

# The Tutte expansion revisited

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With the exception of Introduction, the present paper is self-contained modulo basic concepts related to sets and maps. In particular, no knowledge of the matroid theory or of the graph theory is assumed.

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# Introduction

**The Tutte polynomial and the Tutte expansion.** This paper is a result of a reflection upon a classical theorem of W.T. Tutte [T1]. Let  $G$  be a connected graph. Suppose that a linear order on the set  $E$  of edges of  $G$  is given. W.T. Tutte [T1] defined a polynomial  $\chi(G, \mathbf{x}, \mathbf{y})$  in two variables  $\mathbf{x}, \mathbf{y}$  as the sum

$$\chi(G; \mathbf{x}, \mathbf{y}) = \sum_{T} \mathbf{x}^{r(T)} \mathbf{y}^{s(T)},$$

where  $T$  runs over the set  $\mathcal{T}$  of all spanning trees of  $G$ . The polynomial  $\chi(G; \mathbf{x}, \mathbf{y})$  is called *the Tutte polynomial of  $G$* , and the above sum is called *the Tutte expansion of  $\chi(G; \mathbf{x}, \mathbf{y})$* . The natural numbers  $r(T)$  and  $s(T)$  are the so-called *internal* and *external activities* of a spanning tree  $T$  with respect to the given linear order on  $E$ . The internal and external activities of a spanning tree depend on the choice of a linear order on  $E$ , but the polynomial  $\chi(G; \mathbf{x}, \mathbf{y})$  does not depend on this choice. Therefore  $\chi(G; \mathbf{x}, \mathbf{y})$  is an invariant of the graph  $G$ . This independence on the choice of linear order is a striking result, for which Tutte gave a beautiful and intriguing proof. We will call this result, as also its generalization to matroids (see below), the *Tutte order-independence theorem*.

**The paper.** The present paper is devoted to an elementary, detailed, and self-contained proof of the Tutte order-independence theorem. The remaining part of Introduction is devoted to a discussion of the motivation behind and the novel aspects of this proof.

**The maps  $\varphi, \psi$ .** The internal and external activities  $r(T)$  and  $s(T)$  of a spanning tree  $T$  in a graph are defined as the numbers of *internally* and, respectively, *externally active* edges of  $G$  with respect to a given order on the set  $E$  of edges of  $G$ . The Tutte definition of internally and externally active edges appears to be rather idiosyncratic. In the present paper we do not use these notions at all, and define the Tutte polynomial in terms of the following two maps having spanning trees as values.

Let  $\mathcal{A}$  and  $\mathcal{O}$  be the sets of spanning subgraphs resulting, respectively, from removing an edge from a spanning tree and adding an edge to a spanning tree. A linear order  $<$  on the set of edges allows to define natural maps  $\varphi: \mathcal{A} \rightarrow \mathcal{T}$  and  $\psi: \mathcal{O} \rightarrow \mathcal{T}$ . Namely,

$$\varphi(D) = D + x \quad \text{and} \quad \psi(Q) = Q - y,$$

where  $x$  is the  $<$ -minimal element of  $E \setminus D$  such that  $D + x$  is a spanning tree, and  $y$  is the  $<$ -minimal element of  $Q$  such that  $Q - y$  is a spanning tree, and we denote

by plus and minus the operations of adding an edge to a subgraph and removing an edge from a subgraph, respectively. For a spanning tree  $T$  the numbers of elements in preimages  $\varphi^{-1}(T)$  and  $\psi^{-1}(T)$  are equal to the internal and external activities of  $T$ , respectively, as one can easily check. The maps

$$\varphi: \mathcal{A} \longrightarrow \mathcal{T} \quad \text{and} \quad \psi: \mathcal{O} \longrightarrow \mathcal{T}$$

are the central characters in our approach to the Tutte order-independence theorem.

***The symmetry of the proof.*** The heart of Tutte's proof of the order-independence theorem is an analysis of the effect of replacing a given linear order by a new one differing from it only by the order of two consecutive edges. The non-trivial part of this analysis splits into four cases requiring four similar, but independent, arguments. These four cases exhibit a striking  $\mu_2 \times \mu_2$  symmetry, where  $\mu_2 = \{-1, 1\}$  is the multiplicative group of square roots of  $-1$ .

It turns out that the basic topological tools of the theory of graphs, namely, connected components and cycles, only obscure Tutte's proof and partially destroy its symmetry. Apparently, it is easier to deal with connected components than with cycles, and by this reason connected components are preferred to cycles even when considering cycles is more natural from a logical point of view. At the same time, Tutte proof uses pictures in his proof, and, apparently, prefers more intuitive arguments involving connected components to arguments involving cycles.

***Matroids.*** An attempt to at least partially understand this  $\mu_2 \times \mu_2$  symmetry within our framework inevitably leads to the realization that the Tutte polynomial and the Tutte order-independence theorem are not really about graphs. The right framework for Tutte's results is the theory of matroids.

Let us define a *pre-matroid on a set*  $X$  as a non-empty collection of subsets of  $X$ , called *bases*, such that no basis properly contains another basis. A pre-matroid is called a *matroid* if it satisfies *the exchange property* (see Section 1). Let us denote by  $E(H)$  the set of edges of a graph  $H$ . The matroid associated to the graph  $G$  is a matroid on the set  $E = E(G)$  having as bases all subsets of the form  $E(T) \subset E$ , where  $T \in \mathcal{T}$  is a spanning tree of  $G$ . By the very definition, the Tutte polynomial  $\chi(G; x, y)$  depends only on this matroid associated with  $G$ , but Tutte's proof of his order-independence theorem depends also on the graph  $G$  behind this matroid.

We will deal with matroids from the very beginning. In particular, we will prove the Tutte order-independence theorem for matroids. As a side benefit, our proof disentangles purely combinatorial ideas of Tutte from topological arguments of the theory of graphs. The latter are replaced by topology-independent properties of matroids.

The definition of the Tutte polynomials was extended to matroids long ago by H. Crapo [C], whose motivation, apparently, was quite different. Namely, Crapo's paper is based on his Ph.D. thesis written under supervision of G.-C. Rota and appears to be a part of a far reaching program of G.-C. Rota of integrating combinatorics into the so-called mainstream mathematics. The author hopes that this paper will serve the same goal.

*The symmetry of the proof and linked matroids.* The concept of *matroids duality* (see Appendix 2) explains a half of the  $\mu_2 \times \mu_2$  symmetry of Tutte's proof, but leaves another half unexplained. In order to explain the whole  $\mu_2 \times \mu_2$  symmetry, we introduce notion of *linking* between two matroids on the same set. While every matroid is linked only to itself and to its dual matroid, the notion of a linking identifies the essential features of the theory and allows to replace all four similar arguments by a single one. The resulting theorem is Theorem 9.1 below, the focal point of the present paper.

Only the proof of Theorem 9.1 depends on the exchange property of matroids. All other arguments work for pre-matroids without any modifications (and are usually stated as results about pre-matroids). The exchange property enters the proof of Theorem 9.1 only through Lemma 2.1 and Lemma 2.2. As a result, Lemmas 2.1 and 2.2 emerge as the combinatorial basis of the Tutte theory.

## 1. Pre-matroids and matroids

*Let us fix once and for all a finite set  $X$ .*

*Adding and deleting elements.* For a subset  $Y \subset X$  and an element  $x \in X$  not belonging to  $Y$ , we denote by  $Y + x$  the set  $Y \cup \{x\}$ . Note that  $Y + x$  is defined only if  $x \notin Y$ .

Similarly, for a subset  $Y \subset X$  and an element  $y \in Y$  of this subset, we denote by  $Y - y$  the set  $Y \setminus \{y\}$ . Note that  $Y - y$  is defined only if  $y \in Y$ .

*Pre-matroids.* Let  $\mathcal{P}(X)$  be the set of all subsets of  $X$ . A *pre-matroid*, or a *pre-matroid structure*, on  $X$ , is simply a *non-empty* subset  $\mathcal{B} \subset \mathcal{P}(X)$ . When a subset  $\mathcal{B} \subset \mathcal{P}(X)$  is considered as a pre-matroid, elements  $B \in \mathcal{B}$  are called *bases* of  $\mathcal{B}$ .

*Almost-bases.* Let  $\mathcal{B}$  be a pre-matroid structure on  $X$ . An *almost-basis* of  $\mathcal{B}$  is defined as a subset of  $X$  of the form  $B - x$ , where  $B \in \mathcal{B}$  and  $x \in B$ . A subset  $D \subset X$  is an almost-basis if and only if  $D + x$  is a basis for some element  $x \in X \setminus D$ . For an

almost-basis  $D$  we denote by  $U(D)$  the set of all  $x \in X \setminus D$  such that  $D + x$  is a basis. In other terms,  $x \in U(D)$  if and only if  $x \notin D$  and  $D + x \in \mathcal{B}$ . Suppose that  $C \subset X$ ,  $x, y \notin C$ , and  $x \neq y$ . Then

$$y \in U(C + x) \text{ if and only if } x \in U(C + y).$$

Indeed, both  $y \in U(C + x)$  and  $x \in U(C + y)$  are equivalent to  $C + x + y \in \mathcal{B}$ .

**Over-bases.** An *over-basis* of  $\mathcal{B}$  is defined as a subset of  $X$  of the form  $B + y$ , where  $B \in \mathcal{B}$  and  $y \notin B$ . A subset  $Q \subset X$  is an over-basis if and only if  $Q - y$  is a basis for some element  $y \in Q$ . For an over-basis  $Q$  we denote by  $C(Q)$  the set of all  $y \in Q$  such that  $Q - y$  is a basis. In other terms,  $x \in C(Q)$  if and only if  $x \in Q$  and  $Q - x \in \mathcal{B}$ . Suppose that  $Q \subset X$ ,  $x, y \in Q$ , and  $x \neq y$ . Then

$$y \in C(Q - x) \text{ if and only if } x \in C(Q - y).$$

Indeed, both  $y \in C(Q - x)$  and  $x \in C(Q - y)$  are equivalent to  $Q - x - y \in \mathcal{B}$ .

**Matroids.** A pre-matroid  $\mathcal{B}$  is called a *matroid* if the following *exchange property* holds.

*If  $B_1, B_2$  are bases of  $\mathcal{B}$  and  $x \in B_1 \setminus B_2$ ,  
then  $B_1 - x + y$  is a basis for some  $y \in B_2 \setminus B_1$ .*

Recall that the *symmetric difference*  $P \Delta Q$  of two sets  $P, Q$  is defined as

$$P \Delta Q = (P \setminus Q) \cup (Q \setminus P).$$

If  $B \subset X$ ,  $x, y \in X$ , and  $x \in B$ ,  $y \notin B$ , then, obviously,  $B - x + y = B \Delta \{x, y\}$ .

**1.1. Theorem (Symmetric exchange property).** *If  $\mathcal{B}$  is a matroid,  $B_1, B_2 \in \mathcal{B}$ , and  $x \in B_1 \setminus B_2$ , then there exists  $y \in B_2 \setminus B_1$  such that  $B_1 \Delta \{x, y\} \in \mathcal{B}$  and  $B_2 \Delta \{x, y\} \in \mathcal{B}$ .*

**Proof.** See Appendix 1 for an elementary self-contained proof. ■

**Dual pre-matroids.** For a subset  $Y \subset X$  we denote by  $Y^c$  the complement  $X \setminus Y$  of  $Y$  in  $X$ . The *dual* pre-matroid  $\mathcal{B}^c$  of a pre-matroid  $\mathcal{B}$  is defined as

$$\mathcal{B}^c = \{B^c \mid B \in \mathcal{B}\} \subset \mathcal{P}(X).$$

Obviously,  $\mathcal{B}^{cc} = \mathcal{B}$ .

**1.2. Theorem.** If  $\mathcal{B}$  is a matroid, then the dual pre-matroid  $\mathcal{B}^c$  of  $\mathcal{B}$  is also a matroid.

*Proof.* See Appendix 2. ■

## 2. Triangles

In this section we assume that  $\mathcal{B}$  is a *matroid* structure on  $X$ . The results of this section will be used only in Section 9.

