future value (FV) given interest rate r must be:

$$\mathrm{FV} = \mathrm{PV}\exp(\, r\, t\,) \quad (5)$$

or

$$\mathrm{PV} = \mathrm{FV} / \exp(\, r\, t\,) \quad (6)$$

The interest rate implicit in any pair of PV, FV can be derived from either (5) or (6) and is:

$$r = (1/t)\ln(\,\mathrm{FV} / \mathrm{PV}\,) \quad (7)$$

To detect the origin of the Weitzman-Gollier paradox this relationship has to be analyzed when r is uncertain. For clarity and to provide numerical examples, this will be modeled using a discrete probability distribution of only two values, with the properties shown in Table 1.

Table1

Simple interest rate uncertainty

First rate of interest	1%
Second rate of interest	5%
Probability of 1st rate	50%
Probability of 2nd rate	50%

On this basis we have:

$$\mathrm{FV} = p_1\,\mathrm{PV}\exp(\, r_1\, t\,) + p_2\,\mathrm{PV}\exp(r_2\, t) \quad (8)$$

$$\mathrm{FV} = \mathrm{PV}\,(\, p_1 \exp(\, r_1\, t\,) + p_2 \exp(\, r_2\, t\,)\,) \quad (9)$$

where p_1 and p_2 are the probabilities of r_1 and r_2 being the prevailing market interest rate. Rearranging this we have that

$$\mathrm{PV} = \mathrm{FV} / (p_1 \exp(\, r_1\, t\,) + p_2 \exp(\, r_2\, t\,)\,) \quad (10)$$

Clearly no paradoxical behavior is possible in the above. PV will always be the present value of FV, and FV will always be the future value of PV, no matter what the interest rates or probabilities are. The discount factor $1 / (p_1 \exp(\, r_1\, t\,) + p_2 \exp(\, r_2\, t\,)\,)$ is the reciprocal of the compound factor $p_1 \exp(\, r_1\, t\,) + p_2 \exp(\, r_2\, t\,)$ and vice versa.

If we set PV = 1, to determine what a dollar's future value will be, then

$$FV = p_1 \exp(r_1\ t) + p_2 \exp(r_2\ t) \tag{11}$$

and if we define R to be the certainty-equivalent interest rate, then

$$FV = \exp(R\ t) \tag{12}$$

from which it follows that

$$R = (1 / t) \ln(p_1 \exp(r_1\ t) + p_2 \exp(r_2\ t)) \tag{13}$$

which is exactly the same as R^G in (2) above, adapted to the simple model used here. The certainty-equivalent discount rate can also be derived from the PV definition, when FV is set equal to 1.

$$PV = 1 / (p_1 \exp(r_1\ t) + p_2 \exp(r_2\ t)) \tag{14}$$

and as

$$PV = 1 / \exp(R\ t) \tag{15}$$

it follows that

$$1 / \exp(R\ t) = 1 / (p_1 \exp(r_1\ t) + p_2 \exp(r_2\ t)) \tag{16}$$

and that

$$R = (1 / t) \ln(p_1 \exp(r_1\ t) + p_2 \exp(r_2\ t)) \tag{17}$$

which is exactly the same as (13) and again corresponds exactly to R^G. As R is a function of t, it is clear that as t increases R will tend to the highest possible rate, regardless of whether NPVs or NFVs are being calculated.

How is R^W different from the correct certainty-equivalent value, which is R^G? Its expression for our simple case, see (1) above, is as follows:

$$R^W (t) = - (1 / t) \ln(p_1 \exp(-r_1\ t) + p_2 \exp(-r_2\ t)) \tag{18}$$

The root of the problem lies in the expected discount factor used by Weitzman, which he defined as the expected value of the discount factors corresponding to the possible interest rates:

$$p_1 / \exp(r_1\ t) + p_2 / \exp(r_2\ t) \tag{19}$$

which looks plausible enough at first sight, considering that the deterministic discount factor is $1 / \exp(r\ t)$, but which is unfortunately an error, for the correct expected discount factor is the inverse of the expected compound factor, as shown in expression (10):

$$1 / (p_1 \exp(r_1\ t) + p_2 \exp(r_2\ t)) \tag{20}$$

How can Weitzman's expected discount factor be interpreted? It is actually a compound factor when the product $r \cdot t$ is negative. Expression (19) will convert a PV into a FV correctly if either r or t is negative. See expression (11). Having negative r would correspond to a capital market in which resources are stored for a fee, rather than used productively. Having t negative would imply reversing the flow of time. Both would behave deceptively like discount factors in normal capital markets, thus making it difficult to detect at once that something is amiss[2], but the results are obviously not the same.

