

Elements of Torelli topology: I. The rank of abelian subgroups

Nikolai V. Ivanov

Contents

Preface

1. Surfaces, circles, and diffeomorphisms
2. Dehn twists
3. Action of Dehn twists on homology
4. Dehn multi-twists in Torelli groups
5. The rank of multi-twist subgroups of Torelli groups
6. Pure diffeomorphisms and reduction systems
7. The rank of abelian subgroups of Torelli groups
8. Dehn and Dehn–Johnson twists in Torelli groups: I
9. Dehn and Dehn–Johnson twists in Torelli groups: II

Appendix. Necklaces

References

Preface

By Torelli topology the author understands aspects of the topology of surfaces (potentially) relevant to the study of Torelli groups.

The present paper is devoted to a new approach to the results of W. Vautaw [v1] about Dehn multi-twists in Torelli groups and abelian subgroups of Torelli groups. The main results are a complete description of Dehn multi-twists in Torelli groups and the theorem to the effect that the rank of any abelian subgroup of the Torelli group of a closed surface of genus g is $\leq 2g - 3$. In contrast with W. Vautaw's paper [v1], which heavily relies on the graph-theoretic language, the present paper is based on topological methods. The resulting proofs are more transparent and lead to stronger estimates of the rank of an abelian subgroup when some additional information is available. A key role is played by the notion of a *necklace* of a system of circles on a surface.

As an unexpected application of our methods, in Sections 8 and 9 we give a new proof of the algebraic characterization of the Dehn twist about separating circles and the Dehn-Johnson twists about bounding pairs of circles from the paper [F1] of B. Farb and the author. This proof is much shorter than the original one and bypasses one of the main difficulties, which may be called the *extension problem*, specific to Torelli groups as opposed to the Teichmüller modular groups. This algebraic characterization is also contained in the Ph.D. Thesis of W. Vautaw [v2]. But the proof in [v2] is not quite complete because the extension problem is ignored. The extension problem is discussed in details in the second paper [I3] of this series.

In many respects Torelli groups look similar to Teichmüller modular groups from a sufficient distance. For example, the algebraic characterization of the Dehn twist about separating circles and the Dehn-Johnson twists about bounding pairs of circles in Torelli groups is morally the same as author's [I1] algebraic characterization of Dehn twists in Teichmüller modular groups. But the analogy breaks down to a big extent if one focuses on the details. The extension problem is one of such details.

The new proof this algebraic characterization is simpler thanks not to any technical advances, but to a few shifts in the point of view. It may well be the case that key new insight is in Theorem 5.1 below, which is a trivial corollary of our form of the description of Dehn multi-twists in the Torelli groups (see Theorem 4.4). Another novel aspect of this proof is the shift of emphasis from the canonical reduction systems to the pure reduction systems. See Section 6.

For the experts, one perhaps should point out that the centers of the centralizers, a favorite tool of the author, are replaced by more elegant *bicommutants*. On the technical

level these notions are equivalent in the situation of the present paper, but bicommutants lead to a slightly different mindset.

Sections 1–3 are devoted mostly to a review of terminology and prerequisites, except of an instructive example in Section 3. To a big extent this is also true for Section 6, devoted to the reduction systems, except of rephrasing some well known results in the terms of pure reduction systems. The Appendix is devoted to some motivation for the term *necklace*, and is not used in the main part of the paper. The table of contents should serve as a sufficient guide for the rest of the paper.

1. Surfaces, circles, and diffeomorphisms

Surfaces. By a *surface* we understand a compact orientable 2-manifold with (possibly empty) boundary. The boundary of a surface S is denoted by ∂S , and its genus by $g(S)$. If $\partial S = \emptyset$, then S is called a *closed* surface. By a *subsurface* of a surface S we understand a codimension 0 submanifold Q of S such that each component of ∂Q is either equal to a component of ∂S , or disjoint from ∂S .

For a subsurface Q of S we will denote by cQ the closure of its set-theoretic complement $S \setminus Q$. Clearly, cQ is also a subsurface of S , and $\partial Q \setminus \partial S = \partial cQ \setminus \partial S$. We will say that cQ is the subsurface *complementary to* Q .

