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A CALCULATOR FOR POLYNOMIAL EQUATIONS OF DEGREE FIVE AND ABOVE

ABSTRACT

I present a method that can be used to solve fifth polynomial equations and above. I will propose a general factorization process that will enable the make it possible to reduce a fifth polynomial expression to solvable linear and quartic factors. Quartic equations were first solved by Lodovico Ferari and today they are solvable through many methods presented in various texts.

AUTHORS NAME: SAMUEL BONAYA BUYA

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1. METHOD

Consider the general fifth degree polynomial,

$Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0$. --------------------------- 1

By dividing both sides of the equation by A, it can take the general form,

$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ ----------------------- 2

I propose a method of factorization of the above equation, that will generate auxiliary quartic and linear equations, both of which are solvable. I will introduce a variable **r** that interrelates all the coefficients of the above polynomial such that:

$a_4 = r + k_1r^2$ -------------------- 3

$a_3 = k_1r^3 + k_2r^4$ --------------- 4

$a_2 = k_2r^5 + k_3r^6$ --------------- 5

$a_1 = k_3r^7 + k_4r^8$ --------------- 6

$a_0 = k_4r^9$ ------------------------ 7

The splitting field proposed in the set of equations 3-7 above is meant to help make the polynomial equation 2 above solvable in radicals as will be justified in the next few steps.

From the set of equations 3-7, I will seek to establish a connection between the a_4 and the coefficients a_3 ,a_2, a_1, a_0 .

Using equations 6 and 7:

$a_1 = k_3r^7 + \frac{a_0}{r}$

Or,

$k_3r^6 = (a_1 - \frac{a_0}{r})\frac{1}{r}$ ------------ 8

Using equations 5 and 8 we obtain the relation,

$k_2r^4 = [a_2 - \left(a_1 - \frac{a_0}{r}\right)\frac{1}{r}]\frac{1}{r}$ ----- 9

Using equations 4 and 9 we obtain the relation,

$k_1r^2 = \{a_3 - \left[a_2 - \left(a_1 - \frac{a_0}{r}\right)\frac{1}{r}\right]\frac{1}{r}\}\frac{1}{r}$ -------- 10

Using the equations 3and 10 we obtain the relation connecting the a_4 and the coefficients a_3 ,a_2, a_1, a_0 ,

$a_4 = r + \left[a_3 - (a_2 - \left(a_1 - \frac{a_0}{r}\right)\frac{1}{r}\right]\frac{1}{r}\}\frac{1}{r}$

Or,

$a_4 = r + \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4}$ ----------- 11

When equations 3 – 7 are substituted into equation 2 we obtain the equation below.

$(x + r)(x^4 + k_1r^2x^3 + k_2r^4x^2 + k_3r^6x + a_0) = 0$ ------ 12

Where:

$b_3 = k_1r^2 = \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4}$ --------- 13

$b_2 = k_2r^4 = \frac{a_2}{r} - \frac{a_1}{r^2} + \frac{a_0}{r^3}$ -------- 14

$b_1 = k_3r^6 = \frac{a_1}{r} - \frac{a_0}{r^2}$ ------------------------ 15

$b_0 = k_4r^8 = \frac{a_0}{r}$ ---------------- 16

From equation 12,

Either

$(x + r) = 0$

This implies,

$x_1 = -r$ ------------- 17

Or

$x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 = 0$ ---------18

Using the splitting field proposed above it can be concluded that general polynomial equation of degree 5 has splitting field generated by r, and therefore its Galois Group is abelian. The equation is therefore is solvable by radicals. The Factorization algorithm selected will determine the kind of splitting field obtained. A wrong choice of factorization algorithm will prohibit obtaining a general solution of a quintic equation because the inability to generate a generalized solvable Galois group. The splitting field suggested in this paper can be extended to polynomial equations above degree 5 and generate solvable and generalized Galois groups sufficient for their solvability.