**2.1. Lemma.** Suppose that  $a, z \in X$  and  $a \neq z$ , and suppose the  $C \subset X$  and  $a, z \notin C$ . Suppose that  $\mathcal{B}$  is matroid on  $X$  and  $C + a, C + z$  are almost-bases of  $\mathcal{B}$ . In this situation either  $C + a + z \in \mathcal{B}$ , or  $U(C + a) = U(C + z)$ .

*Proof.* Consider an arbitrary  $e \in U(C + a)$ . Then  $C + a + e \in \mathcal{B}$ . Since  $C + z$  is an almost-basis,  $C + z + v \in \mathcal{B}$  for some  $v \in X$ . If  $v = e$ , then  $e = v \in U(C + z)$ . If  $v \neq e$ , then  $v \in (C + z + v) \setminus (C + a + e)$  and the exchange property implies that

$$C + z + b = (C + z + v) - v + b \in \mathcal{B}$$

for some  $b \in (C + a + e) \setminus (C + z + v)$ . Obviously,  $b = a$  or  $e$ . If  $b = a$ , then  $C + a + z = C + z + b \in \mathcal{B}$ . Otherwise,  $b = e$ , and hence  $C + z + e \in \mathcal{B}$ , i.e.  $e \in U(C + z)$ . Therefore, if  $C + a + z \notin \mathcal{B}$ , then  $e \in U(C + z)$  for every  $e \in U(C + a)$ , i.e.  $U(C + a) \subset U(C + z)$ . Similarly, if  $C + a + z \notin \mathcal{B}$ , then  $e \in U(C + a)$  for every  $e \in U(C + z)$ , i.e.  $U(C + z) \subset U(C + a)$ . The lemma follows. ■

**Triangles.** A *triangle* in  $\mathcal{B}$  is a subset  $C \subset X$  together with three distinct elements  $u, v, w \in X \setminus C$  such that  $C + u + v, C + u + w, C + v + w \in \mathcal{B}$ .

**2.2. Lemma.** Suppose that  $C \subset X$  together with elements  $a, z, d \in X \setminus C$  is a triangle. If  $e \in U(C + z)$ , then either  $e \in U(C + a)$ , or  $e \in U(C + d)$ .

*Proof.* If  $e \in U(C + z)$ , then  $C + z + e \in \mathcal{B}$ . Let us apply the exchange property to the bases  $C + z + e, C + a + d$  and

$$z \in (C + z + e) \setminus (C + a + d).$$

Since  $(C + a + d) \setminus (C + z + e) = \{a, z\}$ , the exchange property implies that either

$$C + a + e, \quad \text{or} \quad C + d + e \in \mathcal{B}.$$

In the first case  $e \in U(C + a)$ , in the second case  $e \in U(C + d)$ . ■

### 3. Permutations and transpositions

**Permutations.** A permutation of  $X$  is simply a bijection  $X \rightarrow X$ . The set of all permutations of  $X$  with the composition  $(\sigma, \tau) \mapsto \sigma \circ \tau$  as the binary operation is well known to be a group. By the very definition, this group acts on  $X$  on the left and therefore acts on other sets canonically related to  $X$ , for example on the set  $\mathcal{P}(X)$  of all subsets of  $X$ . Usually the composition  $\sigma \circ \tau$  is denoted simply by  $\sigma\tau$ .

**Transpositions.** A permutation  $\tau$  of  $X$  is called *transposition* if  $\tau \neq \text{id}_X$  and  $\tau(x) = x$  for all elements  $x \in X$  except two. Then  $\tau$  interchanges these two elements. Clearly, for any two distinct elements  $a, b$  of  $X$  there is a unique transposition interchanging  $a$  and  $b$ . We will denote it by  $\tau_{ab}$ . By the definition,

$$\tau_{ab}(a) = b, \quad \tau_{ab}(b) = a, \quad \text{and} \quad \tau_{ab}(x) = x \quad \text{if} \quad x \neq a, b.$$

The transposition  $\tau_{ab}$  is called the *transposition of*  $a, b$ . Clearly, every transposition  $\tau$  is a non-trivial involution, i.e.  $\tau \circ \tau = \text{id}$  and  $\tau \neq \text{id}$ .

If  $a, b \in X$ ,  $a \neq b$ , and  $\sigma$  is a permutation of  $X$ , then  $\sigma \circ \tau_{ab} \circ \sigma^{-1}$  is the transposition of  $\sigma(a), \sigma(b)$ , as a trivial verification shows. Hence

$$(1) \quad \tau_{\sigma(a)\sigma(b)} = \sigma \circ \tau_{ab} \circ \sigma^{-1} \quad \text{and} \quad \tau_{\sigma(a)\sigma(b)} \circ \sigma = \sigma \circ \tau_{ab}.$$

**Action of transpositions on subsets of  $X$ .** Suppose that  $a, z \in X$  and  $a \neq z$ , and let  $\tau = \tau_{az}$ . Obviously,  $\tau(Y) = Y$  for a subset  $Y \subset X$  if and only if either both  $a, z \in Y$ , or both  $a, z \notin Y$ , and  $\tau(Y) \neq Y$  if and only if exactly one of the elements  $a, z$  belongs to  $Y$ . If  $a \in Y$  and  $z \notin Y$ , then

$$\tau(Y) = Y - a + z,$$

and if  $z \in Y$  and  $a \notin Y$ , then

$$\tau(Y) = Y - z + a.$$

## 4. Linkings of pre-matroids

**Linkings.** Let  $\mathcal{B}$  and  $\mathcal{B}^*$  be two pre-matroids on  $X$ , and let  $L: \bullet \mapsto \bullet^*$  be a bijection  $\mathcal{B} \rightarrow \mathcal{B}^*$ . The bijection  $L$  is said to be a *linking* if for every  $B \in \mathcal{B}$  and every transposition  $\tau: X \rightarrow X$  the following two *linking conditions* hold.

**L1.** If  $\tau(B) \in \mathcal{B}$ , then  $\tau(B^*) \in \mathcal{B}^*$  and  $\tau(B^*) = \tau(B)^*$ .

**L2.** If  $\tau(B^*) \in \mathcal{B}^*$ , then  $\tau(B) \in \mathcal{B}$  and  $\tau(B^*) = \tau(B)^*$ .

Since  $L$  is a bijection, the condition **L2** is equivalent to the condition **L1** for the inverse map  $L^{-1}$ . In particular, a bijection  $L: \mathcal{B} \rightarrow \mathcal{B}^*$  is a linking if and only if  $L^{-1}$  is a linking. Obviously, the identity map  $\text{id}: \mathcal{B} \rightarrow \mathcal{B}$  and the map  $c: \mathcal{B} \rightarrow \mathcal{B}^*$  defined by  $c: B \mapsto B^c$  are linkings. The definition of a linking is motivated not by a desire of greater generality, but by the desire to unify these two examples.

**4.1. Theorem.** If  $L: \mathcal{B} \rightarrow \mathcal{B}^*$  is a linking, then either  $\mathcal{B}^* = \mathcal{B}$  and  $L = \text{id}$ , or  $\mathcal{B}^* = \mathcal{B}^c$  and  $L = c$ .

**Proof.** See Appendix 5. ■

**4.2. Lemma.** Let  $L: \mathcal{B} \rightarrow \mathcal{B}^*$  be a linking. Suppose that  $S$  is an almost-basis of  $\mathcal{B}$  and  $A$  is an almost-basis of  $\mathcal{B}^*$ . Suppose that  $x, y \in U(S)$  and  $x \neq y$ . If  $(S + x)^* = A + y$ , then  $x \in U^*(A)$ .

**Proof.** By Theorem 4.1, it is sufficient to consider only the linkings  $\text{id}: \mathcal{B} \rightarrow \mathcal{B}$  and  $c: \mathcal{B} \rightarrow \mathcal{B}^c$ . See Appendix 4 for a direct proof not relying on Theorem 4.1.

If  $\mathcal{B}^* = \mathcal{B}$  and  $L = \text{id}$ , then  $(S + x)^* = S + x$ . But  $y \notin S + x$  because  $y \neq x$  and  $y \notin S$  (because  $S \cap U(S) = \emptyset$ ). Therefore, in this case  $(S + x)^*$  cannot have the form  $A + y$ , and the lemma is trivially true.

If  $\mathcal{B}^* = \mathcal{B}^c$  and  $L = c$ , then  $A + y = (S + x)^c$  and hence

$$A = (S + x)^c - y = (S + x + y)^c,$$

and  $A + x = (S + y)^c = (S + y)^*$ . But  $S + y \in \mathcal{B}$  because  $y \in U(S)$ . Therefore  $A + x = (S + y)^* \in \mathcal{B}^*$ , and hence  $x \in U^*(A)$ . ■



## 5. Orders

**Orders.** Given an order  $\omega$  on  $X$  and two elements  $x, y \in X$ , both  $x <_\omega y$  and  $x >_\omega y$  will be used as a shorthand for “ $x$  is less than  $y$  with respect to  $\omega$ ”.

We will consider only *linear orders* on  $X$ , i.e. orders  $\omega$  on  $X$  such that for every two elements  $x, y \in X$  either  $x = y$ , or  $x <_\omega y$ , and  $y <_\omega x$ . Given a linear order  $\omega$  on  $X$ , we will denote by  $\min_\omega Y$  the minimal element of a subset  $Y \subset X$ .

**Action of permutations on orders.** Given an order  $\omega$  on  $X$  and a permutation  $\sigma \in \Sigma_X$ , the order  $\sigma \cdot \omega$  is defined as the unique order such that

$$(2) \quad \sigma: (X, \omega) \rightarrow (X, \sigma \cdot \omega)$$

is an isomorphism of ordered sets. In other words, if  $x, y \in X$ , then

$$x <_\omega y \quad \text{if and only if} \quad \sigma(x) <_{\sigma \cdot \omega} \sigma(y).$$

Obviously,  $\sigma \cdot \omega$  is a linear order if and only if  $\omega$  is. The map  $(\omega, \sigma) \mapsto \sigma \cdot \omega$  is a left action of the group of permutation of  $X$  on the set of orders on  $X$ , i.e.

$$(3) \quad \tau \cdot (\sigma \cdot \omega) = (\tau \circ \sigma) \cdot \omega$$

for all pairs  $\tau, \sigma$  of permutations of  $X$  and all orders  $\omega$  on  $X$ .

**Consecutive elements.** Two elements  $x, y \in X$  are called *consecutive elements with respect to the order  $\omega$*  if either  $x <_\omega y$  and there exist no elements  $u \in X$  such that  $x <_\omega u <_\omega y$ , or  $y <_\omega x$  and there exist no elements  $u \in X$  such that  $y <_\omega u <_\omega x$ .

If  $\omega$  is a linear order on  $X$  and  $\varepsilon$  is the transposition of two elements  $a, z$  consecutive with respect to  $\omega$ , then the orders  $\omega$  and  $\varepsilon \cdot \omega$  differ only in order of elements  $a, z$ . In other words,  $x <_\omega y$  is equivalent to  $x <_{\varepsilon \cdot \omega} y$  unless  $\{x, y\} = \{a, z\}$ . At the same time,  $a <_\omega z$  is equivalent to  $z <_{\varepsilon \cdot \omega} a$  by the definition of  $\varepsilon \cdot \omega$ . It follows, in particular, that the elements  $a, z$  are consecutive with respect to the order  $\varepsilon \cdot \omega$ .

**The graph of linear orders on  $X$ .** The graph  $\mathcal{L}_X$  of linear orders on  $X$  has the set of all linear orders on  $X$  as its set of vertices. Two linear order  $\omega, \omega'$  are connected by an edge if  $\omega' = \tau_{ab} \cdot \omega$  for some elements  $a, b \in X$  consecutive with respect to  $\omega$ . If this is the case, then also  $\omega = \tau_{ab} \cdot \omega'$  and  $a, b$  are consecutive with respect to  $\omega'$ . Obviously, the transposition  $\tau_{ab}$  is uniquely determined by the edge connecting

$\omega, \omega'$ , or, what is the same, by the pair  $\omega, \omega'$ . For an edge  $\varepsilon$  we will denote by  $\tau(\varepsilon)$  the corresponding transposition.