Table 2

Errors of Weitzman discounting

	Weitzman discounting					Correct discounting			
Year	**Cert. equiv. Disc. rate**	**PV of FV=1**	**FV of comp. PV**	**Disc. error**		**Cert. equiv. disc rate**	**PV of FV=1**	**FV of comp. PV**	**Disc. error**
1	2.98%	0.97063963	1.000400053	-0.04%		3.02%	0.9702515	1	0.00%
100	1.67%	0.18730869	14.15411642	-2.65%		4.33%	0.0132335	1	0.00%
200	1.35%	0.06769034	745.7395806	-3.31%		4.65%	9.077E-05	1	0.00%
300	1.23%	0.02489369	40689.19786	-3.54%		4.77%	6.118E-07	1	0.00%
400	1.17%	0.00915782	2221528.13	-3.65%		4.83%	4.122E-09	1	0.00%
500	1.14%	0.00336897	121291299.4	-3.72%		4.86%	2.778E-11	1	0.00%

Table 2 compares the results of using both Weitzman's and the correct PV factors with the data of the simple model presented. The explanation for the divergence in certainty-equivalent discount rates is simple. Because compound factors are exponential, future values grow proportionally larger for higher rates of interest, skewing the expected value of the compound factor ever higher with the passage of time. This is a consequence of the facts that (a) interest rates are positive, and (b) a two time-period stochastic model behaves as if annual rates were perfectly autocorrelated. This effect is the same, of course, for both borrowers and lenders, since in any given capital market transaction the interest rate is the same for both parties. It is the sign of their respective payoffs that tells them apart, not the sign of the interest rate used to compute those payoffs.

Weitzman discounting, in contrast, either compounds with negative interest rates or reverses the direction of the flow of time. Both result in the compounding mechanism being put into reverse, which is conceptually different from discounting even if, in the deterministic case, their results are the same. It is now future values (with negative interest rates) or earlier values (when travelling backwards in time) that become proportionally lower due to negative compounding, which is not discounting. Neither of these processes corresponds to the reality that Weitzman intended to model, for he did not postulate negative interest rates, nor did he claim to observe a reversal of the arrow of time.

The PV of a future sum computed with Weitzman discounting can be invested at a profit in the capital market, as shown by the FV of the computed PV in Table 2. An investor who uses Weitzman discounting leaves room for arbitrage, which is evidence of the fact that he has not fully optimized his decisions by reference to the capital market. The return of the arbitrageur is given by the absolute value of the discounting error shown in

[2] This is another example of how tricky computing expected values of non-linear functions can be. The minus sign leading expression (1) might have offered a warning, however. It masks the fact that the interest rate implicit in the expression is negative.

Table 2, which also shows by how much using the Weitzman certainty-equivalent rate will short-change the investor.

If the correct expected discount factor is used, then there is no basis for the assertion that discount rates should be declining. In fact, it follows from Weitzman's model that they should be increasing and will tend to the highest possible rate.

3. The effect of correlation

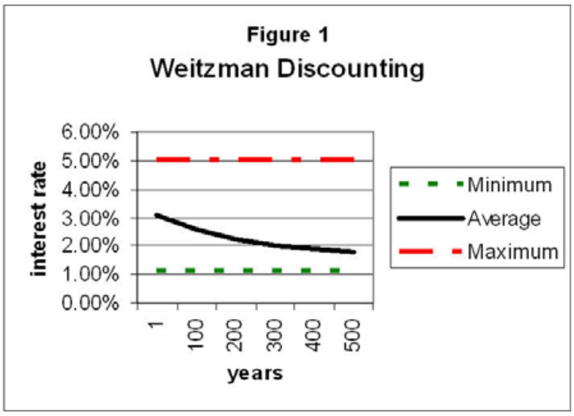
A key premise underlying the Weitzman-Gollier puzzle is the implicit assumption that interest rates are perfectly correlated through time. This is a necessary consequence of having a two period model represent the very long run, in which a single interest rate will prevail unaltered between year 0 and year *t*. Gollier (2009) concedes "that the decreasing nature of the term structure obtained in this framework depends heavily upon the assumption that shocks on the interest rate are permanent. If they are purely transitory, the term structure of discount rates should be flat."