Notice here again from the sets of equations 3-7 and 13-16:

$s_1 = a_4 = r + b_3$ --------- 19

$s_2 = a_3 = b_3r + b_2$ ------ 20

$s_3 = a_2 = b_2r + b_1$ -------- 20

$s_4 = a_1 = b_1r + b_0$ ----------21

$s_5 = a_0 = b_0r$ ------------22

Through choice of r and s_1 to s_5 the values of b_0 to b_3 are determinate.

There are several ways available in mathematics to solve the auxiliary quartic equation 18.

Note also from equations 8-10:

$k_1 = \frac{a_3}{r^3} - \frac{a_2}{r^4} + \frac{a_1}{r^5} - \frac{a_0}{r^6}$ -------------- 23

$k_2 = \frac{a_2}{r^5} + \frac{a_1}{r^6} - \frac{a_0}{r^7}$ ----------- 24

$k_3 = \frac{a_1}{r^7} - \frac{a_0}{r^8}$ ------------ 25

$k_4 = \frac{a_0}{r^9}$ ---------------- 25

2. PROCEDURES OF USING THE CALCUATOR OF DEGREE FIVE POLYNOMIAL EQUATION

1. Input values of the coefficients a_3 ,a_2, a_1 and a_0 of the 5th degree polynomial
2. Choose a value of r and use it to generate the value the value of the coefficient a_4.
3. Use equations 13-16 to generate the coefficients of the auxiliary quartic equation.
4. Use an appropriate quartic equation calculator to generate the four roots of the auxiliary quartic equation
5. The solution of the 5th degree polynomial will be x = -r and the four roots of the auxiliary quartic equation.

Strength of the degree five polynomial calculator

1. It is able to determine all the roots of the generated fifth polynomial equation in radical form.
2. Given a set of the coefficients a_3, a_2, a_1 and a_0, different values of the coefficient a_4 can be generated by varying **r** and solutions of the polynomial simultaneously equation worked out.
3. The degree five polynomial Calculator can be used as a basis for a degree six polynomial Calculator.

Example

Given three coefficients and constant term of a 5th degree polynomial equations are $a_3 = 3$, $a_2 = -2$, $a_1 = -8$, $a_0 = -13$:

a) Determine a_4 when r =1

b) Determine the auxiliary quartic equation

c) Determine the roots of the fifth degree polynomial equation

d) Determine a_4 when r =2

e) Determine a_4 when $r = \sqrt{2}$ and solve the polynomial.

Solution

a) $a_4 = r + \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4} = 1 + 3 + 2 - 8 + 13 = 11$

b) $b_3 = \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4} = 3 + 2 - 8 + 13 = a_4 - 1 = 10$

$b_2 = \frac{a_2}{r} - \frac{a_1}{r^2} + \frac{a_0}{r^3} = -2 + 8 - 13 = -7$

$b_1 = \frac{a_1}{r} - \frac{a_0}{r^2} = -8 + 13 = 5$

$b_0 = \frac{a_0}{r} = -13$

The auxiliary quartic equation is $x^4 + 10x^3 - 7x^2 + 5x - 13 = 0$

c) The roots are:

$x_1 = -1$, $x_2 = 1.1368$, $x_3 = -0.2145 + 1.0109i$, $x_4 = -0.2145 - 1.0109i$, $x_5 = -10.7079$

d) $a_4 = r + \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4} = 2 + \frac{3}{2} + \frac{2}{4} + \frac{-8}{8} + \frac{13}{16} = \frac{61}{16}$

e) $a_5 = r + \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4} = \sqrt{2} + \frac{3}{\sqrt{2}} + \frac{2}{2} + \frac{-8}{2^{3/2}} + \frac{13}{4} = 4.9571$

The coefficients of the auxiliary quartic equation are

$b_3 = \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4} = \frac{3}{\sqrt{2}} + \frac{2}{2} + \frac{-8}{2^{3/2}} + \frac{13}{4} = 3.5429$

$b_2 = \frac{a_2}{r} - \frac{a_1}{r^2} + \frac{a_0}{r^3} = \frac{-2}{\sqrt{2}} + 4 - \frac{13}{2^{3/2}} = -2.0104$