**Remark.** The group of permutations of  $X$  canonically acts on  $\mathcal{L}_X$ . Indeed, this group acts on the set of linear orders, i.e. on the set of vertices of  $\mathcal{L}_X$ . Let  $\sigma$  be a permutation of  $X$ . Suppose that  $\omega' = \tau_{ab} \cdot \omega$  for some elements  $a, b \in X$  consecutive with respect to  $\omega$ . By the second identity in (1)

$$\sigma \cdot \omega' = \sigma \cdot \tau_{\sigma(a)\sigma(b)} \cdot \omega = \tau_{\sigma(a)\sigma(b)} \cdot \sigma \cdot \omega.$$

Obviously, the elements  $\sigma(a), \sigma(b)$  are consecutive with respect to  $\sigma \cdot \omega$ . It follows that  $\sigma$  takes edges to edges. Therefore, the group of permutations of  $X$  acts on  $\mathcal{L}_X$ .

**5.1. Lemma.** *Suppose that  $\omega, \omega'$  are two linear orders on  $X$ . Then there exists a sequence*

$$\omega = \omega_1, \omega_2, \dots, \omega_n = \omega'$$

*of linear orders on  $X$  such that for every  $i = 1, 2, \dots, n-1$  the order  $\omega_{i+1}$  is equal to  $\varepsilon_i \cdot \omega_i$  for some transposition  $\varepsilon_i$  of two elements consecutive with respect to  $\omega_i$ .*

**Proof.** See Appendix 3. ■

**5.2. Corollary.** *The graph  $\mathcal{L}_X$  is connected.* ■

## 6. Multi-sets

**Multi-subsets.** A *multi-subset* of a set  $S$  is defined as a function  $m: S \rightarrow \mathbb{N}$ . The set  $\{s \in S \mid m(s) \neq 0\}$  is called the *support* of the multi-subset. A subset  $Y$  of  $S$  canonically defines a multi-subset of  $S$ , namely, its characteristic function  $\chi_Y: S \rightarrow \mathbb{N}$ . Recall that  $\chi_Y(s) = 1$  if  $s \in Y$  and  $\chi_Y(s) = 0$  if  $s \notin Y$ . Obviously, the support of the multi-set  $\chi_Y$  is nothing else but the subset  $Y$ . By identifying subsets with their characteristic functions, we may consider subsets as multi-subsets.

The values  $m(s)$  of the characteristic function of a multi-subset are interpreted as multiplicities of element  $s \in S$  in this multi-subset. In other words, we think that a multi-subset  $m$  contains  $m(s)$  copies of  $s$  for each  $s \in S$ . Elements  $s$  with  $m(s) = 0$  are treated as elements not contained in the multi-subset  $m$ .

**Multi-sets.** A *multi-set* is defined as a multi-subset of a fixed once and for all universal set  $\mathcal{U}$ . Given a set  $S$ , one can identify the multi-subsets of  $S$  with multi-sets  $m$  such that  $m(x) = 0$  for all  $x \notin S$ .

**Multi-images.** Let  $f: S \rightarrow R$  be a map and  $m: S \rightarrow \mathbb{N}$  be a multi-subset of  $S$ . The multi-subset  $f[m]: R \rightarrow \mathbb{N}$  defined by

$$f[m]: r \mapsto \sum_{f(s)=r} m(s).$$

is called the *multi-image* of  $m$  under the map  $f$ . For a subset  $Y \subset S$ , the *multi-image*  $f[Y]$  of  $Y$  is defined as the multi-image  $f[\chi_Y]$  of the corresponding multi-subset. Clearly, the multiplicity of an element  $r \in R$  in the multi-image  $f[Y]$  is equal to the number of elements of  $(f|_Y)^{-1}(r)$ , where  $f|_Y: Y \rightarrow R$  is the restriction of  $f$  to  $Y$ . The following lemma is obvious.

**6.1. Lemma.** If  $f: S \rightarrow R$  is a map and  $g: S \rightarrow S$  is a bijection, then

$$f[S] = f \circ g[S]. \quad \blacksquare$$

**Multi-subsets of  $\mathbb{N} \times \mathbb{N}$ .** Let  $\mathbf{x}, \mathbf{y}$  be two different variables. One can associate with each multi-subset  $m: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  of  $\mathbb{N} \times \mathbb{N}$  a formal power series  $P_m(\mathbf{x}, \mathbf{y})$  in the variables  $\mathbf{x}, \mathbf{y}$  by the formula

$$P_m(\mathbf{x}, \mathbf{y}) = \sum_{(a,b) \in \mathbb{N} \times \mathbb{N}} m(a,b) \mathbf{x}^a \mathbf{y}^b.$$

Obviously, the map  $m \mapsto P_m(\mathbf{x}, \mathbf{y})$  establishes a one-to-one correspondence between the multi-subsets of  $\mathbb{N} \times \mathbb{N}$  and formal power series with integer non-negative coefficients in the variables  $\mathbf{x}, \mathbf{y}$ . Moreover, the formal power series  $P_m(\mathbf{x}, \mathbf{y})$  is a polynomial if and only if the multi-subset  $m$  has finite support. The following lemma follows directly from the definitions.

**6.2. Lemma.** Let  $S$  be a finite set and  $f: S \rightarrow \mathbb{N} \times \mathbb{N}$  be a map. Let  $\mathbf{a}, \mathbf{b}: S \rightarrow \mathbb{N}$  be the two components of  $f$ , i.e.  $f(s) = (\mathbf{a}(s), \mathbf{b}(s))$  for every  $s \in S$ . Then the polynomial corresponding to the multi-image  $f[S]$  is equal to

$$P_{f[S]}(\mathbf{x}, \mathbf{y}) = \sum_{s \in S} \mathbf{x}^{\mathbf{a}(s)} \mathbf{y}^{\mathbf{b}(s)}. \quad \blacksquare$$

## 7. The Tutte polynomials and the Whitney multi-sets

Let  $\mathcal{B}$  be a pre-matroid on  $X$  and let  $\omega$  be a linear order on  $X$ . Let  $\mathcal{A}$  and  $\mathcal{O}$  be the sets of all almost-bases and over-bases of  $\mathcal{B}$ , respectively.

**The Tutte polynomial of a pre-matroid.** Let  $u_\omega: \mathcal{A} \rightarrow X$ ,  $c_\omega: \mathcal{O} \rightarrow X$  be the maps defined, respectively, by

$$u_\omega(D) = \min_\omega U(D), \quad c_\omega(Q) = \min_\omega C(Q),$$

Let  $\varphi_\omega: \mathcal{A} \rightarrow \mathcal{B}$ ,  $\psi_\omega: \mathcal{O} \rightarrow \mathcal{B}$  be the maps defined, respectively, by

$$\varphi_\omega(D) = D + u_\omega(D), \quad \psi_\omega(Q) = Q - c_\omega(Q).$$

For a basis  $B \in \mathcal{B}$ , let  $i_\omega(B)$  and  $e_\omega(B)$  be the numbers of elements in the pre-images  $(\varphi_\omega)^{-1}(B)$  and  $(\psi_\omega)^{-1}(B)$  respectively.

Let  $\mathbf{x}$ ,  $\mathbf{y}$  be two different variables. The *Tutte polynomial* of the pre-matroid  $\mathcal{B}$  with respect to the order  $\omega$  is defined as

$$T_\omega(\mathcal{B})(\mathbf{x}, \mathbf{y}) = \sum_{B \in \mathcal{B}} \mathbf{x}^{i_\omega(B)} \mathbf{y}^{e_\omega(B)}.$$

It turns out that the Tutte polynomial  $T_\omega(\mathcal{B})(\mathbf{x}, \mathbf{y})$  of  $\mathcal{B}$  does not depend on the order  $\omega$  if  $\mathcal{B}$  is a matroid.

**Tutte's activities.** For a basis  $B \in \mathcal{B}$ , let  $\mathcal{J}_\omega(B)$  and  $\mathcal{E}_\omega(B)$  be the sets

$$\mathcal{J}_\omega(B) = \{u_\omega(D) \mid D \in (\varphi_\omega)^{-1}(B)\},$$

$$\mathcal{E}_\omega(B) = \{c_\omega(Q) \mid Q \in (\psi_\omega)^{-1}(B)\}.$$

By the very definition, the sets  $\mathcal{J}_\omega(B)$  and  $\mathcal{E}_\omega(B)$  are nothing else but, respectively, the sets of *internally active* and *externally active* elements of  $X$  with respect to the basis  $B$  and the linear order  $\omega$  in the sense of Tutte. The maps  $u_\omega$  and  $c_\omega$  induce bijections

$$(\varphi_\omega)^{-1}(B) \longrightarrow \mathcal{J}_\omega(B), \quad (\psi_\omega)^{-1}(B) \longrightarrow \mathcal{E}_\omega(B),$$

and hence  $i_\omega(B)$  and  $e_\omega(B)$  are nothing else but, respectively, the *internal activity* and the *external activity* of the basis  $B$  with respect to the order  $\omega$  in the sense of Tutte. Therefore, the polynomial  $T_\omega(\mathcal{B})$  is nothing else but the classical Tutte polynomial.

**The Tutte polynomial of a linking.** Let  $\mathcal{B}^*$  be another pre-matroid on  $X$  and let  $\bullet \mapsto \bullet^*$  be a linking  $\mathcal{B} \rightarrow \mathcal{B}^*$ . Let  $\mathcal{A}^*$  be the set of all almost-bases of  $\mathcal{B}^*$ .

As before, let  $\mathbf{x}, \mathbf{y}$  be two different variables. The *Tutte polynomial* of the linking  $\mathcal{B} \rightarrow \mathcal{B}^*$  with respect to the order  $\omega$  is defined as

$$T_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)(\mathbf{x}, \mathbf{y}) = \sum_{B \in \mathcal{B}} \mathbf{x}^{i_\omega(B)} \mathbf{y}^{i_\omega(B^*)}.$$

**The Whitney multi-set of a linking.** The Tutte polynomial  $T_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)(\mathbf{x}, \mathbf{y})$  has a natural interpretation as a multi-subset of  $\mathbb{N} \times \mathbb{N}$ . Consider the map

$$\mathfrak{W}_\omega(\mathcal{B} \rightarrow \mathcal{B}^*) : \mathcal{B} \rightarrow \mathbb{N} \times \mathbb{N}$$

defined by  $B \mapsto (i_\omega(B), i_\omega(B^*))$ . The multi-image of  $\mathcal{B}$  under this map is a multi-subset of  $\mathbb{N} \times \mathbb{N}$ . We call this multi-image the *Whitney multi-set* of the linking  $\mathcal{B} \rightarrow \mathcal{B}^*$  with respect to the order  $\omega$  and denote it by  $W_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)$ .

**7.1. Theorem (Order-independence of the Whitney multi-sets).** *If  $\mathcal{B}$  is a matroid, then the Whitney multi-set  $W_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)(\mathbf{x}, \mathbf{y})$  does not depend on the order  $\omega$ .*

Sections 8 – 10 are devoted to a proof of this theorem.

**7.2. Corollary (Order-independence for linkings of matroids).** *If  $\mathcal{B}$  and  $\mathcal{B}^*$  are matroids, then the Tutte polynomial  $T_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)(\mathbf{x}, \mathbf{y})$  does not depend on the order  $\omega$ .*

**Proof.** Since  $\mathcal{B} \subset \mathcal{P}(X)$  is finite together with  $X$ , the multi-image  $W_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)$  of the map  $\mathfrak{W}_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)$  has finite support, and hence the corresponding power series is a polynomial. Lemma 6.2 implies that this polynomial is equal to the Tutte polynomial  $T_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)(\mathbf{x}, \mathbf{y})$ . It remains to apply Theorem 7.1. ■

**7.3. Corollary (Order-independence for matroids).** *If  $\mathcal{B}$  is a matroid, then the Tutte polynomial  $T_\omega(\mathcal{B})(\mathbf{x}, \mathbf{y})$  does not depend on the order  $\omega$ .*

**Proof.** Recall that  $\mathbf{c}: \mathcal{B} \mapsto \mathcal{B}^c$  is a linking  $\mathcal{B} \rightarrow \mathcal{B}^c$ . An immediate application of the matroid duality (see Appendix 2, Lemma A.2.2) shows that  $T_\omega(\mathcal{B} \rightarrow \mathcal{B}^c)(\mathbf{x}, \mathbf{y})$  is equal to the Tutte polynomial  $T_\omega(\mathcal{B})(\mathbf{x}, \mathbf{y})$ . It remains to apply Corollary 7.3. ■

## 8. Branching and balance

*The framework.* Let  $\mathcal{B}$  be pre-matroid on  $X$ , and let  $\mathcal{A}$  be the set of almost-bases of  $\mathcal{B}$ . Let  $\mathcal{E}$  be an edge of  $\mathcal{L}_X$ , and let  $\varepsilon = \tau(\mathcal{E})$  be the corresponding transposition.