Teichmüller modular groups and Torelli groups. The *Teichmüller modular group* $\text{Mod}(S)$ of an orientable surface S is the group of isotopy classes of orientation-preserving diffeomorphisms $S \rightarrow S$. Both diffeomorphisms and isotopies are required to preserve the boundary ∂S only set-wise (this is automatic for diffeomorphisms and standard for isotopies). For closed surfaces S the subgroup of elements $\text{Mod}(S)$ acting trivially on the homology group $H_1(S, \mathbf{Z})$ is called the *Torelli group* of S and is denoted by $\mathcal{J}(S)$. Usually we will denote the homology group $H_1(S, \mathbf{Z})$ simply by $H_1(S)$.

There are several candidates for the definition of the Torelli groups of surfaces with boundary, but none of them is completely satisfactory. It seems that this is so not for the lack of trying to find a “right definition”, but because there is no such definition.

By technical reasons we will need also the subgroups $\mathcal{J}_m(S)$ of $\text{Mod}(S)$ consisting of the isotopy classes of diffeomorphisms acting trivially on $H_1(S, \mathbf{Z}/m\mathbf{Z})$, where m is an integer. Clearly, $\mathcal{J}(S) \subset \mathcal{J}_m(S)$ for every $m \in \mathbf{Z}$.

Circles. A *circle* on a surface S is defined as a submanifold of S diffeomorphic to the standard circle S^1 and disjoint from ∂S . A circle in S is called *non-peripheral* if does

not bound an annulus together with a component of the boundary ∂S . If S is a closed surface, then, obviously, all circles in S are non-peripheral. A circle on S is said to be *non-trivial* in S if it does not bound an annulus together with a component of the boundary ∂S and does not bound a disc in S .

Separating circles. A circle D in a connected surface S is called *separating* if the set-theoretic difference $S \setminus D$ is not connected. In this case $S \setminus D$ consist of two components. The closures of these components are subsurfaces of S having D as a boundary component. The other boundary components of these subsurfaces are at the same time boundary components of S . We will call these subsurfaces *the parts into which D divides S* . If Q is one of these parts, then, obviously, cQ is the other part.

Bounding pairs of circles. A *bounding pair of circles* on a connected surface S is defined as an unordered pair C, C' of disjoint non-isotopic circles in S such that both circles C, C' are non-separating, but $S \setminus (C \cup C')$ is not connected. In this case the difference $S \setminus (C \cup C')$ consist of two components. The closures of these components are subsurfaces of S having both C and C' as their boundary components. The other boundary components of these subsurfaces are at the same time boundary components of S . We will call these subsurfaces *the parts into which the bounding pair of circles C, C' divides S* . If Q is one of these parts, then, obviously, cQ is the other part.

Cutting surfaces and diffeomorphisms. Every one-dimensional closed submanifold c of a surface S leads to a new surface $S//c$ obtained by cutting S along c . The components of $S//c$ are called the *parts* into which c divides S . The canonical map

$$p//c: S//c \longrightarrow S.$$

induces a diffeomorphism $(p//c)^{-1}(S \setminus c) \longrightarrow S \setminus c$ and a double covering map

$$(p//c)^{-1}(c) \longrightarrow c.$$

We will treat the diffeomorphism $(p//c)^{-1}(S \setminus c) \longrightarrow S \setminus c$ as an identification. If $p//c$ is injective on a component Q of $S//c$, then we will treat as an identification also the induced map $Q \longrightarrow p//c(Q)$, and will treat Q as a subsurface of S .

Since S is orientable, the covering $(p//c)^{-1}(c) \longrightarrow c$ is actually trivial. Namely, for every component C of c its preimage $(p//c)^{-1}(C)$ consists of two boundary circles of $S//c$, and each of these circles is mapped by $p//c$ diffeomorphically onto C .