$b_1 = \frac{a_1}{r} - \frac{a_0}{r^2} = \frac{-8}{\sqrt{2}} + \frac{13}{2} = 0.8431$

$b_0 = \frac{a_0}{r} = \frac{-13}{\sqrt{2}} = -9.1924$

The roots are: $x_1 = 1.4142$, $x_2 = 1.3368$, $x_3 = -0.3425 + 1.2337i$, $x_4 = -0.3425 - 1.2337i$, $x_5 = -4.1946$

3. SOLUTION OF A GENERALIZED POLYNOMIAL EQUATION OF DEGREE N

Consider the general polynomial equation;

$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + - - - + a_0 = 0$ ----------------- 1

By the above factorization algorithm, the equation factorizes to:

$(x + r_1)(x + r_2) - - - (x + r_{n-4})(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) = 0$ -------- 2

For factorization of equation 1 to a linear factor and a polynomial factor of degree n- 1

$a_{n-1} = r_1 + k_1r_1^2 = r_1 + \frac{a_{n-2}}{r_1} - \frac{a_{n-3}}{r_1^2} + - - - \frac{a_0}{r_1^{n-1}}$ ------------------ 2

$a_{n-2} = k_1r_1^3 + k_2r_1^4 = r_1(a_{n-1} - r_1 - \frac{a_{n-3}}{r_1^2} + - - - - + \frac{a_0}{r_1^{n-1}})$ --- 3

And so on up to

$a_0 = r_1^{n-1}(a_{n-1} - r_1 - \frac{a_{n-3}}{r_1^2} + - - - \frac{a_1}{r_1^{n-2}})$

So that:

$(x + r_1)(x^{n-1} + c_{n-2}x^{n-2} + c_{n-3}x^{n-3} + - - + c_1x + c_0) = 0$ ------- 4

Where $c_{n-2} = k_1r_1^2 = (a_{n-1} - r_1)$--------- 5

$c_{n-3} = k_2r_2^4 = a_{n-2} - r_1c_{n-2}$ ------- 6 and so on and

$c_0 = a_1 - r_1c_1$ ------------- 7

The auxiliary polynomial equation of degree n-1 can similarly be factorized by a similar process and so on.

Polynomial equations of order n can be split into solvable Galois groups depending on the factorizing criteria used to generate the splitting fields.

4. SOLVING QUINTIC EQUATIONS- OTHER PURSPECTIVES

Consider a set **S** containing the roots of a quintic equation

$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ ----------------------1

Such that:

$\boldsymbol{S} = \{x_1, x_2, x_3, x_4, x_5\} = \{-r, -\alpha_1r^2, -\alpha_2r^2, -\alpha_3r^2, -\alpha_4r^2)$ ------------ 2

The quintic equation 1 above can be expressed in terms of the roots to include the roots in the set **S**:

$(x+r)(x+\alpha_1 r^2)(x+\alpha_2 r^2)(x+\alpha_3 r^2)(x+\alpha_4 r^2)=0$ --------------- 2

Where:

$x_1=-r$ ----------------- 3

$x_2=-\alpha_1 r^2$ -------------4

$x_3=-\alpha_2 r^2$ -------------- 5

$x_4=-\alpha_3 r^2$ -------------- 6

$x_5=-\alpha_4 r^2$ ------------- 6

The expansion of 2 gives the elementary symmetric functions of x_i:

$S_1=-(x_1+x_2+x_3+x_4+x_5)=r+(\alpha_1+\alpha_2+\alpha_3+\alpha_4)r^2=r+k_1 r^2$ ----------- 7

$S_2=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)r^3+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4)r^4=k_1 r^3+k_2 r^4$ ---- 8

$S_3=(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4)r^5+(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4)r^6$
Or

$S_3=k_2 r^5+k_3 r^6$ -------------------- 9

$S_4=(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4)r^7+\alpha_1\alpha_2\alpha_3\alpha_4 r^8=k_3 r^7+k_4 r^8$ ------------------ 10