Let  $\omega, \pi$  be the linear orders on  $X$  connected by  $\mathcal{E}$ , and let  $a, z$  be the elements of  $X$  interchanged by  $\varepsilon$ . Then the orders  $\omega$  and  $\pi$  differ only by the order of elements  $a, z$ , and the elements  $a, z$  are consecutive with respect to both  $\omega$  and  $\pi$ . Without any loss of generality one may assume that  $a <_\omega z$  and  $z <_\pi a$ .

If  $A \subset X$  and  $a, z \notin A$ , then  $\varepsilon(A + a) = A + z$  and  $\varepsilon(A + z) = A + a$ . In particular,  $\varepsilon(A + a) \neq A + a$  and  $\varepsilon(A + z) \neq A + z$ .

*Branching almost-bases.* An almost-basis  $A \in \mathcal{A}$  is said to be  $\mathcal{E}$ -branching if

$$\varphi_\omega(A) \neq \varphi_\pi(A).$$

Clearly, if  $A$  is  $\mathcal{E}$ -branching, then the orders  $\omega$  and  $\pi$  differ on  $U(A)$  and hence  $a, z \in U(A)$ . In particular,  $a, z \notin A$  and hence  $\varepsilon(A) = A$ . Moreover, one of the elements  $a, z$  should be equal to  $\min_\omega U(A)$ , and the other to  $\min_\pi U(A)$ . Since  $a <_\omega z$  and  $z <_\pi a$ , in this case

$$(4) \quad \min_\omega U(A) = a \quad \text{and} \quad \varphi_\omega(A) = A + a,$$

$$(5) \quad \min_\pi U(A) = z \quad \text{and} \quad \varphi_\pi(A) = A + z.$$

Conversely, suppose that  $a, z$  are the two smallest elements of  $U(A)$  with respect to  $\omega$ . Since  $a <_\omega z$  and  $z <_\pi a$ , in this case  $a = \min_\omega U(A)$  and  $z = \min_\pi U(A)$ . This implies (4) and (5). Since  $a \neq z$ , it follows that  $A$  is  $\mathcal{E}$ -branching.

*Balanced almost-bases.* An almost-basis  $Q \in \mathcal{A}$  is said to be  $\omega$ -balanced with respect to  $\mathcal{E}$  if  $\varepsilon(Q)$  is also an almost-basis, i.e.  $\varepsilon(Q) \in \mathcal{A}$ , and

$$\varepsilon(\varphi_\omega(Q)) = \varphi_\pi(\varepsilon(Q)).$$

Clearly,  $Q \in \mathcal{A}$  is  $\pi$ -balanced with respect to  $\mathcal{E}$  if and only if  $\varepsilon(Q) \in \mathcal{A}$  and

$$\varepsilon(\varphi_\pi(Q)) = \varphi_\omega(\varepsilon(Q)).$$

It follows that  $Q \in \mathcal{A}$  is  $\pi$ -balanced with respect to  $\mathcal{E}$  if and only if  $\varepsilon(Q) \in \mathcal{A}$  and  $\varepsilon(Q)$  is  $\omega$ -balanced with respect to  $\mathcal{E}$ .

An almost-basis  $Q \in \mathcal{A}$  is said to be  $\mathcal{E}$ -balanced if  $Q$  is both  $\omega$ -balanced and  $\pi$ -balanced with respect to  $\mathcal{E}$ . In view of the previous paragraph,  $Q \in \mathcal{A}$  is  $\mathcal{E}$ -balanced if and only if  $\varepsilon(Q) \in \mathcal{A}$  and both  $Q$  and  $\varepsilon(Q)$  are  $\omega$ -balanced with respect to  $\mathcal{E}$ . In particular, if  $Q, \varepsilon(Q) \in \mathcal{A}$ , then  $Q$  is  $\mathcal{E}$ -balanced if and only if  $\varepsilon(Q)$  is.

**8.1. Lemma.** *If  $A \in \mathcal{A}$  is  $\mathcal{E}$ -branching, then  $A$  is  $\mathcal{E}$ -balanced.*

**Proof.** If  $A \in \mathcal{A}$  is  $\mathcal{E}$ -branching, then  $\varepsilon(\varphi_\omega(A)) = \varepsilon(A + \mathfrak{a}) = A + z$ , and

$$\varphi_\pi(\varepsilon(A)) = \varphi_\pi(A) = A + z.$$

It follows that  $A$  is  $\omega$ -balanced. By a similar argument,  $A$  is  $\pi$ -balanced. ■

**Non-branching almost-bases.** An almost-basis  $Q \in \mathcal{A}$  is said to be  $\mathcal{E}$ -non-branching if

$$\varphi_\omega(Q) = \varphi_\pi(Q).$$

In this case we will denote by  $\varphi(Q)$  the coinciding images  $\varphi_\omega(Q)$  and  $\varphi_\pi(Q)$ .

**8.2. Lemma.** *Suppose that  $Q, \varepsilon(Q) \in \mathcal{A}$  and both almost-bases  $Q$  and  $\varepsilon(Q)$  are  $\mathcal{E}$ -non-branching. Then  $Q$  is  $\mathcal{E}$ -balanced if and only if*

$$(6) \quad \varepsilon(\varphi(Q)) = \varphi(\varepsilon(Q)),$$

*and if and only if  $\varepsilon(Q)$  is  $\mathcal{E}$ -balanced.*

**Proof.** The first equivalence is obvious. In order to prove the second equivalence, let us replace  $Q$  by  $\varepsilon(Q)$  in (6) and apply  $\varepsilon$  to the result. Since  $\varepsilon \circ \varepsilon = \text{id}$ , the resulting condition is equivalent to  $\varphi(\varepsilon(Q)) = \varepsilon(\varphi(Q))$  and hence to (6). ■

**8.3. Lemma.** *Suppose that  $Q$  is an  $\mathcal{E}$ -non-branching almost-basis and  $\varphi(Q) = Q + \mathfrak{d}$ . If  $\varepsilon(Q + \mathfrak{d}) \neq Q + \mathfrak{d}$  and  $\varepsilon(Q + \mathfrak{d}) \in \mathcal{B}$ , then  $\mathfrak{d} \neq \mathfrak{a}, z$ .*

**Proof.** Suppose that  $\mathfrak{d} = \mathfrak{a}$  or  $z$ . Then  $\{\mathfrak{d}, \varepsilon(\mathfrak{d})\} = \{\mathfrak{a}, z\}$ , and since  $\varepsilon(Q + \mathfrak{d}) \neq Q + \mathfrak{d}$ , exactly one of the elements  $\mathfrak{a}, z$  belongs to  $Q + \mathfrak{d}$ .

Therefore, if  $\mathfrak{d} = \mathfrak{a}$  or  $z$ , then  $\mathfrak{a}, z \notin Q$  and hence

$$\varepsilon(Q) = Q, \quad \varepsilon(Q + \mathfrak{d}) = Q + \varepsilon(\mathfrak{d}).$$

Since  $\varepsilon(Q + \mathfrak{d}) \in \mathcal{B}$ , in this case  $\varepsilon(\mathfrak{d}) \in \mathcal{U}(Q)$ , and hence  $\mathfrak{a}, z \in \mathcal{U}(Q)$ .

By the definition,  $\mathfrak{d}$  is the minimal element of  $\mathcal{U}(Q)$  with respect to both  $\omega$  and  $\pi$ . It follows that if  $\mathfrak{d} = \mathfrak{a}$  or  $z$ , then  $\mathfrak{a}, z \in \mathcal{U}(Q)$  and one of the elements  $\mathfrak{a}, z$  is the minimal element of  $\mathcal{U}(Q)$  with respect to both  $\omega$  and  $\pi$ . But this contradicts to the fact that  $\mathfrak{a} <_\omega z$  and  $z <_\pi \mathfrak{a}$ . ■

**Balanced bases.** A basis  $B$  is said to be  $\omega$ -balanced if  $\varepsilon(B) \in \mathcal{B}$  and every almost-basis  $Q$  such that

$$(7) \quad \text{either } \varphi_\omega(Q) = B, \quad \text{or } \varphi_\pi(Q) = \varepsilon(B)$$

is  $\mathcal{E}$ -balanced. A basis is said to be  $\mathcal{E}$ -balanced if it is both  $\omega$ -balanced and  $\pi$ -balanced. By interchanging the roles of  $\omega$  and  $\pi$ , we see that  $B$  is  $\pi$ -balanced if and only if  $\varepsilon(B) \in \mathcal{B}$  and every almost-basis  $Q$  such that

$$(8) \quad \text{either } \varphi_\pi(Q) = B, \quad \text{or } \varphi_\omega(Q) = \varepsilon(B)$$

is  $\mathcal{E}$ -balanced. It follows that  $B$  is  $\omega$ -balanced if and only if  $\varepsilon(B)$  is  $\pi$ -balanced, and that  $B$  is  $\mathcal{E}$ -balanced if and only if  $\varepsilon(B)$  is  $\mathcal{E}$ -balanced.

**8.4. Lemma.** If  $B \in \mathcal{B}$  is  $\omega$ -balanced with respect to  $\mathcal{E}$ , then  $\varepsilon$  induces a bijective map

$$(9) \quad (\varphi_\omega)^{-1}(B) \longrightarrow (\varphi_\pi)^{-1}(\varepsilon(B)).$$

**Proof.** If  $\varphi_\omega(Q) = B$ , then  $\varepsilon(Q) \in \mathcal{A}$ ,

$$\varepsilon(\varphi_\omega(Q)) = \varphi_\pi(\varepsilon(Q)),$$

and hence  $\varphi_\pi(\varepsilon(Q)) = \varepsilon(B)$ . Similarly, if  $\varphi_\pi(Q) = \varepsilon(B)$ , then  $\varepsilon(Q) \in \mathcal{A}$ ,

$$\varepsilon(\varphi_\pi(Q)) = \varphi_\omega(\varepsilon(Q)),$$

and hence  $\varphi_\omega(\varepsilon(Q)) = \varepsilon(\varepsilon(B)) = B$ . It follows that  $\varepsilon$  maps  $(\varphi_\omega)^{-1}(B)$  into  $(\varphi_\pi)^{-1}(\varepsilon(B))$  and maps  $(\varphi_\pi)^{-1}(\varepsilon(B))$  into  $(\varphi_\omega)^{-1}(B)$ . Since  $\varepsilon \circ \varepsilon = \text{id}$ , the two maps induced by  $\varepsilon$  are mutually inverse, and hence both of them are bijections. ■

**$\mathcal{E}$ -branching images.** A basis  $B$  is said to be an  $\omega$ -branching image if  $B = \varphi_\omega(A)$  for some  $\mathcal{E}$ -branching  $A \in \mathcal{A}$ . By (4), in this case  $B = A + \alpha$  and

$$(10) \quad \varepsilon(B) = A + z = \varphi_\pi(A) \in \mathcal{B}.$$

Trivially,  $B$  is  $\pi$ -branching image if and only if  $B = \varphi_\pi(A)$  for some  $\mathcal{E}$ -branching  $A \in \mathcal{A}$ . By (5), in this case  $B = A + z$  and

$$(11) \quad \varepsilon(B) = A + \alpha = \varphi_\omega(A) \in \mathcal{B}.$$

It follows that  $B$  is an  $\pi$ -branching image if and only if  $\varepsilon(B)$  is an  $\omega$ -branching image.