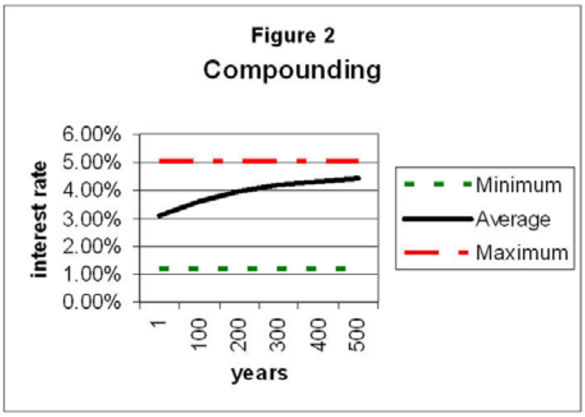
Despite this admission, the real world policy recommendation remained, albeit with a mild caveat. Gollier and Weitzman (2009) state: "When future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value. Empirically, this important feature can have significant ramifications for climate-change CBA — by weighting the distant future much more heavily than is done by standard exponential discounting at a constant rate."

Richard Newell and William Pizer (2001) empirically examined the question of autocorrelation of interest rates and their reversion to the mean (meaning the return to a long-run average value that is stochastic) and found that "[the] inconsistency between the mean-reverting forecasts and the realized interest rate is particularly troubling because we know that the lower range of possible interest rates ultimately determines the future certainty-equivalent rate. Because the random walk model does a better job of predicting this possibility, we find it more compelling for our application, even though evidence based on standard statistical tests is ambiguous."

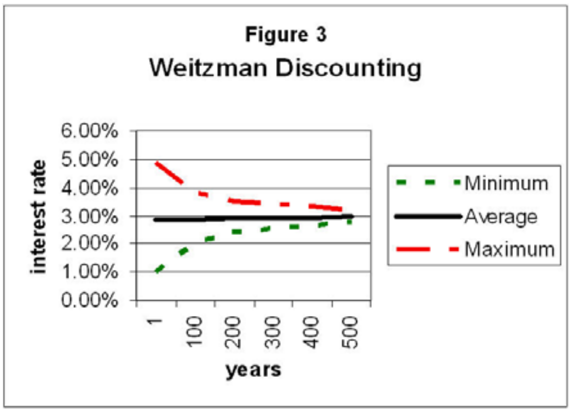
Those who advocate the use of declining certainty-equivalent discount rates must nonetheless believe that there is sufficient autocorrelation between annual interest rates to justify their position. Lacking any firm evidence for this, it might at least be useful to find out just how strong such autocorrelation should be for the purported effect to be detected. To find out, I conducted a simple Monte Carlo experiment, in which annual interest rates were chosen from a uniform distribution between 1% and 5%. The evolution of R^W and R^G in case of perfect year to year autocorrelation is shown in Figure 1 and Figure 2.[3]

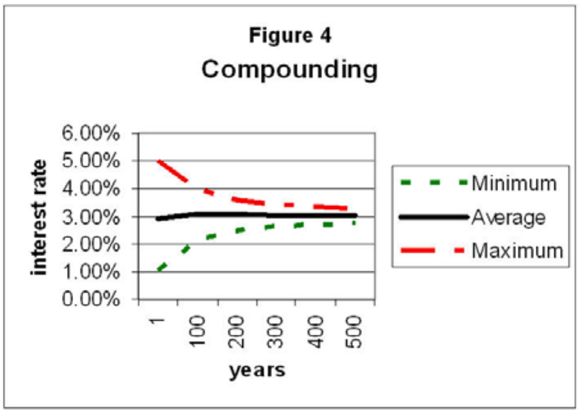
[3] These results are easy to reproduce in Excel, even without special add-in software. Create a matrix of, say, 500 rows by 100 columns, populated by random numbers. This will allow 100 simulations of discount and compound factors for 500 years. Use the random numbers to create correlated random numbers in a second worksheet of similar structure. Serial correlation can be simulated by transforming each random number into a weighted average of the random number of the preceding year, multiplied by the correlation coefficient, and itself, multiplied by the complement to one of the correlation coefficent. From this, yearly random interest rates can be