$S_5=\alpha_1\alpha_2\alpha_3\alpha_4 r^9=k_4 r^9$ -------------- 11

From equations 7 to 10:

$S_1=r+\frac{S_2}{r}-\frac{S_3}{r^2}+\frac{S_4}{r^3}-\frac{S_5}{r^4}$ ----------------- 12

Also from equations 7 – 11

$k_1=\alpha_1+\alpha_2+\alpha_3+\alpha_4=\frac{a_3}{r^3}-\frac{a_2}{r^4}+\frac{a_1}{r^5}-\frac{a_0}{r^6}$ ------------------- 13

$k_2=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4=\frac{a_2}{r^5}-\frac{a_1}{r^6}+\frac{a_0}{r^7}$ ------------------- 14

$k_3=\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_2\alpha_3\alpha_4=\frac{a_1}{r^7}-\frac{a_0}{r^8}$ ------------- 15

$k_4=\alpha_1\alpha_2\alpha_3\alpha_4=\frac{a_0}{r^9}$ --------------------- 16

The determination of the values of α_1, α_2, α_3 and α_4 is necessary to obtain the roots of the quintic equation.

Multiplying k_1 by α_1 and substituting into k_2:

$k_2 = \alpha_1 k_1 - \alpha_1^2 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$ ------------------- 15

Multiplying k_2 by α_1 and substituting into 16 and rearranging the relation:

$\alpha_1 k_2 = k_3 - \alpha_2\alpha_3\alpha_4 + \alpha_1^2 k_1 - \alpha_1^3$ ------------------------- 16

Multiplying equation 16 by α_1 , simplifying and rearranging we obtain the equation:

$\alpha_1^4 - k_1\alpha_1^3 + k_2\alpha_1^2 - k_3\alpha_1 + k_4 = 0$ ------------------------- 17

The equation 17 is solvable.

We could using similar steps obtain the quintic equation for the root coefficient α_4 as

$\alpha_4^4 - k_1\alpha_4^3 + k_2\alpha_4^2 - k_3\alpha_4 + k_4 = 0$ ------------------- 18

By similar manipulations we can obtain cubic and quartic equations for the coefficients α_2 and α_3 respectively:

$\alpha_2^3 - k_1\alpha_2^2 + k_2\alpha_2 - k_3\alpha_1 + \frac{k_4}{\alpha_1} = 0$ ---------------------- 19

$\alpha_3^4 - (k_1 + \alpha_2)\alpha_3^3 + k_2\alpha_3^2 - (k_3 - \frac{k_4}{\alpha_1})\alpha_3 + k_4 = 0$ ----------- 20

Determination of the coefficients of the roots of quintic equations

Consider the general cubic equation;

$x^3 + b_2x^2 + b_1x + b_0 = 0$ ----------------- 21

For determination of α_2 in equation 19, $\alpha_2 = x$, $b_2 = -k_1$, $b_1 = k_2$ and $b_0 = \frac{k_4}{\alpha_1}$

If we take $x = y - \frac{b_2}{3}$ the equation takes the solvable form:

$y^3 + py + q = 0$ ----------------- 23

If $q = u - t = 2\frac{b_2^3}{27} + b_0 - \frac{b_0 b_1}{3}$ --------- 24

And $p = 3(ut)^{\frac{1}{3}} = b_1 - \frac{b_2^2}{3}$ ------------------ 25

Equation 23 when factorized takes the form:

$\left(y - t^{\frac{1}{3}} + u^{\frac{1}{3}}\right)(y^2 + ay + b) = 0$ ------------- 26

Where $a = t^{\frac{1}{3}} - u^{\frac{1}{3}} = -y_1$ ---------------- 27

And $b = \frac{u-t}{u^{\frac{1}{3}}-t^{\frac{1}{3}}} = \frac{-q}{y_1}$ ----------------------- 28

