Beyond Sperner's lemma

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Preface

The present note is devoted to a recent beautiful and ingenious proof of Brouwer's fixed point theorem due to mathematical economists H. Petri and M. Voorneveld [PV]. It bears a strong resemblance to one of the most popular proofs of Brouwer's fixed point theorem, namely to the proof based on Sperner's lemma [S] and the Knaster–Kuratowski–Mazurkiewicz [KKM] reduction of Brouwer's fixed point theorem to Sperner's lemma. The latter proof is the standard cohomological proof in a disguise (see [I] for an explanation). In contrast with this proof and many other, the Petri–Voorneveld proof seems to have no cohomological interpretation.

Apparently motivated by the Shapley–Scarf [SS] classical model of a market of agents with preferences over indivisible goods, H. Petri and M. Voorneveld [PV] discovered an analogue of Sperner's lemma in this context. This analogue is more general than Sperner's lemma and has a weaker, but subtler conclusion. In order to apply it one needs to modify the Knaster–Kuratowski–Mazurkiewicz reduction.

The Shapley–Scarf model is the source of the most unusual feature of the Petri–Voorneveld lemma, the simultaneous presence of many linear orders on the same set. In the Shapley–Scarf model these orders reflect the preferences of different agents on the market.

The goal of this note is to present a version of the Petri–Voorneveld proof which makes both its similarities and its differences with the Sperner–Knaster–Kuratowski–Mazurkiewicz proof transparent. This version closely follows the outline of the latter proof and allows to isolate the ideas of Petri and Voorneveld which have no classical analogues. They are, first of all, the notion of a *dominant set* (see the beginning of Section 1), the proofs of Lemmas 5 – 7 (the classical analogues of these lemmas are obvious) and Lemma 8.

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1. Linear orders

Linear orders and dominant sets. Let T be a finite set. Suppose that a family of linear orders $<_i$ on T, labeled by elements i of a finite set I, is given. For a non-empty subset $X \subset T$ let $\min_i X$ be the minimal element of X with respect to the order $<_i$. Let C be a non-empty subset of I. A subset $X \subset T$ is said to be *dominant with respect to* C if

(1) there is no element $y \in T$ such that $\min_i X <_i y$ for all $i \in C$.

It is convenient to agree that $\varnothing \subset T$ is dominant with respect to every non-empty $C \subset I$. The condition (1) is the central notion of [PV], called there the *"No Bullying"* condition.

1. Theorem (Petri–Voorneveld). *For every map $c : T \longrightarrow I$ there exists a non-empty subset $X \subset T$ such that X is dominant with respect to $c(X)$.*

In order to stress the analogy with Sperner's lemma, as it is usually presented nowadays, the maps $c : T \longrightarrow I$ will be called *colorings*, and the labels $i \in I$ will be called *colors*. The rest of this section is devoted to a proof of Theorem 1.

2. Lemma. *If $X \subset T$ is dominant with respect to $C \subset I$, then* $X = \{ \min_i X \mid i \in C \}$.

Proof. Clearly, X contains all minima $\min_i X$. Suppose that $x \in X$ is different from all $\min_i X$ with $i \in C$. Then $\min_i X <_i x$ for all $i \in C$, contrary to the assumption. ■

3. Corollary. *If $X \subset T$ is dominant with respect to $C \subset I$, then* $|X| \leqslant |C|$. ■

Cells. We will consider pairs (X, C) of subsets $X \subset T$ and $C \subset I$. Such a pair (X, C) is said to be a *cell* if $C \neq \varnothing$, the set X is dominant with respect to C, and $|C \smallsetminus c(X)| \leqslant 1$. If (X, C) is a cell, then $|C| \leqslant |c(X)| + 1 \leqslant |X| + 1$ and $|X| \leqslant |C|$ by Corollary 3. Therefore

$$|C| = |X| \quad \text{or} \quad |X| + 1.$$

A cell (X, C) is is said to be a *codimension* 0 cell if $|C| = |X|$, and a *codimension* 1 cell if $|C| = |X| + 1$. We will call them respectively *0-cells* and *1-cells* for short. They are analogues of codimension 0 and 1 simplices in the proof of Sperner's lemma.