A basis  $B$  is said to be an  $\mathcal{E}$ -branching image if it is either an  $\omega$ -branching image, or a  $\pi$ -branching image. In view of the previous paragraph,  $B$  is an  $\mathcal{E}$ -branching image if and only if either  $B$ , or  $\varepsilon(B)$  is equal to  $\varphi_\omega(A)$  for some  $\mathcal{E}$ -branching  $A \in \mathcal{A}$ . Therefore  $B$  is an  $\mathcal{E}$ -branching image if and only if  $\varepsilon(B)$  is.

**8.5. Lemma.** *If  $B$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B) \in \mathcal{B}$  and  $\varepsilon(B)$  is an  $\mathcal{E}$ -branching image. Moreover,  $\varepsilon(B) \neq B$  and  $B$  contains exactly one of the elements  $a, z$ .*

**Proof.** The first statement of the lemma follows from (10) and (11). If  $B$  is an  $\mathcal{E}$ -branching image, then  $B = A + a$  or  $A + z$  for some almost-basis  $A$  not containing  $a, z$ . This implies the second statement of the lemma. ■

**8.6. Lemma.** *If  $B$  is not an  $\mathcal{E}$ -branching image, then  $(\varphi_\omega)^{-1}(B) = (\varphi_\pi)^{-1}(B)$ .*

**Proof.** If  $B$  is not an  $\mathcal{E}$ -branching image, then  $B$  is neither an  $\omega$ -branching, nor  $\pi$ -branching image with respect to  $\mathcal{E}$ . Therefore, if either  $Q \in (\varphi_\omega)^{-1}(B)$ , or  $Q \in (\varphi_\pi)^{-1}(B)$ , then  $Q$  is  $\mathcal{E}$ -non-branching almost-basis, and hence  $\varphi_\omega(Q) = \varphi_\pi(Q)$ . ■

## 9. Forced balance

**The linking framework.** As before, we assume that  $\omega, \pi$  are two linear orders connected by an edge  $\mathcal{E}$  of  $\mathcal{L}_X$ , denote by  $\varepsilon$  the corresponding transposition  $\tau(\mathcal{E})$ , and by  $a, z$  be the elements interchanged by  $\varepsilon$ . We may assume that  $a <_\omega z$  and  $z <_\pi a$ . In the rest of this section the edge  $\mathcal{E}$  will be omitted from the notations.

Let  $\mathcal{B}$  and  $\mathcal{B}^*$  be two *matroids* on  $X$ , and let  $\mathcal{B} \mapsto \mathcal{B}^*$  be a linking  $\mathcal{B} \rightarrow \mathcal{B}^*$ . Adjusting the notations for  $\mathcal{B}$ , we will denote by  $\mathcal{A}^*$  the set of almost-bases of  $\mathcal{B}^*$ , and for  $Q \in \mathcal{A}^*$  we will denote by  $U^*(Q)$  the set of all  $x \in X$  such that  $x \notin Q$  and  $Q + x \in \mathcal{B}^*$ . The same orders  $\omega, \pi$  and the same edge  $\mathcal{E}$  will be used for both pre-matroids  $\mathcal{B}$  and  $\mathcal{B}^*$ .

**9.1. Theorem.** *Suppose that  $Q$  is a non-branching almost-basis of  $\mathcal{B}$  and*

$$\varphi(Q)^* = A + a \quad \text{or} \quad A + z$$

*for some branching almost-basis  $A$  of  $\mathcal{B}^*$ . Then  $\varepsilon(Q)$  is also a non-branching almost-basis, and both almost-bases  $Q$  and  $\varepsilon(Q)$  are balanced.*

**Proof.** Let  $d = \min_{\omega} U(Q)$ . Then  $d \in U(Q)$  and  $\varphi(Q) = Q + d$ . Moreover,  $(Q + d)^* = A + a$  or  $A + z$  and hence

$$\varepsilon((Q + d)^*) \in \mathcal{B}^* \quad \text{and} \quad \varepsilon((Q + d)^*) \neq (Q + d)^*.$$

Now the linking condition **L2** and the injectivity of the linking map imply that

$$\varepsilon(Q + d) \in \mathcal{B} \quad \text{and} \quad \varepsilon(Q + d) \neq Q + d.$$

Therefore we can apply Lemma 8.3 and conclude that  $d \neq a, z$ .

In turn,  $d \neq a, z$  implies that  $\varepsilon(d) = d$  and hence  $\varepsilon(Q + d) = \varepsilon(Q) + d$ . Since  $\varepsilon(Q + d) \in \mathcal{B}$ , it follows that  $\varepsilon(Q) + d \in \mathcal{B}$ . Therefore  $\varepsilon(Q) \in \mathcal{A}$  and  $d \in U(Q)$ . In addition,  $\varepsilon(Q + d) \neq Q + d$  together with  $d \neq a, z$  implies that  $\varepsilon(Q) \neq Q$ , and hence exactly one of the elements  $a, z$  is in  $Q$ .

Suppose now that not only  $d = \min_{\omega} U(Q)$ , but also  $d = \min_{\omega} U(\varepsilon(Q))$ . Because  $d \neq a, z$ , in this case the almost-basis  $\varepsilon(Q)$  is non-branching and

$$\varphi(\varepsilon(Q)) = \varepsilon(Q) + d.$$

And since  $\varepsilon(Q + d) = \varepsilon(Q) + d$ , in this case  $\varepsilon(\varphi(Q)) = \varepsilon(Q) + d = \varphi(\varepsilon(Q))$ . It follows that  $Q$  is balanced. By Lemma 8.2 this implies that  $\varepsilon(Q)$  is also balanced. It remains to prove that  $d = \min_{\omega} U(\varepsilon(Q))$ .

Let  $b$  be the element of the pair  $\{a, z\}$  contained in  $Q$ , and let  $w$  be the other element. Clearly, the elements  $b, w$  are consecutive. Let  $C = Q - b$ . Then

$$C + b = Q \in \mathcal{A}, \quad C + w = \varepsilon(Q) \in \mathcal{A}, \quad C + d = (Q + d) - b \in \mathcal{A}.$$

Suppose that  $C + b + w \notin \mathcal{B}$ . Then  $U(C + b) = U(C + w)$  by Lemma 2.1. In other terms,  $U(Q) = U(\varepsilon(Q))$  and hence

$$d = \min_{\omega} U(Q) = \min_{\omega} U(\varepsilon(Q)).$$

It follows that in this case indeed  $d = \min_{\omega} U(\varepsilon(Q))$ .  $\square$

**The triangular case.** Suppose now that  $C + b + w \in \mathcal{B}$ . Since

$$C + b + d = Q + d \in \mathcal{B},$$

$$C + w + d = \varepsilon(Q) + d = \varepsilon(Q + d) \in \mathcal{B},$$

in this case  $C$  together with  $b, w, d$  forms a triangle in the sense of Section 2.

Note that in this case  $w \in \mathcal{U}(Q)$  because  $Q + w = C + b + w \in \mathcal{B}$ . This implies, in particular, that  $d <_{\omega} w$ . Since the elements  $b, w$  are consecutive, in this case  $d <_{\omega} b$  also. It follows that  $d <_{\omega} a, z$ .

Let us prove that  $(Q + d)^* = A + b$ . Recall that  $(Q + d)^* = A + a$  or  $A + z$ . In other terms,  $(Q + d)^* = A + b$  or  $A + w$ . Suppose that  $(Q + d)^* = A + w$ . Since  $w \in \mathcal{U}(Q)$ , in this case one can apply Lemma 4.2 to  $S = Q$ ,  $x = d$ ,  $y = w$ , and conclude that  $d \in \mathcal{U}^*(A)$ . On the other hand,  $A$  is branching and hence  $a, z$  are the two smallest elements of  $\mathcal{U}(A)$ . Since  $d \in \mathcal{U}^*(A)$ , this contradicts to  $d <_{\omega} a, z$ . Therefore  $(Q + d)^* \neq A + w$  and hence  $(Q + d)^* = A + b$ .

Let us consider an arbitrary  $e \in \mathcal{U}(\varepsilon(Q)) = \mathcal{U}(C + w)$  and prove that  $d \leq_{\omega} e$ . By Lemma 2.2 either  $e \in \mathcal{U}(C + a) = \mathcal{U}(Q)$ , or  $e \in \mathcal{U}(C + d)$ . If  $e \in \mathcal{U}(Q)$ , then  $d \leq_{\omega} e$  by the definition of  $d$ . Suppose now that  $e \in \mathcal{U}(C + d)$ . Since  $d <_{\omega} b$ , we may assume that  $d \neq e$ . Note that

$$(C + d) + b = (C + b) + d = Q + d$$

and hence  $b \in \mathcal{U}(C + d)$ . Since, as we saw above,  $(Q + d)^* = A + b$ , in this case one can apply Lemma 4.2 to  $S = C + d$ ,  $x = e$ ,  $y = b$ , and conclude that  $e \in \mathcal{U}^*(A)$ . Since  $a, z$  are the smallest elements of  $\mathcal{U}^*(A)$ , it follows that  $a <_{\omega} e$ . Together with  $d <_{\omega} a$  this implies that  $d <_{\omega} e$ . This completes the proof of the inequality  $d \leq_{\omega} e$  for all  $e \in \mathcal{U}(\varepsilon(Q))$ .

It follows that in this case also  $d = \min_{\omega} \mathcal{U}(\varepsilon(Q))$ . The theorem follows.  $\square$   $\blacksquare$

**9.2. Corollary.** *Let  $B \in \mathcal{B}$ . If  $B^*$  is an  $\mathcal{E}$ -branching image in  $\mathcal{B}^*$ , then  $B$  is a balanced.*

*Proof.* If  $B^*$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B^*) \in \mathcal{B}$  by Lemma 8.5. Therefore the linking property L2 implies that  $\varepsilon(B) \in \mathcal{B}$  and  $\varepsilon(B^*) = \varepsilon(B)^*$ .

It remains to prove that for every  $Q \in \mathcal{A}$  each of the two conditions (7) and (8) implies that  $Q$  is balanced. In view of Lemma 8.1, we may assume that  $Q$  is non-branching. In this case both conditions (7) and (8) mean that  $\varphi(Q)$  is equal either to  $B$ , or to  $\varepsilon(B)$ . It follows that  $\varphi(Q)^*$  is equal either to  $B^*$ , or to  $\varepsilon(B)^* = \varepsilon(B^*)$ . By Lemma 8.5  $\varepsilon(B^*)$  is an  $\mathcal{E}$ -branching image together with  $B^*$ . By Theorem 9.1 this implies that  $Q$  is  $\mathcal{E}$ -balanced.  $\blacksquare$

**9.3. Lemma.** *If either  $B$ , or  $B^*$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon$  induces a bijective map*

$$(\varphi_{\omega})^{-1}(B) \longrightarrow (\varphi_{\pi})^{-1}(\varepsilon(B)).$$

*Proof.* By applying Corollary 9.2 either to the identity linking  $\mathcal{B} \mapsto \mathcal{B}$  or to the linking  $\mathcal{B} \mapsto \mathcal{B}^*$  we see that  $\mathcal{B}$  is balanced. It remains to apply Lemma 8.4. ■

## 10. Coda: the order-independence

We continue to work under assumptions described at the beginning of Section 9.

*The map  $\mathcal{B} \rightarrow \mathcal{B}$  induced by the edge  $\varepsilon$ .* For a basis  $B \in \mathcal{B}$ , let  $\varepsilon_B = \varepsilon$  if either  $B$ , or  $B^*$  is an  $\mathcal{E}$ -branching image, and let  $\varepsilon_B = \text{id}_X$  otherwise. Let

$$\sigma(B) = \varepsilon_B(B).$$

By Lemma 8.5, if  $B \in \mathcal{B}$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B) \in \mathcal{B}$ , and if  $B^*$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B^*) \in \mathcal{B}^*$ . The linking property **L2** implies that in the latter case  $\varepsilon(B) \in \mathcal{B}$  also. It follows that  $\sigma$  is a map  $\mathcal{B} \rightarrow \mathcal{B}$ .