From equations 24 and 25:

$$u = \frac{q \pm \sqrt{q^2+4\frac{p^3}{27}}}{2} \text{ ---------------------- 29}$$

$$t = \frac{-q \pm \sqrt{q^2+4\frac{p^3}{27}}}{2} \text{ --------------------- 30}$$

From equation 30:

$$y_1 = t^{\frac{1}{3}} - u^{\frac{1}{3}} = \sqrt[3]{\frac{-q \pm \sqrt{q^2+4\frac{p^3}{27}}}{2}} - \sqrt[3]{\frac{q \pm \sqrt{q^2+4\frac{p^3}{27}}}{2}} \text{ ------------ 31}$$

$$y_2 = \frac{y_1 + \sqrt{y_1^2 - \frac{4q}{y_1}}}{2} \text{ -------------- 32}$$

$$y_3 = \frac{y_1 - \sqrt{y_1^2 - \frac{4q}{y_1}}}{2} \text{ ---------------- 33}$$

Therefore:

$$x_1 = \sqrt[3]{\frac{-q \pm \sqrt{q^2+4\frac{p^3}{27}}}{2}} - \sqrt[3]{\frac{q \pm \sqrt{q^2+4\frac{p^3}{27}}}{2}} - \frac{b_2}{3} \text{ -------------- 34}$$

$$x_2 = \frac{y_1 + \sqrt{y_1^2 + \frac{4q}{y_1}}}{2} - \frac{b_2}{3} \text{ ------------------- 35}$$

$$x_3 = \frac{y_1 - \sqrt{y_1^2 + \frac{4q}{y_1}}}{2} - \frac{b_2}{3} \text{ ---------------- 36}$$

Consider the general quartic equation:

$x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0$ ------------- 37

By substituting $x = y - \frac{c_3}{4}$,

It reduces to the form:

$y^4 + ey^2 + fy + g = 0$ ------------ 38

Where:

$e = c_2 - \frac{3{c_3}^2}{8}$ ------------------ 39a

$f = c_1 + \frac{c_3^3}{8} - + \frac{c_3 c_2}{2}$ -------------- 40a

$g = c_0 - \frac{3{c_3}^4}{256} + \frac{c_3^2 c_2}{16} - \frac{c_3 c_1}{4}$ -------- 41a

For determination of α_1 and α_4 in equations 17 and 8

$e = k_2 + \frac{3{k_3}^2}{8}$ ------------- 39b

$f = -k_3 - \frac{k_1^3}{8} + \frac{k_1 k_2}{2}$ --------------- 40b

$g = k_4 - \frac{3{k_1}^4}{256} + \frac{k_1^2 k_2}{16} - \frac{k_1 k_3}{4}$ ---------------- 41b

For the determination of α_3:

$e = k_2 + \frac{3(k_2+\alpha_2)^2}{8}$ --------------------- 39c

$f = \frac{k_4}{\alpha_1} - k_3 - \frac{(k_1+\alpha_2)^3}{8} + k_1 + \alpha_2 + \frac{k_2}{2}$ ------------------ 40 c

$g = k_4 + \frac{3(k_1+\alpha_2)}{256} + \frac{(k_1+\alpha_2)^2}{16} - \frac{(k_1+\alpha_2)(\frac{k_4}{\alpha_1}-k_3)}{4}$ --------- 41 c

The equation 38 is can be factorized to the form:

$y^4 + ey^2 + fy + g = (y^2 + hy + j)(y^2 - hy + \frac{g}{j})$ ---------------- 42

Where:

$e = \frac{g}{j} + j - h^2$ --------------- 43a

$f = h(\frac{g}{j} - j)$ ------------- 44

From equations 43 and 44 we obtain the relations:

$\frac{g}{j} + j = h^2 + e$ ----------- 45

$\frac{2g}{j} - j = \frac{f}{h}$ ----------------- 46

Adding and subtracting the equations 45 and 46:

$\frac{g}{j} = h^2 + e + \frac{f}{h}$ ----------------- 47

$2j = h^2 + e - \frac{f}{h}$ ------------------ 48

Multiplying together equations 47 and 48 and rearranging:

$h^6 + 2eh^4 + (e^2 - 4g)h^2 - f^2 = 0$ ----------------- 49

Letting $w = h^2$ we obtain the relation:

$w^3 + 2ew^2 + (e^2 - 4g)w - f^2 = 0$ ---------------- 50

Let $w = z - \frac{2e}{3}$ we obtain the relation:

$z^3 + z\left(-\frac{e^2}{3} - 4g\right) + \frac{28e^3}{3} + \frac{8ge}{3} - f^2 = 0$ --------------- 51

Letting $u - t = q$ and $ut = (\frac{p}{3})^3$

We obtain:

$z_1 = \sqrt[3]{\frac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2}} - \sqrt[3]{\frac{q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2}}$ ------- 52

$w_1 = \sqrt[3]{\frac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2}} - \sqrt[3]{\frac{q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2}} - \frac{e}{3}$

$w_1 = (\sqrt[3]{\sqrt[3]{\frac{-\frac{28e^2}{3} - \frac{8ge}{3} + f^2 \pm \sqrt{(\frac{28e^2}{3} + \frac{8ge}{3} - f^2)^2 + \frac{4}{27}(-4g - \frac{e^2}{g})^3}}{2}} - \sqrt[3]{\frac{\frac{28e^2}{3} + \frac{8ge}{3} - f^2 \pm \sqrt{(\frac{28e^2}{3} + \frac{8ge}{3} - f^2)^2 + \frac{4}{27}(-4g - \frac{e^2}{g})^3}}{2}}}) - \frac{e}{3}$ ---
53

$$h = \sqrt{\left(\sqrt[3]{\sqrt[3]{\frac{-\frac{28e^2}{3}-\frac{8ge}{3}+f^2 \pm \sqrt{(\frac{28e^2}{3}+\frac{8ge}{3}-f^2)^2+\frac{4}{27}(-4g-\frac{e^2}{g})^3}}{2}} - \sqrt[3]{\frac{\frac{28e^2}{3}+\frac{8ge}{3}-f^2 \pm \sqrt{(\frac{28e^2}{3}+\frac{8ge}{3}-f^2)^2+\frac{4}{27}(-4g-\frac{e^2}{g})^3}}{2}}} - \frac{e}{3}\right)}$$ --- 54

From equations 47- 48:

$$j = \frac{-\frac{4f}{h} \pm \sqrt{\frac{16h^2}{h^2}-16g}}{2}$$ ------ 55

From equation 42:

$$y_{1,2} = \frac{-h \pm \sqrt{h^2-4j}}{2}$$ -------- 56

$$y_{3,4} = \frac{-h \pm \sqrt{h^2-\frac{4g}{j}}}{2}$$ ------- 57

$$x_{1,2} = y_{1,2} - \frac{c_3}{4}$$ ------------ 58

$$x_{3,4} = y_{3,4} - \frac{c_3}{4}$$ ------------- 59

Example 1

In a polynomial equation r = 2 $a_3 = 7$, $a_2 = 9$, $a_1 = 3$, $a_0 = 12$.

a) Determine a_4
b) Solve the polynomial equation

Solution

Method 1

$a_4 = r + \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4} = 2.875$

$b_3 = \frac{a_3}{r} - \frac{a_2}{r^2} + \frac{a_1}{r^3} - \frac{a_0}{r^4} = 1.625$

$b_2 = \frac{a_2}{r} - \frac{a_1}{r^2} + \frac{a_0}{r^3} = 5.25$

$b_1 = \frac{a_1}{r} - \frac{a_0}{r^2} = -1.5$

$b_0 = \frac{a_0}{r} = 6$

b_3 to b_0 are coefficients of the auxiliary quartic equation

The quintic equation is $x^5 + 2.875x^4 + 7x^3 + 9x^2 + 3x + 12 = 0$

$x_1 = -2$, $x_2 = -1.15039 + 2.107235i$, $x_3 = 0.33787 + 0.962708i$, $x_4 = 0.337887 - 0.92708i$, $x_5 = -1.1503871 - 2.10723489i$