4. Lemma. *If (X, C) is a 0-cell, then either $C = c(X)$, or $|C \smallsetminus c(X)| = |c(X) \smallsetminus C| = 1$. If (Y, D) is a 1-cell, then $c(Y) \subset D$ and $|D \smallsetminus c(Y)| = 1$.*

Proof. Let (X, C) be a 0-cell. If $|C \smallsetminus c(X)| = 0$, then $C = c(X)$. If $|C \smallsetminus c(X)| = 1$, then $|c(X) \smallsetminus C| = 1$. If (Y, D) is a 1-cell, then

$$|Y| \geqslant |c(Y)| \geqslant |c(Y) \cap D| \geqslant |D| - 1 = |Y|.$$

It follows that $|c(Y)| = |c(Y) \cap D|$ and hence $c(Y) = c(Y) \cap D$. Therefore $c(Y) \subset D$. Since $|D| = |Y| + 1$, it follows that $|D \smallsetminus c(Y)| = 1$. ■

Types of cells. A cell (X, C) is called *balanced* if $C = c(X)$. By Lemma 4 if (X, C) is not balanced, then $C \smallsetminus c(X)$ consists of one element. This element is called the *type* of (X, C). The balanced cells are the analogues of simplices such that all colors are used to colors of their vertices in Sperner's lemma. The cells of type i are the analogues of simplices with no vertex colored by i.

Deleting and adding elements. For a set A and an element $a \in A$ we will denote by $A - a$ the set $A \smallsetminus \{a\}$. Similarly, for $b \notin A$ we will denote by $A + b$ the set $A \cup \{b\}$. The set $A - a$ is defined only if $a \in A$, and the set $A + b$ is defined only if $b \notin A$.

Faces. A 1-cell (Y, D) is said to be a *face* of a 0-cell (X, C) if either

$$D = C \quad \text{and} \quad Y = X - x \quad \text{for some} \quad x \in X, \quad \text{or}$$

$$X = Y \quad \text{and} \quad D = C + i \quad \text{for some} \quad i \in I \smallsetminus C.$$

5. Lemma. *Let (X, C) be a 0-cell. If (X, C) is not balanced, then there is a unique element $y \in X$ such that $c(y) \notin C$, and the 0-cell (X, C) has exactly two faces, namely*

$$(X - y,\ C) \quad \textit{and} \quad (X,\ C + c(y)).$$

These faces have the same type as (X, C). If (X, C) is balanced, then all pairs of the form

$$(X - x,\ C) \quad \textit{and} \quad (X,\ C + i)$$

are faces of (X, C), and for every $i \in I$ the cell (X, C) has exactly one face of the type i.

Proof. If (X, C) is not balanced, then Lemma 4 implies that $|c(X) \smallsetminus C| = 1$ and hence there exists a unique $y \in X$ such that $c(y) \notin C$.

Let us consider pairs of the form $(X - x, C)$. If $x \in X$ and $c(x) \in C$, then

$$C \smallsetminus c(X - x) = \big(C \smallsetminus c(X)\big) + c(x)$$

and hence $|C \smallsetminus c(X - x)| = |C \smallsetminus c(X)| + 1 = 2$. Therefore $(X - x, C)$ can be a cell only if $c(x) \notin C$, i.e. only if $x = y$. Since $c(y) \notin C$, we see that

$$(2) \qquad C \smallsetminus c(X - y) = C \smallsetminus c(X)$$

and hence $|C \smallsetminus c(X - y)| = |C \smallsetminus c(X)| \leqslant 1$. On the other hand, if $x \in X$, then

$$(3) \qquad \min_i X \leqslant_i \min_i (X - x)$$

for every $i \in I$ and hence $X - x$ is dominant with respect to C together with X. It follows that $X - y$ is dominant with respect to C and hence $(X - y, C)$ is a cell and is a face of (X, C). By (2) it has the same type as (X, C).