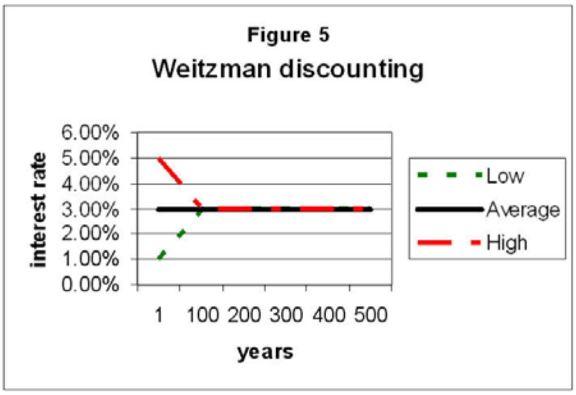


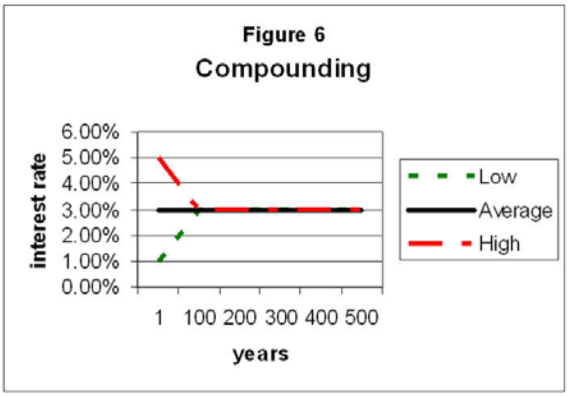
However, if the degree of autocorrelation drops by just 0.02 to 0.98, the expected term structure becomes flat for both Weitzman discounting and compounding. See Figure 3 and Figure 4.





I also tried another simple way of modeling the evolution of interest rates. I assumed that they would follow a sinusoidal path between the highest and lowest interest rates, with a wavelength in years that could be varied. I used Monte Carlo simulation to choose the phase of the cycle for year 1. The results obtained were the same for both Weitzman discounting and compounding, as shown in Figure 5 and Figure 6, for any wavelength tried:





computed, which can then be converted into both compound or discount factors in a third worksheet. Their expected values will yield the certainty equivalents for all years.

It is important to observe that doing Monte Carlo simulations of present values with stochastic interest rates automatically results in Weitzman discounting, which was done here for testing purposes, but which should otherwise be avoided. This is the insidious trap into which Weitzman unwittingly fell.

Two conclusions can be derived from these calculations:

1. Unless the autocorrelation of interest rates can be shown to be nearly perfect, the Weitzman-Gollier puzzle is moot.

2. The lack of perfect autocorrelation of interest rates means that the farther into the future we look, the lower the uncertainty about the certainty-equivalent rate.

This also means that the correct conclusion of the Weitzman model, namely that discount rates should be increasing, need not become the new norm.

5. Conclusions

Judging investment projects by their present or future values must always lead to the same conclusion, even if the relevant market interest rate is stochastic. This is so by definition, for the present value of a future sum is defined as the amount that compounds to said sum. Consequently discount and compound factors derived from a given transaction must be each others' reciprocals. The "Weitzman-Gollier puzzle" arises because Weitzman did not use the correct expected discount factor and therefore computed one of the certainty-equivalent interest rates incorrectly. Correcting for this, Weitzman's simple two time-period model yields the conclusion that certainty-equivalent discount rates will increase with time and will tend to the highest possible rate.

The entire question is most likely moot, however. Experimental calculations show that unless annual interest rates are nearly perfectly autocorrelated, the justification for increasing certainty-equivalent discount rates disappears. It is also interesting to note that if annual interest rates are not perfectly autocorrelated and are bound in a finite range, then uncertainty about their certainty-equivalent declines with time.

What is disturbing in this story is not that an error went undetected for some time. Errors are found sooner or later. What is disturbing is how quickly a notion based on a simplistic model with unrealistic implications became a guide for policy, with real resource allocation consequences. "The prompt uptake of [declining discount rates] in policy circles can be understood as a consequence of their ability to salve policy makers' chief unease with conventional discounting: that generations in the deep future count for virtually nothing, in a rigorous way." (Mark Freeman and Ben Groom, 2010.) This is a view widely shared among economists. "With a constant [discount] rate, the costs and benefits accruing to generations in the distant future appear relatively unimportant in present value terms." (Ben Groom, Cameron Hepburn, Phoebe Koundouri, and David Pearce, 2005.)

I disagree with this view, which confuses discounting with valuation. The relative value of the costs and benefits accruing to future generations should be expressed in the relevant welfare function, in the form of a pure rate of time preference[4]. Increasing time preference for the future would make us shift more resources from present day consumption to investments that will benefit future generations. This will happen with any discount rate. The role of discounting is not primarily to determine how much we should set aside for future generations, but to help us decide how to invest efficiently what we do set aside. It helps us choose between alternative ways of transferring resources between the present and the future, by pointing out what the opportunity cost of investing in a given project is. The widely shared view would have merit if the alternative to a given project were present day consumption. But it is not. The alternative is to transfer the same resources to the same future generations in a manner that might benefit them more. "To evaluate climate mitigation policy with a lower rate of return unnecessarily harms either current or future generations, or both. Future generations would not thank us for investing in a low-return project" (Gary S. Becker *et al*, 2010).

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[4] This will not change the discount rate to be used, despite the fact that the pure rate of time preference affects the present value of welfare much like a discount rate does. Optimal inter-temporal resource allocation will only be achieved if discounting is done at the interest rate of the capital market that is able to trasfer resources from the present to the future. See Szekeres (2011).

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