**10.1. Lemma.**  $\sigma(B) \neq B$  if and only if either  $B$ , or  $B^*$  is an  $\mathcal{E}$ -branching image.

*Proof.* If  $B$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B) \neq B$  by Lemma 8.5. Similarly, if  $B^*$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B^*) \neq B^*$ . Moreover, in this case  $\varepsilon(B)^* = \varepsilon(B^*)$  by the linking property **L2** and hence  $\varepsilon(B)^* \neq B^*$ . By the injectivity of the linking map, in this case  $\varepsilon(B) \neq B$  also. Since in both cases  $\sigma(B) = \varepsilon(B)$ , the “if” part of the lemma follows. The “only if” part follows immediately from the definitions. ■

**10.2. Lemma.** If  $\sigma(B) \neq B$ , then  $\varepsilon_B = \varepsilon$ ,  $\varepsilon_{\varepsilon(B)} = \varepsilon$ , and

$$\sigma(B) = \varepsilon(B), \quad \sigma(\varepsilon(B)) = B.$$

*Proof.* If  $\sigma(B) \neq B$ , then Lemma 10.1 implies that either  $B$ , or  $B^*$  is an  $\mathcal{E}$ -branching image. It follows that  $\varepsilon_B = \varepsilon$ , and hence  $\sigma(B) = \varepsilon(B)$ .

By Lemma 8.5, if  $B$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B)$  is also an  $\mathcal{E}$ -branching image. Therefore, in this case  $\varepsilon_{\varepsilon(B)} = \varepsilon$ , and hence  $\sigma(\varepsilon(B)) = \varepsilon(\varepsilon(B)) = B$ .

By Lemma 8.5 applied to  $\mathcal{B}^*$  in the role of  $\mathcal{B}$ , if  $B^*$  is an  $\mathcal{E}$ -branching image, then  $\varepsilon(B^*) \in \mathcal{B}^*$  and  $\varepsilon(B^*)$  is an  $\mathcal{E}$ -branching image. Since  $\varepsilon(B^*) \in \mathcal{B}^*$ , the linking property **L2** implies that  $\varepsilon(B^*) = \varepsilon(B)^*$ . It follows that  $\varepsilon(B)^*$  is an  $\mathcal{E}$ -branching image. Therefore  $\varepsilon_{\varepsilon(B)} = \varepsilon$  and  $\sigma(\varepsilon(B)) = \varepsilon(\varepsilon(B)) = B$ . ■

**10.3. Corollary.**  $\sigma$  is an involution, i.e.  $\sigma \circ \sigma = \text{id}$ . In particular,  $\sigma$  is a bijection. ■

**10.4. Lemma.** For every basis  $B$  of  $\mathcal{B}$ , the map  $\varepsilon_B$  induces bijections

$$(12) \quad (\varphi_\omega)^{-1}(B) \longrightarrow (\varphi_\pi)^{-1}(\sigma(B)),$$

$$(13) \quad (\varphi_\omega^*)^{-1}(B^*) \longrightarrow (\varphi_\pi^*)^{-1}(\sigma(B^*)),$$

*Proof.* If either  $B$ , or  $B^*$  is an  $\varepsilon$ -branching image, then  $\sigma(B) \neq B$  by Lemma 10.1, and hence  $\varepsilon_B = \varepsilon$  and  $\sigma(B) = \varepsilon(B)$  by Lemma 10.2. Therefore, Lemma 9.3 implies that  $\varepsilon_B$  induces a bijection (12). If neither  $B$ , nor  $B^*$  is a branching image, then  $\varepsilon_B = \text{id}_X$  and  $\sigma(B) = B$ . Therefore, in this case Lemma 8.6 implies that  $\varepsilon_B$  induces a bijection (12).

Note that replacing the linking  $\mathcal{B} \rightarrow \mathcal{B}^*$  by the inverse linking  $\mathcal{B}^* \rightarrow \mathcal{B}$  does not affect neither  $\varepsilon_B$ , nor  $\sigma$ . Therefore, by applying the already proved part of the lemma to the inverse linking  $\mathcal{B}^* \rightarrow \mathcal{B}$ , we see that  $\varepsilon_B$  induces a bijection (13). ■

**10.5. Corollary.** For every basis  $B$  of  $\mathcal{B}$  the following equalities hold.

$$\text{card} (\varphi_\omega)^{-1}(B) = \text{card} (\varphi_\pi)^{-1}(\sigma(B)).$$

$$\text{card} (\varphi_\omega^*)^{-1}(B^*) = \text{card} (\varphi_\pi^*)^{-1}(\sigma(B^*)). \quad \blacksquare$$

Let us restate the last corollary in terms of the Tutte activities, and then in terms of the maps  $\mathfrak{W}_\bullet(\mathcal{B} \rightarrow \mathcal{B}^*)$  (see Section 7).

**10.6. Corollary.**  $i_\omega(B) = i_\pi(\sigma(B))$  and  $i_\omega(B) = i_\pi(\sigma(B)^*)$  for every  $B \in \mathcal{B}$ . ■

**10.7. Corollary.**  $\mathfrak{W}_\omega(\mathcal{B} \rightarrow \mathcal{B}^*) = \mathfrak{W}_\pi(\mathcal{B} \rightarrow \mathcal{B}^*) \circ \sigma$ . ■

*Proof of Theorem 7.1.* Suppose now that  $\omega$  and  $\pi$  are two arbitrary linear orders on  $X$ . In view of Lemma 6.1, the last corollary implies that

$$(14) \quad \mathbb{W}_\omega(\mathcal{B} \rightarrow \mathcal{B}^*) = \mathbb{W}_\pi(\mathcal{B} \rightarrow \mathcal{B}^*)$$

if the orders  $\omega$  and  $\pi$  are connected by an edge of  $\mathcal{L}_X$ . In view of Lemma 5.1, the equality (14) for linear orders  $\omega, \pi$  connected by an edge implies the equality (14) for arbitrary two linear orders  $\omega, \pi$ . It follows that the Whitney multi-set  $\mathbb{W}_\omega(\mathcal{B} \rightarrow \mathcal{B}^*)$  of  $\mathcal{B} \rightarrow \mathcal{B}^*$  with respect to  $\omega$  does not depend on the choice of  $\omega$ . ■

## Appendix 1. The symmetric exchange property

In this appendix  $\mathcal{B}$  is assumed to be a *matroid* on  $X$ .

**A.1.1. Lemma.** *No basis of  $\mathcal{B}$  is properly contained in another basis. In particular, no almost-basis is a basis and no over-basis is a basis.*

**Proof.** If  $B_1, B_2 \in \mathcal{B}$ ,  $B_1 \subset B_2$ , and  $B_1 \neq B_2$ , then  $B_2 \setminus B_1 \neq \emptyset$ . By applying the exchange property to any  $x \in B_2 \setminus B_1$ , we see, in particular, that  $B_1 \setminus B_2$  is non-empty. The contradiction with  $B_1 \subset B_2$  proves the first statement of the lemma, which immediately implies the second one. ■

**A.1.2. Lemma.** *Suppose that  $J \subset S \subset X$ . If  $S$  contains a basis and  $J$  is contained in a basis, then there exist a basis  $B$  such that  $J \subset B \subset S$ .*

**Proof.** Consider all bases containing  $J$  and choose among them a basis  $B$  such that  $B \cap S$  is maximal with respect to the inclusion. Note that  $J \subset B \cap S$  because  $J \subset B$  and  $J \subset S$ . It is sufficient to prove that  $B \subset S$ .

Suppose that  $B \not\subset S$ . Then  $B \setminus S \neq \emptyset$ . Let  $x \in B \setminus S$ . Consider some basis  $\beta \subset S$ . Clearly,  $x \in B \setminus \beta$ . By the exchange property,  $B - x + y \in \mathcal{B}$  for some  $y \in \beta \setminus B$ . Since  $x \notin S$  and, obviously,  $y \in S \setminus B$ , we see that

$$(B - x + y) \cap S = (B \cap S) + y.$$

and hence  $(B - x + y) \cap S$  properly contains  $B \cap S$ , and at the same time  $J \subset B \cap S \subset (B - x + y) \cap S$ . But this contradicts to the choice of  $B$ . It follows that  $B \subset S$ . ■

**Independent sets and circuits.** A subset  $Y \subset X$  is called *independent* if  $Y \subset B$  for some  $B \in \mathcal{B}$ , and *dependent* if  $Y$  is not contained in any  $B \in \mathcal{B}$ . Clearly, any subset of an independent set is independent. A subset  $C \subset X$  is called a *circuit* if  $C$  is dependent, but  $C - x$  is independent for every  $x \in C$ .

**A.1.3. Lemma.** *For every over-basis  $Q$  the set  $C(Q)$  is a circuit.*

**Proof.** By the definition,  $x \in C(Q)$  if and only if  $Q - x \in \mathcal{B}$ . Therefore, if  $x \in C(Q)$ , then  $C(Q) - x \subset Q - x \in \mathcal{B}$ . It follows that  $C(Q) - x$  is independent for every  $x \in C(Q)$ . It remains to prove that  $C(Q)$  is dependent.

Suppose that  $C(Q)$  is independent. Then  $C(Q)$  is contained in some basis. Since  $Q$  is an over-basis,  $Q$  contains a basis. By applying Lemma A.1.2 to  $J = C(Q)$  and  $S = Q$ , we see that  $C(Q) \subset B \subset Q$  for some  $B \in \mathcal{B}$ . By Lemma A.1.1  $Q$  is not a basis, and hence  $B \neq Q$ . It follows that  $Q \setminus B \neq \emptyset$ .

Let  $q \in C(Q)$ . Then  $q \in B$  and  $Q - q \in \mathcal{B}$ . Consider some  $x \in Q \setminus B = (Q - q) \setminus B$ . By the exchange property,

$$(Q - q) - x + y \in \mathcal{B}$$

for some  $y \in B \setminus (Q - q)$ . But  $B \setminus (Q - q) = \{q\}$  because  $B \subset Q$  and  $q \in B$ . It follows that  $y = q$  and hence

$$(Q - q) - x + y = Q - q - x + q = Q - x.$$

It follows that  $Q - x \in \mathcal{B}$  and hence  $x \in C(Q)$ . But  $x \notin B$  and hence  $x \notin C(Q) \subset B$ . Therefore, the assumption that  $C(Q)$  is independent leads to a contradiction. It follows that  $C(Q)$  is a dependent set. ■

**A.1.4. Lemma.** *Let  $A$  be an almost-basis, and let  $Y \subset X$  be a subset disjoint from  $A$ . If  $A + y \notin \mathcal{B}$  for all  $y \in Y$ , then  $A \cup Y$  does not contain any basis.*

**Proof.** Since  $A$  is an almost-basis,  $A + a \in \mathcal{B}$  for some  $a \notin A$ . Then, in particular,  $a \notin Y$ , and hence  $a \notin A \cup Y$ .

Suppose that  $B \in \mathcal{B}$  and  $B \subset A \cup Y$ . Then  $a \notin B$  and hence

$$B \setminus (A + a) = B \setminus A, \quad (A + a) \setminus B = (A \setminus B) + a.$$

In particular,  $a \in (A + a) \setminus B$ . By the exchange property of matroids, there exists an element  $y \in B \setminus (A + a) = B \setminus A$  such that  $(A + a) - a + y = A + y$  is a basis. But  $B \setminus A \subset Y$  and hence  $y \in Y$ . This contradicts to the assumption that  $A + y$  is not a basis for all  $y \in Y$ . It follows that  $A \cup Y$  cannot contain a basis. ■

**A.1.5. Lemma.** *Let  $A$  be an almost-basis, and let  $Y \subset X$  be a subset disjoint from  $A$ . Suppose that  $A + y \notin \mathcal{B}$  for all  $y \in Y$ . If  $C$  is a circuit such that  $C - c \subset A \cup Y$  for some  $c \in C$ ,  $c \notin A \cup Y$ , then  $A \cup (Y + c)$  does not contain any basis.*

**Proof.** Suppose that  $A \cup (Y + c)$  contains a basis. Since  $C$  is a circuit, the set  $C - c$  is independent, i.e. is contained in a basis. By applying Lemma A.1.2 to  $J = C - c$  and  $S = A \cup (Y + c)$ , we see that  $C - c \subset B \subset A \cup (Y + c)$  for some  $B \in \mathcal{B}$ .