Let us consider now the pairs of the form $(X, C + i)$. If $i \in I \smallsetminus C$ and $i \notin c(X)$, then

$$(C + i) \smallsetminus c(X) = \big(C \smallsetminus c(X)\big) + i$$

and hence $|(C + i) \smallsetminus c(X)| = |C \smallsetminus c(X)| + 1 = 2$. Therefore $(X, C + i)$ can be a cell only if $i \in c(X) \smallsetminus C$, i.e. only if $i = c(y)$. Since $c(y) \in c(X)$, we see that

$$(4) \qquad C + c(y) \smallsetminus c(X) = C \smallsetminus c(X)$$

and hence $|C + c(y) \smallsetminus c(X)| = |C \smallsetminus c(X)| \leqslant 1$. Since the set X is dominant with respect to C, it is dominant with respect to $C + c(y)$ also. It follows that $(X, C + c(y))$ is a cell and hence is a face of (X, C). By (4) it has the same type as (X, C).

If (X, C) is balanced and $x \in X$, then $|C \smallsetminus c(X - x)| = 1$ and (3) implies that $X - x$ is dominant with respect to C. Hence $(X - x, C)$ is a face of (X, C). The case of $(X, C + i)$ is even simpler. Obviously, the type of $(X - x, C)$ is $c(x)$, and the type of $(X, C + i)$ is i. This implies the last statement of the lemma. ■

6. Lemma. *A 1-cell of the form $(\varnothing, D)$ is a face of exactly one 0-cell. If this 0-cell is not balanced, then it has the same type as $(\varnothing, D)$.*

Proof. If $(\varnothing, D)$ is a 1-cell, then $|D| = 1$ and hence $D = \{i\}$, where i is the type of $(\varnothing, D)$. If $(\varnothing, \{i\})$ is a face of (X, C), then $C = D$ (otherwise $|C| \geqslant |D| + 1 = 2$ and $|X| = 0$) and $X = \{x\}$ for some $x \in T$. The set $\{x\}$ is dominant with respect to $\{i\}$ if and only if x is the maximal element of T with respect to $<_i$. Hence $(\varnothing, \{i\})$ is the face of exactly one cell and type of this cell is i. ■

7. Lemma. *A 1-cell (Y, D) such that $Y \neq \varnothing$ is a face of exactly two 0-cells. Each of these 0-cells is either balanced, or has the same type as (Y, D).*

Proof. If (Y, D) is a face of (X, C), then either $(\mathrm{X}, \mathrm{C}) = (\mathrm{Y}+x, \mathrm{D})$ for some $x \notin \mathrm{Y}$, or $(\mathrm{X}, \mathrm{C}) = (\mathrm{Y}, \mathrm{D}-i)$ for some $i \in \mathrm{C}$. Let us find out when a pair of the form $(\mathrm{Y}+x, \mathrm{D})$ or $(\mathrm{Y}, \mathrm{D}-i)$ is a cell.

Since $|\mathrm{D} \smallsetminus c(\mathrm{Y})| \leqslant 1$, the inclusions

$$\mathrm{D} \smallsetminus c(\mathrm{Y}+x) \subset \mathrm{D} \smallsetminus c(\mathrm{Y}) \quad \text{and} \quad (\mathrm{D}-i) \smallsetminus c(\mathrm{Y}) \subset \mathrm{D} \smallsetminus c(\mathrm{Y})$$

imply that $|\mathrm{D} \smallsetminus c(\mathrm{Y}+x)| \leqslant 1$ and $|(\mathrm{D}-i) \smallsetminus c(\mathrm{Y})| \leqslant 1$. In addition, these inclusions imply that if $(\mathrm{Y}+x, \mathrm{D})$ or $(\mathrm{Y}, \mathrm{D}-i)$ is a cell, then it is either balanced, or has the same type as the cell (Y, D). Therefore, it remains only to find out when Y is dominant with respect to $\mathrm{D}-i$, and when $\mathrm{Y}+x$ is dominant with respect to D.