Any diffeomorphism $F: S \longrightarrow S$ such that $F(c) = c$ induces a diffeomorphism

$$F//c: S//c \longrightarrow S//c.$$

If $F//c$ leaves a component Q of $S//c$ invariant, then $F//c$ induces a diffeomorphism

$$F_Q : Q \longrightarrow Q,$$

called the *restriction* of F to Q .

Systems of circles. A one-dimensional closed submanifold c of a surface S is called a *system of circles on S* if the components of c are all non-trivial circles on S and are pair-wise non-isotopic. As usual, we will denote by $\pi_0(c)$ the set of components of c . If c is a system of circles, then the elements of $\pi_0(c)$ are pair-wise disjoint and pair-wise non-isotopic circles, and if \mathcal{U} is a set of pair-wise disjoint and non-isotopic circles, then the union of these circles is a system of circles.

Reduction systems. A system of circles c on S is called a *reduction system for a diffeomorphism $F: S \longrightarrow S$* if $F(c) = c$. A system of circles c on S is called a *reduction system for an element $f \in \text{Mod}(S)$* if c is a reduction system for some diffeomorphism in the isotopy class f , i.e. if the isotopy class f can be represented by a diffeomorphism $F: S \longrightarrow S$ such that $F(c) = c$.

Reducible and pseudo-Anosov elements. A non-trivial element $f \in \text{Mod}(S)$ is said to be *reducible* if there exists a non-empty reduction system for f , and *irreducible* otherwise. An irreducible element of infinite order is called a *pseudo-Anosov element*. This is an easy, but hardly enlightening way to define pseudo-Anosov elements. The original definition of Thurston [T] looks more like a theory than a short definition.

2. Dehn twists

Twist diffeomorphisms of an annulus. Let A be an annulus, i.e. a surface diffeomorphic to $S^1 \times [0, 1]$. As is well known, the group of diffeomorphisms of A , fixed in a neighborhood of ∂A and considered up to isotopies fixed in a neighborhood ∂A , is an infinite cyclic group. A diffeomorphism of A is called a *twist diffeomorphism* of A if it is fixed in a neighborhood of ∂A and its isotopy class is a generator of this group.

Fixing an orientation of A allows to choose a preferred generator of this group. A twist diffeomorphism of $A \longrightarrow A$ is called a *left twist diffeomorphism* if its isotopy class is the preferred generator, and a *right twist diffeomorphism* otherwise. Up to an isotopy, the right twist diffeomorphism is the inverse of the left one.

Twist diffeomorphisms of a surface. If A is annulus contained in S as a subsurface, then any twist diffeomorphism of A can be extended by the identity to a diffeomor-

phism of S . Such extensions are called *twist diffeomorphisms* of S .

If S is oriented, then the extension of a left (respectively, right) twist diffeomorphism of an annulus in S with the induced orientation is called a *left* (respectively, *right*) twist diffeomorphism of S .

Dehn twists. Suppose that S is oriented. Let A be an annulus contained in S , and let C be a circle in S contained in A as a deformation retract. Such a circle C is unique up to isotopy.

It turns out that isotopy class of a left (respectively, right) twist diffeomorphism of S obtained by extension of a left (respectively, right) twist diffeomorphism of A depends only on the isotopy class of C in S . In particular, it is uniquely determined by C . This isotopy class is called the *left* (respectively, *right*) *Dehn twist* of S about the circle C . It is an element of the group $\text{Mod}(S)$.

The left Dehn twist about a circle C is denoted by t_C . The right Dehn twist about C is the inverse of the left one and hence is equal to t_C^{-1} .

The Dehn twist t_C is equal to $1 \in \text{Mod}(S)$ if and only if the circle C is trivial. Let G be a diffeomorphism of S and let $g \in \text{Mod}(S)$ be its isotopy class. Then

$$g t_C g^{-1} = t_{G(C)}.$$

In particular, if $G(C) = C$, or if $G(C)$ is isotopic to C , then t_C and g commute.

