A mathematical formula obtained by Evans triangle

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Abstract: R.J Evans in 1977 in the American Mathematical Monthly ask such a question: Find all the integer side of the triangle, Make it a high and the bottom edge of the ratio is an integer, This triangle is only \( d = 3 \). At the same time, Evans has given such a triangle, The length of 3,8,53. In 2009 the side Hin constructs a kind of the triangle, And that such a triangle has infinite. In this paper, a formula are obtained by Evans, the definition of the triangle, By this formula obtained Evans triangle.

Key words: Integers, rational Numbers.

The Improved Method

As shown in figure: In the \( O \) circle with a \( R \), Triangle \( ABC \) for round \( O \) inscribed triangle lines, The \( BC = a \). At the edge of the \( BC \) high as \( n \cdot a \). \( O \) do \( OD \) is perpendicular to the \( BC \), duo to \( BC \) in \( D \). By the Pythagorean theorem \( OD = \sqrt{CE^2 + AE^2} \), Point \( A \) do \( BC \) of parallel lines \( BC \). Delay \( DO \) Hand in \( L \) in \( H \). The \( OH \) is perpendicular to the \( BC \). Point \( A \) do \( AE \) vertical \( BC \) due \( BC \) to \( E \). Then: \( AE = n \cdot a \).

So: \( DH = n \cdot a \), \( OH = n \cdot a - OD = n \cdot a - \sqrt{CE^2 + AE^2} \).
\[ CE = |HA - \frac{a}{2}| \]
\[ = \left| \sqrt{R^2 - (n \cdot a - \sqrt{R^2 - \left(\frac{a}{2}\right)^2})^2} - \frac{a}{2} \right| \]

then:
\[ AC = \sqrt{CE^2 + AE^2} \]
\[ = \sqrt{R^2 - (n \cdot a - \sqrt{R^2 - \left(\frac{a}{2}\right)^2})^2 - \left(\frac{a}{2}\right)^2} + (n \cdot a)^2 \]

The same available:
\[ BA = \sqrt{R^2 - (n \cdot a - \sqrt{R^2 - \left(\frac{a}{2}\right)^2})^2 + \left(\frac{a}{2}\right)^2} + (n \cdot a)^2 \]

For the triangle, We only need to meet the trilateral is rational, And for the rational number can. That is when \( n = (3, 4, 5, 6, 7, 8, 9, \ldots) \). Side \( a, AC, BA \) is rational. Then the triangle line of nature, At the same time expand \( k \) times, Until meet the trilateral all rational Numbers for integer.

Lemma: If \( a, AC, BA \) are rational, Then \( \frac{AC}{a}, \frac{BA}{a} \) Also for the rational Numbers, If \( y = \frac{AC}{a} + \frac{BA}{a} \), \( y \) will be as rational Numbers.

Then:
\[ \frac{AC}{a} = \sqrt{\left(\frac{R}{a}\right)^2 - (n - \sqrt{\left(\frac{R}{a}\right)^2 - \left(\frac{1}{2}\right)^2})^2} + (\frac{a}{2})^2 + n^2 \]
\[ \frac{BA}{a} = \sqrt{\left(\frac{R}{a}\right)^2 - (n - \sqrt{\left(\frac{R}{a}\right)^2 - \left(\frac{1}{2}\right)^2})^2 + \left(\frac{1}{2}\right)^2} + n^2 \]

Make \( \frac{R}{a} = x \), Then
\[ \frac{AC}{a} = \sqrt{x^2 - (n - \sqrt{x^2 - \left(\frac{1}{2}\right)^2})^2} + (\frac{1}{2})^2 + n^2 \]
Known \( R, a \) into \( x \), This is the key to the problem solving.

The following order:

\[
y = \frac{AC}{a} - \frac{BA}{a}
\]

Then, start to equation. After finishing:

\[
x = \frac{1}{2} \left[ \frac{y^{2} - 1}{4} + \frac{n}{y^{2} - 1} \right], \quad x > \frac{n^{2} + \frac{1}{4}}{4n}, \quad y > \sqrt{4n^{2} + 1}
\]

Solution here. We will think of this problem has been solved. So we make \( n = 4 \). \( y \) assignment is given. Get the \( x \) value, then get the

value of the \( \frac{AC}{a}, \frac{BA}{a} \). Found that it is not rational. So we will thinking about \( x \) generation into the \( \frac{AC}{a}, \frac{BA}{a} \), the generations:

\[
\frac{AC}{a} = \frac{y}{2} - \frac{1}{2} \sqrt{1 - \frac{4n^{2}}{y^{2} - 1}}, \quad \frac{BA}{a} = \frac{y}{2} + \frac{1}{2} \sqrt{1 - \frac{4n^{2}}{y^{2} - 1}}
\]

We through the above formula, We can know that we need only ensure

\[
\sqrt{1 - \frac{4n^{2}}{y^{2} - 1}} \quad \text{for mileage. To make this type is equal to the number}
\]

\( k \).

The \( 4n^{2} = (1 - k^{2})(y^{2} - 1) \)'s formula can be obtained. So through the \( 4n^{2} = (1 - k^{2})(y^{2} - 1) \) formula, We just make sure \( k, y \)
to have the mileage, \( n \) as an integer.

References:


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