Since $|\mathrm{D}| = |\mathrm{Y}|+1$, Lemma 2 implies that there is a unique pair $\{a, b\} \subset \mathrm{D}$ such that

$$\min_a \mathrm{Y} = \min_b \mathrm{Y}$$

and $a \neq b$. For $i = a$ or b let

$$\mathcal{M}_i = \left\{\, y \in \mathrm{T} \;\middle|\; \min_k \mathrm{Y} <_k y \text{ for all } k \in \mathrm{D}-i \,\right\}.$$

If $\mathcal{M}_i \neq \varnothing$, then we will denote by m_i the maximal element of $\mathcal{M}_i$ with respect to $<_i$. The lemma immediately follows from the next two sublemmas.

7.1. Sublemma. *The set* Y *is dominant with respect to* $\mathrm{D}-i$ *if and only if* $i \in \{a, b\}$ *and* $\mathcal{M}_i = \varnothing$.

Proof. If $i \neq a, b$, then the set $\{\min_k \mathrm{Y} \mid k \in \mathrm{D}-i\}$ has $\leqslant |\mathrm{D}|-2 = |\mathrm{Y}|-1$ elements and hence Y is not dominant with respect to $\mathrm{D}-i$ by Lemma 2. If $i = a$ or b, then Y is dominant with respect to $\mathrm{D}-i$ if and only if $\mathcal{M}_i = \varnothing$. $\square$

7.2. Sublemma. $\mathrm{Y}+x$ *is dominant with respect to* D *if and only if* $x = m_i$ *for a color* $i \in \{a, b\}$ *such that* $\mathcal{M}_i \neq \varnothing$.

Proof. To begin with, let us observe that

(5) $\quad \min_i(\mathrm{Y}+x) = x \quad$ if $\quad x <_i \min_i \mathrm{Y}, \quad$ and

(6) $\quad \min_i(\mathrm{Y}+x) = \min_i \mathrm{Y} \quad$ if $\quad \min_i \mathrm{Y} <_i x.$

In particular, $\min_i(\mathrm{Y}+x) = \min_i \mathrm{Y}$ or x for every $i \in \mathrm{D}$.

Lemma 2 implies that $\{ \min_i Y \mid i \in D \} = Y$ and that

$$\left\{ \min_i (Y + x) \,\middle|\, i \in D \right\} = Y + x$$

if $Y + x$ is dominant with respect to D. This may happen only if $\min_i (Y + x) = \min_i Y$ for all $i \in D \smallsetminus \{a, b\}$ and for i equal to one of the elements of the pair $\{a, b\}$, and if $\min_i (Y + x) = x$ for i equal to the other element of $\{a, b\}$. We may assume that

(7) $\quad \min_i (Y + x) = \min_i Y \quad$ for all $\quad i \in D - a \quad$ and $\quad \min_a (Y + x) = x.$

By (5) and (6) in this case $\min_i Y <_i x$ for all $i \in D - a$ and $x <_a \min_a Y$. It follows that $x \in \mathscr{M}_a$ and that $Y + x$ can be dominant with respect to D only if x is the maximal element of $\mathscr{M}_a$ with respect to $<_a$, i.e. only if $x = m_a$.

Conversely, if, say, $\mathscr{M}_a \neq \varnothing$ and $x \in \mathscr{M}_a$, then

$$\min_i Y <_i x \quad \text{for all} \quad i \in D - a.$$

If also $\min_a Y <_a x$, then Y is not dominant with respect to D, contrary to the assumption. Therefore $x <_a \min_a Y$. By applying (5) and (6) again, we see that (7) holds. It follows that $Y + x$ is dominant with respect to D if $x = m_a$. □ ■

Proof of Theorem 1. Let us fix an arbitrary color $i \in I$. Let e be the number of balanced 0-cells and f be the number of not balanced 0-cells having the type i. There is only one 1-cell of the form $(\varnothing, D)$ having the type i, namely, the cell $(\varnothing, \{i\})$. Let g be the number of the other 1-cells having the type i.