If  $c \in B$ , then  $C = (C - c) + c \subset B$  and hence  $C$  is independent. Since  $C$  is a circuit, this is impossible. It follows that  $c \notin B$  and hence  $B \subset A \cup Y$ . But the last inclusion contradicts to Lemma A.1.4. Therefore,  $A \cup (Y + c)$  cannot contain a basis. This proves the lemma. ■

**A.1.6. Lemma.** *Suppose that  $A$  is an almost-basis and  $Q$  is an over-basis. If the intersection  $U(A) \cap C(Q)$  is not empty, then it contains at least 2 elements.*

**Proof.** Suppose that  $U(A) \cap C(Q)$  consists of only one element  $c$ . Then  $A + y \notin \mathcal{B}$  for all  $y \in C(Q) - c$ ,  $y \notin A$ . Let  $C = C(Q)$  and  $Y = (C \setminus A) - c$ . Then  $A$ ,  $Y$ ,  $C$ , and  $c$  satisfy the assumptions of Lemma A.1.5. This lemma implies that  $A \cup (Y + c)$  does not contain any basis. But  $Y + c = C \setminus A$  and hence  $A \cup (Y + c) = A \cup C$ . It follows that  $c \in A \cup (Y + c)$  and hence  $A \cup (Y + c)$  contains  $A + c$ . But  $c \in U(A)$  and hence  $A + c$  is a basis. The contradiction shows that  $U(A) \cap C(Q)$  cannot consist of only one element. The lemma follows. ■

**Proof of Theorem 1.1.** Suppose that  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \setminus B_2$ . Let

$$E_1 = \{y \in B_2 \setminus B_1 \mid B_2 - y + x \in \mathcal{B}\}$$

$$E_2 = \{y \in B_2 \setminus B_1 \mid B_1 - x + y \in \mathcal{B}\}$$

It is sufficient to prove that  $E_1 \cap E_2 \neq \emptyset$ .

Note that  $B_2 + x$  is an over-basis, and the condition  $B_2 - y + x \in \mathcal{B}$  is equivalent to  $(B_2 + x) - y \in \mathcal{B}$ . Note also that  $(B_2 + x) \setminus B_1 = B_2 \setminus B_1$  because  $x \in B_1$ . It follows that  $E_1 = C(B_2 + x) \setminus B_1$ . Similarly,  $B_1 - x$  is an almost-basis, and the condition  $B_1 - x + y \in \mathcal{B}$  is equivalent to  $(B_1 - x) + y \in \mathcal{B}$ . Note also that  $(B_1 - x) \cap B_2 = B_1 \cap B_2$  because  $x \notin B_2$ . It follows that  $E_2 = U(B_1 - x) \cap B_2$ .

Note that  $x \in C(B_2 + x)$  and  $x \in U(B_1 - x)$ , and hence the intersection  $U(B_1 - x) \cap C(B_2 + x)$  is not empty. By Lemma A.1.6 it consists of at least 2 elements. Let  $y$  be some element of this intersection different from  $x$ .

Then  $y \notin B_1$  because  $y \neq x$  and  $U(B_1 - x)$  is disjoint from  $B_1 - x$ . Similarly,  $y \in B_2$  because  $C(B_2 + x) \subset B_2 + x$  and  $y \neq x$ . It follows that  $y \in E_1 = C(B_2 + x) \setminus B_1$  and  $y \in E_2 = U(B_1 - x) \cap B_2$ . Therefore  $y \in E_1 \cap E_2$  and hence  $E_1 \cap E_2 \neq \emptyset$ . The theorem follows. ■

**A.1.7. Remark.** *For every over-basis  $Q$  the set  $C(Q)$  is the only circuit contained in  $Q$ .*



*Proof.* Let  $C \subset Q$  be an arbitrary circuit in  $Q$ . If  $x \in C(Q)$ , but  $x \notin C$ , then  $C \subset Q - x$  and  $Q - x$  is a basis. In this case  $C$  is independent, in contradiction with being a circuit. It follows that  $C(Q) \subset C$ .

If  $C(Q) \neq C$ , then  $C(Q) \subset C$  implies that  $C(Q) \subset C - x$  for some  $x \in C$ . Since  $C$  is a circuit,  $C - x$  is independent, and hence  $C(Q)$  is also independent. But  $C(Q)$  is a circuit by Lemma A.1.3, and hence is a dependent set. It follows that  $C = C(Q)$ . ■

## Appendix 2. Duality

*The dual of a pre-matroid.* As usual, let  $\mathcal{B}$  be a pre-matroid on  $X$ , and let  $\mathcal{B}^c$  be its dual pre-matroid (see Section 1). Let  $\mathcal{A}^c$  and  $\mathcal{O}^c$  be, respectively, the sets of all almost-bases and all over-bases of  $\mathcal{B}^c$ . For  $D \in \mathcal{A}^c$  we will denote by  $U^c(D)$  the set of all  $x \in X$  such that  $x \notin D$  and  $D + x \in \mathcal{B}^c$ . Similarly, for  $Q \in \mathcal{O}^c$  we will denote by  $C^c(Q)$  the set of all  $x \in Q$  such that  $Q - x \in \mathcal{B}^c$ .

*Proof of Theorem 1.2.* Let  $\mathcal{B}$  be a matroid. By Theorem 1.1  $\mathcal{B}$  has the symmetric exchange property. By taking complements the symmetric exchange property of  $\mathcal{B}$  translates into the following property of  $\mathcal{B}^c$ .

*If  $B_1, B_2$  are bases of  $\mathcal{B}^c$  and  $x \in B_2 \setminus B_1$ , then there exists  $y \in B_1 \setminus B_2$  such that both  $B_2 \triangle \{x, y\}$  and  $B_1 \triangle \{x, y\}$  are bases of  $\mathcal{B}^c$ .*

This property turns into the symmetric exchange property of  $\mathcal{B}^c$  after interchanging the roles of  $B_1$  and  $B_2$ . It follows that the dual  $\mathcal{B}^c$  of  $\mathcal{B}$  also satisfies the symmetric exchange property. Since the symmetric exchange property trivially implies the exchange property, the dual  $\mathcal{B}^c$  has the exchange property. Therefore  $\mathcal{B}^c$  is a matroid. ■

*Remark.* The symmetric exchange property cannot be replaced in the proof of Theorem 1.2 by the exchange property. Indeed, taking complements turns the exchange property of  $\mathcal{B}$  into the following property of  $\mathcal{B}^c$ .

*If  $B_1, B_2$  are bases of  $\mathcal{B}^c$  and  $x \in B_2 \setminus B_1$ , then  $B_1 + x - y$  is a basis of  $\mathcal{B}^c$  for some  $y \in B_1 \setminus B_2$ .*

This property is different from the exchange property, and does not turn into the exchange property after interchanging the roles of  $B_1$  and  $B_2$ .

**A.2.1. Lemma** Let  $Q \subset X$ . Then  $Q \in \mathcal{O}$  if and only if  $Q^c \in \mathcal{A}^c$ . If  $Q \in \mathcal{O}$ , then

$$(15) \quad C(Q)^c = U^c(Q^c). \quad \blacksquare$$

**Maps defined by a linear order.** Suppose that a linear order  $\omega$  on  $X$  is fixed. Let

$$\varphi_\omega^c: \mathcal{A}^c \longrightarrow \mathcal{B}^c, \quad \psi_\omega^c: \mathcal{O}^c \longrightarrow \mathcal{B}^c$$

be the maps defined as the maps  $\varphi_\omega, \psi_\omega$  defined in Section 7, but with  $\mathcal{B}^c$  playing the role of  $\mathcal{B}$ . For  $B \in \mathcal{B}^c$  let  $i_\omega^c(B)$  be the number of elements in  $(\varphi_\omega)^{-1}(B)$ . Lemma A.2.1 immediately implies the following lemma.

**A.2.2. Lemma.** If  $Q \in \mathcal{O}$ , then  $\psi_\omega(Q)^c = \varphi_\omega^c(Q^c)$ .  $\blacksquare$

**A.2.3. Lemma.** If  $B \in \mathcal{B}$ , then  $e_\omega(B) = i_\omega^c(B^c)$ .

**Proof.** Let  $B \in \mathcal{B}$ . Lemma A.2.2 implies that  $B = \psi(Q)$  if and only if  $B^c = \varphi_\omega^c(Q^c)$ . Therefore, the map  $Q \mapsto Q^c$  induces a bijection

$$\psi^{-1}(B) \longrightarrow (\varphi_\omega^c)^{-1}(B^c),$$

and hence these two sets have the same number of elements. The lemma follows.  $\blacksquare$

## Appendix 3. Permutations and transpositions

**A.3.1. Lemma.** Suppose that  $n \in \mathbb{N}$  and  $n \geq 2$ . Let  $X_n = \{1, 2, \dots, n\}$ , and for each  $i = 1, 2, \dots, n-1$  let  $\varepsilon_i$  be the transposition of elements  $i, i+1$ . Then every permutation of  $X_n$  is equal to a composition of several transpositions of the form  $\varepsilon_i$ .

**Proof.** We use induction by  $n$ . There are only two permutations of  $X_2$ , namely,  $\text{id}_X$  and  $\varepsilon_1$ . Let  $m \in \mathbb{N}$  and  $m > 2$ . Suppose that the lemma is true for all  $n < m$ , and consider a permutation  $\sigma$  of the set  $X_m = \{1, 2, \dots, m\}$ .

If  $\sigma(m) = m$ , then  $\sigma$  induces a permutation of  $X_{m-1}$ . By the inductive assumption the induced permutation is equal to a composition of several transpositions of  $X_{m-1}$  of the required form. Then  $\sigma$  is equal to the composition of transpositions of the same elements, but considered as elements of  $X_m$  (note that all of them leave  $m$  fixed).

It remains to consider the case when  $\sigma(m) < m$ . Let  $a = \sigma(m)$ . Then

$$\varepsilon_{m-1} \circ \dots \circ \varepsilon_{a+1} \circ \varepsilon_a \circ \sigma(m) = m.$$

By the previous paragraph the permutation  $\tau = \varepsilon_{m-1} \circ \dots \circ \varepsilon_a \circ \sigma$  is equal to a composition of transpositions  $\varepsilon_i$ . Since every transposition is equal to its own inverse,

$$\sigma = \varepsilon_a \circ \varepsilon_{a+1} \circ \dots \circ \varepsilon_{m-1} \circ \tau,$$

and hence  $\sigma$  is also equal to a composition of transpositions  $\varepsilon_i$ . ■

**Proof of Lemma 5.1.** We may assume that  $X = X_n = \{1, 2, \dots, n\}$  for some  $n \in \mathbb{N}$ , and that the order  $\omega$  is induced by the standard order on  $\mathbb{N}$ . Any other linear order on  $X$  has the form  $\sigma \cdot \omega$  for some permutation  $\sigma$  of  $X$ .

Let  $\varepsilon_i$  be the transpositions defined in Lemma A.3.1. If  $\tau$  is a permutation of  $X$ , then  $\tau_i = \tau \circ \varepsilon_i \circ \tau^{-1}$  is the transposition of elements  $\tau(i)$  and  $\tau(i+1)$ , which are consecutive with respect to the order  $\tau \cdot \omega$ . At the same time  $\tau_i \circ \tau = \tau \circ \varepsilon_i$  and hence

$$(16) \quad \tau_i \cdot (\tau \cdot \omega) = (\tau_i \circ \tau) \cdot \omega = (\tau \circ \varepsilon_i) \cdot \omega.$$

By Lemma A.3.1 every permutation  $\sigma$  of  $X$  is equal to a composition of several transpositions of the form  $\varepsilon_i$ . It follows that one can get from  $\text{id}_X$  to  $\sigma$  by a sequence of right compositions with  $\varepsilon_i$ . In view of (16), this implies that one can connect the standard order  $\omega$  with the order  $\sigma \cdot \omega$  by a sequence of orders of the form

$$\omega = \tau_0 \cdot \omega, \quad \tau_1 \cdot \omega, \quad \tau_2 \cdot \omega, \quad \dots, \quad \tau_m \cdot \omega = \sigma \cdot \omega$$

(where, of course,  $\tau_0 = \text{id}$  and  $\tau_m = \sigma$ ) such that  $\tau_{i+1} \cdot \omega$  is the image of  $\tau_i \cdot \omega$  under the transposition of two elements consecutive with respect to  $\tau_i \cdot \omega$  for all  $i = 0, 1, 2, \dots, m-1$ . The lemma follows. ■

## Appendix 4. A direct proof of Lemma 4.2

Let  $\mathcal{B}, \mathcal{B}^*$  be pre-matroids on  $X$ , and let  $L: \bullet \mapsto \bullet^*$  be a linking  $\mathcal{B} \mapsto \mathcal{B}^*$ .

