Ramanujan’s Factorial Conjecture

By.-Alberto Durán-.

otreblam@gmail.com

University J.M.Vargas. Faculty Engineering/Education

Caracas/Venezuela

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Abstract. We attempt to show that, Ramanujan’s Conjecture has negative solution.

For this we shall use the known method of sketch proof, another basic results of

Arithmetic, Algebra and elementary Number Theory.

Reduction ad absurdum, which Euclid loved so much, is one of a mathematician’s
finest weapons.

G.H. Hardy (London 1941).

1. Introduction.

In the current literature\([1]\), the Problem of finding all integer solutions to the equation:

\(*\) \(n! + 1 = a^2\), it’s also named, a Brocard’s Problem\([12]\). In this article, we describe

details, that no other solutions different to that found by S.Ramanujan (1887-1920),
can be calculated from \(*\), without contradicting some arithmetical principle\([7]\).

2. Notation and Preliminary Lemmas.

2.1 Before going on, we start by establishing the conventional symbols below:

As is usual in Number Theory, Prime numbers shall be denoted by using the letter \(\mathbb{P}\).

At the same time, \(\forall n \in \mathbb{N}\); the factorial function shall be represented by employing,

\[(1) \quad n! = \prod_{k=1}^{n} k = 1.2.3...(n-2).(n-1)n\]

Continuing our task, also we need set the following arithmetical statements:

Lemma 1. Let \(n \in \mathbb{N}\), if \(n \geq 5 \Rightarrow \exists \alpha, \beta\); such that \(\alpha \in \mathbb{N}\) and \(\beta \in \mathbb{N}\), satisfying

the relation,
(2) $\pi_{k=1}^{k=n} k = n! = \alpha (10)^\beta$, where $\alpha \geq 12$ and simultaneously $\beta \geq 1, [14].$ This arithmetical assertion, is of course a cornerstone in our approach. An aside, It’s needed point out the following.

Zeros at the end of a number, indicate the number of factors of ten this number has. Indeed when $n!$ increases, there is a plenty of them.

2.2 With regard to Ramanujan’s equation: What is the nature of the Problem to be solved? According to several historical references $[2, 4]$, S.Ramanujan conjectured that, his equation $(*)$ above, only has solution, if the values of $n$ and $a$ are: $4, 5, 7$ and $5, 11, 71$ respectively $[6]$.

Modestly, as far our current understanding of the conjecture goes, in the next sections, our strategy to solve the Problem, is based upon relation $(2)$ above.

3. Analyzing equation $n!+1 = a^2$

3.1 Since $\forall n \geq 2$, the quantity $n!$ always is even, consequently we may write Ramanujan’s equation by convenience, employing that equivalent form:

$\pi_{k=1}^{k=n} k = n! = (a+1).(a-1)$; from this equation we can see that, $a$ is to be an odd natural number necessarily, because in principle, $n!$ must be written (or factored) as a product of two consecutive integer even numbers. And $(3)$ or $(*)$ are satisfied correctly if and only if, by construction(taking in to account our hypothesis) we may rewrite that important equation $(3)$ through that manner:

$\pi_{k=1}^{k=n} k = n! = (a+1).(a-1) = (2k).(2k + 2)$, obviously for some natural number $k$, its value must be sufficiently large, the important thing is that, the quantity $n!$; may be represented (factored) according to equation $(4)$.

Although, it should be observe that, this arithmetical fact, is a crucial condition to prove that, the proposed guess has no other integral solutions, being $n \succ 7$.
3.2 For the sake of completeness, we show how our hypothesis works, by testing exhaustively, all values their well known solutions: In fact, take a generic pair \((n, a)\), now by employing the roots of \((*)\), it’s plane that, realizing counting thence we have this immediate system:

\[
\begin{align*}
(4!)^2 + 1 &= 5^2 \Rightarrow 25 - 1 = 4! = (4)(6) \\
(5!)^2 + 1 &= (11)^2 \Rightarrow 121 - 1 = 5! = (10)(12),
\end{align*}
\]

Notice that, up to this point, all agree exactly with our Ramanujan’s equation[9].