Let N be the number of pairs (σ, τ) such that σ is a 0-cell which is either balanced or has the type i, and τ a 1-cell of type i which is a face of σ. By Lemma 5 the number of such pairs with a given σ is equal to 1 if σ is balanced and is equal to 2 otherwise. Hence $N = e + 2f$. On the other hand, by Lemmas 6 and 7 the number of such pairs with a given τ is equal to 1 if $\tau = (\varnothing, \{i\})$ and is equal to 2 otherwise. Hence $N = 1 + 2g$. It follows that

$$e + 2f = 1 + 2g.$$

Therefore e is a odd natural number and hence $e \neq 0$. This means that there exists a balanced 0-cell, i.e. a pair of the form $(X, c(X))$ such that X is dominant with respect to $c(X)$. ■

Remark. The numbers e, f, g from the proof of Theorem 1 correspond to the numbers e, f, g in Sperner's proof [S] of his lemma (see also [I]). The number 1 corresponds to the Sperner's h, and the double counting argument is exactly the same as Sperner's one. The analogues of Lemmas 5 – 7 in the context of Sperner's lemma are completely trivial.

2. From linear orders to fixed points

Lattice points in the standard simplex. Let us fix a nonnegative integer $d \geqslant 0$. For a point $x \in \mathbf{R}^d$ we will denote by x_i its i-th coordinate, so that $x = (x_1, x_2, \ldots, x_d)$. Let

$$\Delta^{d-1} = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d_{\geqslant 0} \;\middle|\; x_1 + x_2 + \ldots + x_d = 1 \right\}.$$

be the standard $(d-1)$-simplex. For a natural number $n \geqslant 1$ and let $\mathrm{T} = \mathrm{T}_n \subset \Delta^{d-1}$ be the set of all $x \in \Delta^{d-1}$ such that their coordinates x_i are multiples of $1/n$. In other terms, T is the intersection of the simplex Δ^{d-1} with the lattice $(1/n)\,\mathbf{Z}^d \subset \mathbf{R}^d$.

Let $\mathrm{I} = \{1, 2, \ldots, d\}$ be the set of colors. For each $i \in \mathrm{I}$ let us choose a linear order $<_i$ on T such that

(8) $\qquad x_i < y_i \quad$ implies $\quad x <_i y$

for every $x, y \in \mathrm{T}$ (obviously, such orders exist). Let $\mathrm{X} \subset \mathrm{T}$ and $\mathrm{C} \subset \mathrm{I}$. For $i \in \mathrm{I}$ let

$$x(i) = \min\nolimits_i \mathrm{X}.$$

Let $m = m(\mathrm{X}, \mathrm{C}) \in (1/n)\,\mathbf{Z}^d$ be the point

$$m = (m_1, m_2, \ldots, m_n)$$

with the coordinates $m_i = x(i)_i$ for $i \in \mathrm{C}$ and $m_i = 0$ for $i \notin \mathrm{C}$.

8. Lemma. *If* X *is dominant with respect to* C, *then* $0 \leqslant x_i - m_i < d/n$ *for every* $x \in \mathrm{X}$ *and* $i \in \mathrm{I}$.

Proof. By (8) the point $x(i) \in \mathrm{X}$ has the smallest i-th coordinate among all points of X. Since m_i is equal to the i-th coordinate of $x(i)$, this implies that $0 \leqslant x_i - m_i$ if $i \in \mathrm{C}$. If $i \notin \mathrm{C}$, then $m_i = 0$ and hence $0 \leqslant x_i - m_i$. This proves the left inequalities of the lemma.

As the first (and the main) step in the proof of the right inequalities, let us prove that

(9) $$\sum_{i \in \mathrm{C}} m_i > 1 - (d/n).$$

If this is not the case, then $\displaystyle\sum_{i \in \mathrm{C}} \big(m_i + (1/n) \big) \leqslant 1.$

Since m_i is a multiple of $1/n$ for all $i \in \mathrm{I}$, in this case there exists $\mathrm{M} \in \mathrm{T}$ such that

$$\mathrm{M}_i \geqslant m_i + (1/n)$$

for $i \in \mathrm{C}$ and $\mathrm{M}_i = 0$ for $i \notin \mathrm{C}$. In particular,

$$\min_i \mathrm{X} <_i \mathrm{M}$$

for all $i \in \mathrm{C}$, contrary to X being dominant with respect to C. The inequality (9) follows.