Method 2:

$k_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{a_3}{r^3} - \frac{a_2}{r^4} + \frac{a_1}{r^5} - \frac{a_0}{r^6} = 0.21875$

$k_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = \frac{a_2}{r^5} - \frac{a_1}{r^6} + \frac{a_0}{r^7} = 0.32813$

$k_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4 = \frac{a_1}{r^7} - \frac{a_0}{r^8} = 0$

$k_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{a_0}{r^9} = 0.023438$

Using equation 17 of chapter 4

a) $\alpha_1^4 - k_1\alpha_1^3 + k_2\alpha_1^2 - k_3\alpha_1 + k_4 = 0$
The values of α_1 are 0.1589411+0.498888i; -0.0495661+0.288161i; -0.0495661-0.288161i; 0.1589411-0.498888i
The roots of the quintic equation obtained from this quartic equation are:
$x_{11} = -2; x_{21} = -4(0.1589411 + 0.498888\text{i})$;
$x_{31} = -4(-0.0495661 + 0.288161\text{i}); x_{41} = -4(-0.0495661 - 0.288161\text{i});$
$x_{51} = -4(0.1589411 - 0.498888\text{i})$

Solving for equation 18 of chapter 4:

$\alpha_2^3 - k_1\alpha_2^2 + k_2\alpha_2 - k_3\alpha_1 + \frac{k_4}{\alpha_1} = 0$
b) Taking $\alpha_{21} = 0.1589411 + 0.498888i$
$\alpha_2^3 - 0.21875\,\alpha_2^2 + 0.32813\,\alpha_2 + 0.013588331 - 0.0426514i = 0$
Three roots of the quintic equation and be obtained by obtained by solving this cubic equation.
c) Taking $\alpha_{22} = -0.0495661 + 0.288161\text{i}$, the cubic equation obtained is:
$\alpha_2^3 - 0.21875\,\alpha_2^2 + 0.32813\,\alpha_2 - 0.013588331 - 0.07899914i = 0$
Thee roots of the quintic equation are obtained by solving this cubic equation
d) Taking $\alpha_{23} = -0.0495661 - 0.288161i$, the cubic equation obtained is:
$\alpha_2^3 - 0.21875\,\alpha_2^2 + 0.32813\,\alpha_2 - 0.0135885 + 0.07899914i = 0$
Three roots of the quintic equation are obtained by solving this cubic equation.
e) Taking $\alpha_{24} = 0.1589411 - 0.498888i$, the quadratic equation obtained is:
$\alpha_2^2 - 0.21875\,\alpha_2 + 0.32813 = 0$
The two coefficients are, $0.10938 + 0.56229i$; $0.10938 - 0.56229i$
The roots of the polynomial are $-4(0.10938 + 0.56229i)$, $-4(0.10938 - 0.56229i)$

Solving for equation 20 this chapter:

$$\alpha_3^4 - (k_1 + \alpha_2)\alpha_3^3 + k_2\alpha_3^2 - (k_3 - \frac{k_4}{\alpha_1})\alpha_3 + k_4 = 0$$

The coefficient equation can be used to derive up to 64 roots of the quintic equation. Generally a quintic equation has up to 5! possible roots on the upper limit.

Reference:

Galoise, Evariste, Journal des mathématiques pures et appliqués XI 381-441

R. Bruce King, *Behind the Quartic Equation*, Boston, 1996.
Sigeru Kabayashi, Hiroshi Nakagawa, Resolution of equation, *Math. Japonica*, 5 (1992), pp. 882–886.

J. Pierpont, Zur Geschichte der Gleichung V. Grades (bis 1858), *Monatshefte f¨ur Mathematik und Physik*, 6 (1895), pp. 18–19.

Buya, S. B. (2013). A Calculator for Polynomial Equations of Degree Five and Above. Open Science Repository Mathematics, Online(open-access), e23050424. doi:10.7392/openaccess.23050424