3.3 However, returning to the question asked, we can restates Brocard’s Problem most formally, by employing this equivalent statement:

**Proposition 1.3.1** Demonstrate that, for \(n > 7\), equation \((*)\) hasn’t solution.

**Proof.** In order to solve the problem, we proceed to apply indirect method[11].

Assuming that there exists, at least a pair \((n, a)\) for some \(n \in \mathbb{N}\), belonging to the range \([8, \infty)\), thus it follows that; for **any odd number** \(a \in \mathbb{N}\), coherently with \((*)\) and our constructed equation \((3)\) at 3.1 above; hence we may write,

\[
(6) \quad n! = a^2 + 1 \Rightarrow n! = (a + 1)(a - 1), \text{ but relation } (2) \text{ tell us likely that,}
\]

\[
(7) \quad \pi_{k=1}^{\infty} k = n! = \alpha(10)^\beta, \text{ where } \alpha, \beta \text{ are both natural numbers. Can we use that arithmetical feature ?}, \text{ yes this we do. Next, combining equations } (4), (6)
\]

and \((7)\), obviously we obtain the immediate relations below:

\[
(8) \quad n! = \alpha(10)^\beta = (2k)(2k + 2) \Rightarrow \alpha(2.5)^\beta = (2k)(2k + 2), \text{ whence}
\]

\[
(9) \quad (\alpha.2^\beta)(5)^\beta = (2k)(2k + 2). \text{This last equation can be interpreted in the following way:}
\]

in virtue of the fact that, we are looking for some **integer solution**, from \((9)\) follows at once, these new systems of Diophantine equations:
As the reader can see, the first equation inside both systems would have solution; but, unfortunately second equation, does contain (in particular) that “mandatory” condition:

\[
\begin{align*}
\alpha \cdot 2^\beta &= 2k, \\
5^\beta &= 2k + 2, \\
\end{align*}
\]

or we have only another possibility,

\[
\begin{align*}
\alpha \cdot 2^\beta &= 2k + 2, \\
5^\beta &= 2k.
\end{align*}
\]

As the reader can see, the first equation inside both systems would have solution; but, unfortunately second equation, does contain (in particular) that “mandatory” condition:

\[
\begin{align*}
\text{either } 5^\beta &= 2k + 2 \quad \text{or } 5^\beta &= 2k,
\end{align*}
\]

which isn’t possible, because clearly number five, raised to any integer power \( \beta \), greater than or equal to one: never is even.

3.4 We conclude that our assumption made previously, therefore is false; so, our equation \((*)\), has not solution whenever \( n > 7 \), this completes the proof of the Proposition 1.3.1 above. Ramanujan’s conjecture is true, as was to be proved. And after elapsed ninety four years, we conclude that: Ramanujan was right.

4. Historical Comments.

Honestly, Biography of the Indian Mathematician Srinivas Ramanujan(1887-1920), It’s well known inside scientific world, that we don’t need repeat here. As matter of fact, exactly, Dr. Robert Kanigel,(1991) wrote;“The Man Who Knew Infinite”: A Life of the Genius Ramanujan. His Mathematical achievements and his mathematical Papers, in collaboration with English Mathematician G. H. Hardy\[10\]. The book also reviews the life of Hardy and the academic culture of Cambridge University during early twentieth century\[13\].

Referred to \((*)\), several Authors, particularly: Number Theorist have studied this classic and so tantalizing Problem of Arithmetic, Erdős Paul conjectured that no other solutions exist\[5\]. On the other hand, Berndt and Galway, performed calculations for \( n \) up to, this amazing number, \( 1000000000 \); and it’s well known that, found no further solutions\[8\].
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6. References


[5] Erdös Paul conjectured that no other solutions exist. Overholt (1993) showed that there are only finitely many solutions provided that the abc conjecture is true.


[9] Overholt, Marius, "The Diophantine equation $n! + 1 = m^2$",


    (From English translation),(2004); pages.245-305.


[14] Ziegler, Günter & Aigner, Martin; *Proofs from THE BOOK ,*