Now we are ready to prove the right inequalities. Let $x \in \mathrm{X}$ and $k \in \mathrm{I}$. If $k \in \mathrm{C}$, then

$$x_k - m_k \leqslant \sum_{i \in \mathrm{C}} (x_i - m_i) \leqslant \sum_{i \in \mathrm{I}} x_i - \sum_{i \in \mathrm{C}} m_i = 1 - \sum_{i \in \mathrm{C}} m_i \,.$$

If $k \notin \mathrm{C}$, then $m_k = 0$ and hence

$$x_k - m_k = x_k \leqslant \sum_{i \notin \mathrm{C}} x_i = 1 - \sum_{i \in \mathrm{C}} x_i \leqslant 1 - \sum_{i \in \mathrm{C}} m_i \,.$$

But $1 - \sum_{i \in \mathrm{C}} m_i < d/n$ by (9), and hence $x_k - m_k < d/n$ for all $k \in \mathrm{I}$. ■

9. Corollary. *If* X *is dominant with respect to* C, *then* $|x_i - y_i| < 2d/n$ *for all* $x, y \in \mathrm{X}$ *and all* $i \in \mathrm{I}$.

Proof. It is sufficient to combine Lemma 8 with the triangle inequality. ■

Knaster–Kuratowski–Mazurkiewicz colorings. Let $f\colon \Delta^{d-1} \longrightarrow \Delta^{d-1}$ be a continuous map.

A coloring $c\colon \mathrm{T} \longrightarrow \mathrm{I}$ is said to be a *Knaster–Kuratowski–Mazurkiewicz coloring* for f if $f(x)_i \geqslant x_i$ for every $x \in \mathrm{T}$ and $i = c(x)$. If $x \in \Delta^{d-1}$, then

$$\sum_{i \in \mathrm{I}} x_i = \sum_{i \in \mathrm{I}} f(x)_i = 1$$

and hence $f(x)_i \geqslant x_i$ for some $i \in \mathrm{I}$. It follows that Knaster–Kuratowski–Mazurkiewicz colorings of T exist.

A modification of the Knaster–Kuratowski–Mazurkiewicz reduction. By Theorem 1 for every $n \geqslant 1$ there exists $\mathrm{X}_n \subset \mathrm{T}_n$ dominant with respect to $c_n(\mathrm{X}_n)$. By Corollary 9 the diameter of X_n tends to 0 as $n \to \infty$. Hence by passing to a subsequence, still denoted by X_n, we may assure that all elements of X_n converge to the same $z \in \Delta^{d-1}$ when $n \to \infty$.

Since there is only a finite number of subsets of I, by passing to a further subsequence we can assure that $c_n(X_n) = C$ for some C independent of n. Then for each $i \in C$ there is a point $z(i, n) \in X_n$ such that $c_n(z(i, n)) = i$. Since c_n is a Knaster–Kuratowski–Mazurkiewicz coloring, passing to the limit $n \to \infty$ shows that $f(z)_i \geqslant z_i$ for all $i \in C$.

On the other hand, Lemma 8 implies that $x_i < d/n$ for every $x \in X_n$ and $i \notin C$. By passing to the limit $n \to \infty$ we conclude that $z_i = 0$ for $i \notin C$. Therefore

$$\sum_{i \in C} z_i = 1.$$

Since $f(z)_i \geqslant z_i$ for all $i \in C$, this inequality implies that $\sum_{i \in C} f(z)_i \geqslant 1$. Since

$$\sum_{i \in I} f(z)_i = 1,$$

the last two inequalities imply that $f(z)_i = 0 = z_i$ for all $i \notin C$ and $f(z)_i = z_i$ for all $i \in C$. Therefore $f(z) = z$, i.e. z is a fixed point of f.

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