**A.4.1. Lemma.** *Suppose that  $S$  is an almost-basis of  $\mathcal{B}$  and  $x, y \in \mathcal{U}(S)$ ,  $x \neq y$ . Let  $D = (S + x)^*$ . Then  $\tau_{xy}(D) \in \mathcal{B}^*$  and exactly one of the elements  $x, y$  is in  $D$ .*

*Proof.* Since  $S + x \in \mathcal{B}$  and  $\tau_{xy}(S + x) = S + y \in \mathcal{B}$ , we see that

$$(17) \quad \tau_{xy}((S + x)^*) = (\tau_{xy}(S + x))^* = (S + y)^* \in \mathcal{B}^*.$$

by the condition **L1**. Obviously,  $S + x \neq S + y$ . Therefore

$$(18) \quad (S + y)^* \neq (S + x)^*$$

by the injectivity of the linking map. Since  $D = (S + x)^*$ , (17) implies that  $\tau_{xy}(D) \in \mathcal{B}^*$ , and (17) and (18) together imply that  $\tau_{xy}(D) \neq D$ . Finally,  $\tau_{xy}(D) \neq D$  implies that exactly one of the elements  $x, y$  is in  $D$ . ■

*Proof of Lemma 4.2.* Let  $D = (S + x)^* = A + y$ . By Lemma A.4.1  $\tau_{xy}(D) \in \mathcal{B}^*$  and exactly one of the elements  $x, y$  is in  $D$ . Since  $y \in A + y = D$ , in fact  $x \notin D$  and  $y \in D$ . It follows that  $A + x = \tau_{xy}(A + y) = \tau_{xy}(D) \in \mathcal{B}^*$ , and hence  $x \in U^*(A)$ . ■

## Appendix 5. Classification of linkings

Let  $\mathcal{B}, \mathcal{B}^*$  be pre-matroids on  $X$ , and let  $L: \bullet \mapsto \bullet^*$  be a linking  $\mathcal{B} \mapsto \mathcal{B}^*$ . Recall that by  $\tau_{ab}$  we denote the transposition of distinct elements  $a, b \in X$ .

**A.5.1. Lemma.** *Suppose that  $B \in \mathcal{B}$  and  $x, y$  are two different elements of  $X$ . If  $x \in B^*$  and  $\tau_{xy}(B) = B$ , then  $y \in B^*$ .*

*Proof.* By applying  $\bullet \mapsto \bullet^*$  to  $\tau_{xy}(B) = B$ , we see that  $\tau_{xy}(B)^* = B^*$ . Since  $\tau_{xy}(B) = B \in \mathcal{B}$ , the property **L1** implies that  $\tau_{xy}(B^*) = \tau_{xy}(B)^* = B^*$ . Since  $x \in B^*$  and  $\tau_{xy}(x) = y$ , this implies  $y \in B^*$ . ■

**A.5.2. Lemma.** *For every  $B \in \mathcal{B}$  either  $B^* = B$ , or  $B^* = B^c$ .*

*Proof. Case 1:*  $B \cap B^* \neq \emptyset$ . Let us fix an element  $x \in B \cap B^*$ , and let  $y$  be an arbitrary element of  $B - x$ . Then  $\tau_{xy}(B) = B$ , and hence Lemma A.5.1 implies that  $y \in B^*$ . It follows that  $B - x \subset B^*$ , and hence  $B \subset B^*$ . By applying this argument to the inverse linking  $\mathcal{B}^* \rightarrow \mathcal{B}$ , we see that  $B^* \subset B$  also, and hence  $B^* = B$ .

**Case 2:**  $B \cap B^* = \emptyset$ . Then  $B^* \subset B^c$ . Let us fix an element  $x \in B^*$ , and let  $y$  be an arbitrary element of  $B^c - x$ . The both  $x, y \notin B$ , and hence  $\tau_{xy}(B) = B$ . Now Lemma A.5.1 implies that  $y \in B^*$ . As in Case 1, it follows that  $B^c - x \subset B^*$  and hence  $B^c \subset B^*$ . Since  $B^* \subset B^c$ , it follows that in this case  $B^* = B^c$ . ■

**A.5.3. Lemma.** Suppose that  $\tau$  is a transposition. If  $B^* = B$ , then  $\tau(B)^* = \tau(B)$ , and if  $B^* = B^c$ , then  $\tau(B)^* = \tau(B)^c$ .

**Proof.** Note that  $\tau(B)^* = \tau(B^*)$  by the linking property L1, and  $\tau(B^c) = \tau(B)^c$  because  $\tau$  is a bijection. Therefore, if  $B^* = B$ , then

$$\tau(B)^* = \tau(B^*) = \tau(B),$$

and if  $B^* = B^c$ , then  $\tau(B)^* = \tau(B^*) = \tau(B^c) = \tau(B)^c$ . ■

**A.5.4. Lemma.** Suppose that  $B, B'$  are two bases of  $\mathcal{B}$ . Then there exists a sequence

$$B = B_1, B_2, \dots, B_n = B'$$

of bases of  $\mathcal{B}$  such that  $B_{i+1} = \tau_i(B_i)$  for some transposition  $\tau_i$  for every  $i = 1, 2, \dots, n-1$ .

**Proof.** If  $B' = B$ , there is nothing to prove. If  $B' \neq B$ , then the symmetric difference  $B \Delta B'$  is not empty. By Theorem 1.1, in this case there are two elements  $x \in B \setminus B', y \in B' \setminus B$  such that  $B \Delta \{x, y\} \in \mathcal{B}$ . Let

$$B_1 = B \Delta \{x, y\} = B - x + y.$$

Then  $B_1 \cap B' = (B \cap B') + y$  because  $B \Delta \{x, y\} = B - x + y$ . It follows that the intersection  $B_1 \cap B'$  contains one element more than  $B \cap B'$ . An application of the induction completes the proof. ■

**Proof of Theorem 4.1.** Let  $B_1 \in \mathcal{B}$ . By Lemma A.5.2 for every basis  $B$  of  $\mathcal{B}$  either  $B^* = B$ , or  $B^* = B^c$ .

Lemma A.5.3 together with Lemma A.5.4 imply that if  $B^* = B$  for one basis  $B \in \mathcal{B}$ , then  $B^* = B$  for all bases  $B \in \mathcal{B}$ , and if  $B^* = B^c$  for one basis  $B \in \mathcal{B}$ , then  $B^* = B^c$  for all bases  $B \in \mathcal{B}$ . ■

## Note historique

W.T. Tutte was uncommonly generous in sharing the route which lead him to his discoveries. He was equally generous in giving the credit to his predecessors. In particular, he described the way which lead him to the definition of the Tutte polynomial and to the Tutte order-independence theorem in Chapter 5 of his book [T3], and, with additional details and from a somewhat different perspective, in a remarkable paper [T4].

Together with his coauthors on the paper [B...T], Tutte observed that the number  $C(G)$  of the spanning trees (or forests) of a graph  $G$  satisfies the relation

$$(19) \quad C(G) = C(G'_A) + C(G''_A),$$

where  $A$  is an arbitrary edge of  $G$ ,  $G'_A$  is the result of the deleting  $A$  from  $G$ , and  $G''_A$  is the result of contracting  $A$  together with its two endpoints into a single vertex. Tutte discovered his polynomial while looking for invariants of graphs satisfying relations similar to (19). As Tutte wrote in [T4], he “...come across one such in a footnote to one of Hassler Whitney’s papers” [W2].

This footnote is, in fact, the note added in proofs at the very end of [W2]. Whitney [W1] introduced an invariant  $m_{ij}$  of graphs for every pair of non-negative integers  $i, j$ . The paper [W1] is devoted to the proof of what is now known as the inclusion-exclusion principle, but was called by Whitney *the logical expansion*, and to various applications of it. One of the application is to the number of colorings of a graph. The inclusion-exclusion principle applied to colorings inevitably leads to the Whitney invariants  $m_{ij}$ . In [W2] Whitney continued to study these invariants and their applications to the colorings of graphs. These invariants satisfy the relation

$$(20) \quad m_{ij}(G) = m_{ij}(G'_A) + m_{i-1,j}(G''_A).$$

Apparently, this is an observation of R.M. Foster, who used it to calculate  $m_{ij}$  for large number of graphs, according to [W2].

In [T1], Tutte developed a general theory of graph invariants satisfying recursion relations similar to (19), (20). He called them *W-functions*. Let us now quote [T4].

Playing with my *W-functions* I obtained a two-variable polynomial...

... In my papers I called this function the dichromate, but it is now generally known as the Tutte polynomial. This may be unfair to Hassler Whitney who knew and used analogous coefficients without bothering to affix them to two variables [W2].

From the point of view adopted in the present paper, there is no reason to attach the coefficients to a polynomial. Since not even the addition of polynomials is used, it is more natural to deal directly with the coefficients. It is convenient to arrange these coefficients (which are non-negative integers) into a single entity, namely, into a multi-set. Multi-sets are a special case of *generalized sets*, introduced by Whitney in the paper [W3], closely related to [W1].

Tutte observed that the value of his polynomial  $\chi(G; \mathbf{x}, \mathbf{y})$  at  $(\mathbf{x}, \mathbf{y}) = (1, 1)$  is equal to the number of spanning trees of  $G$ . This led him to the hypothesis that  $\chi(G; \mathbf{x}, \mathbf{y})$  can be presented as a sum of “*something simple*” over all spanning trees of  $G$ . Initially, this seemed to be impossible because even for very simple graphs  $G$  the symmetries of  $G$  are not reflected in  $\chi(G; \mathbf{x}, \mathbf{y})$ . The way out, found by Tutte, was to break any potential symmetry by enumerating the edges of  $G$ . This led him to his remarkable definition of  $\chi(G; \mathbf{x}, \mathbf{y})$  in terms of the internal and external activities (see Section 7), and to his order-invariance theorem. Let us quote Tutte [T3] again.

I marvelled that all the different possible enumerations should give rise to the same polynomial  $\chi(G; \mathbf{x}, \mathbf{y})$ , even though different enumerations usually gave different internal and external activities for a given spanning tree. But I recalled that Hassler Whitney, giving the chromatic polynomial in terms of broken circuits, had encountered a similar phenomenon [W1].

It seems that the role of contributions of Hassler Whitney to the modern combinatorics in general, and to theory of polynomial invariants of graphs and matroids in particular, is at the very least under-appreciated. Whitney also is one of discoverers of matroids [W4], which provide the proper context for the latter theory. On the other hand, the beauty and originality of the definition of the Tutte polynomial in terms of the enumerations of edges, as well as the Tutte order-independence theorem, are also under-appreciated.

The relations of the form (19), (20), apparently, dominated the field from the very beginning. This domination was later reinforced by a somewhat superficial analogy with the Grothendieck construction of  $K$ -groups in terms of generators and relations. In fact, similar constructions were used before Grothendieck. The easiest example is the standard construction of the integers  $\mathbb{Z}$  from the natural numbers  $\mathbb{N}$ . More importantly, in the framework of this analogy graphs and matroids correspond to vector bundles on a *fixed* algebraic variety, and the abelian group  $\mathbb{Z}[\mathbf{x}, \mathbf{y}]$  corresponds to the  $K$ -group of this variety. Grothendieck  $K$ -groups form a functor on the category of algebraic varieties, but there is no corresponding category in the theory of the Tutte polynomials.

In more recent times, relations similar to (19), (20) come to prominence in topology and led to new invariants of knots and 3-dimensional manifolds. The analogy with these invariants is much closer than with the Grothendieck  $K$ -groups. As in the theory of the

Tutte polynomial, these invariants do not lead to a functor similar to the Grothendieck K-functor. Moreover, these invariants turned out to be directly related with the Tutte polynomial. See, for example, M.B. Thistlethwaite's paper [Th]. Unfortunately, these remarkable developments moved the ideas of Tutte [T2] even further into the shadows.

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