3. Action of Dehn twists on homology

The explicit formula. From now on we will assume that the surface S is closed and oriented. The orientation of S allows to define a skew-symmetric pairing on $H_1(S)$, known as the *intersection pairing*. It is denoted by

$$(a, b) \longmapsto \langle a, b \rangle.$$

Let C be a circle on S . Let us orient C and denote by $[C]$ the image of the fundamental class of C in $H_1(S)$. Then t_C acts on $H_1(S)$ by the formula

$$(3.1) \quad (t_C)_*(a) = a + \langle a, [C] \rangle [C].$$

Changing the orientation of C replaces $[C]$ by $-[C]$ and hence does not change the

right hand side of (3.1). The powers of t_C act on $H_1(S)$ by the formula

$$(3.2) \quad (t_C^m)_*(a) = a + m\langle a, [C] \rangle [C].$$

If $[C] \neq 0$, then $\langle a, [C] \rangle \neq 0$ for some $a \in H_1(S)$, and hence

$$a + \langle a, [C] \rangle [C] \neq a.$$

On the other hand, if $[C] = 0$, then $a + \langle a, [C] \rangle [C] = a$. Therefore, $t_C \in \mathcal{J}(S)$ if and only if $[C] = 0$. i.e. if and only if C is a separating circle.

Dehn–Johnson twists and multi-twists. Let C, D be a pair of disjoint circles on S such that the union $C \cup D$ is equal to the boundary ∂Q of some subsurface Q of S . Such pairs of circles are called *bounding pairs*. The circles C and D can be oriented in such a way that $[C] = [D]$. Therefore (3.1) implies that the maps

$$(t_C)_*, (t_D)_* : H_1(S) \longrightarrow H_1(S)$$

are equal, and hence $t_C t_D^{-1}, t_D t_C^{-1} \in \mathcal{J}(S)$. Both these elements of $\mathcal{J}(S)$ are called the *Dehn–Johnson twists* about the bounding pair C, D .

Let c be a one-dimensional closed submanifold of S . A *Dehn multi-twist about* c is defined as a product t of the form

$$(3.3) \quad t = \prod_O t_O^{m_O},$$

where O runs over all components of c , and m_O are integers.

A siren song. Suppose that Q is a subsurface of S and $c = \partial Q$. The orientation of S defines an orientation of Q , which, in turn, defines an orientation of $c = \partial Q$. Let t be defined by (3.3). If c has ≤ 2 components, then $t \in \mathcal{J}(S)$ if and only if

$$(3.4) \quad \sum_O m_O [O] = 0,$$

where O runs over components of c , and these components are considered with orientations induced from $c = \partial Q$. It is tempting to believe that this is true in general.

In fact, this is very far from being true. The following example, explained to the author in few words by R. Hain in 1992, illustrates the reason behind this, and Theorem 4.4 below provides a simple necessary and sufficient condition for $t \in \mathcal{J}(S)$.

It is much more difficult to find reasons for such a belief.

Example. Suppose that both Q and the complementary subsurface cQ are connected and that $c = \partial Q$ consists of three circles C, D, E . Let us orient these circles as boundary components of Q . Then the only relation between $[C], [D], [E]$ is

$$[C] + [D] + [E] = 0.$$

In particular, the classes $[C], [D], [E]$ are non-zero and any two of them are linearly independent. Let A be a circle in S disjoint from E and intersecting each of the circles C, D transversely at one point. Let us orient A and let $a = [A]$. By the choice of A the intersection number $\langle a, [E] \rangle = 0$, one of the intersection numbers $\langle a, [C] \rangle, \langle a, [D] \rangle$ is equal to 1 , and the other is equal to -1 . We may assume that $\langle a, [C] \rangle = 1$ and $\langle a, [D] \rangle = -1$. Let $m_C, m_D, m_E \in \mathbf{Z}$ and

$$t = t_C^{m_C} t_D^{m_D} t_C^{m_D}.$$

$$\begin{aligned} \text{Then } t_*(a) &= a + m_C \langle a, [C] \rangle [C] + m_D \langle a, [D] \rangle [D] + m_E \langle a, [E] \rangle [E] \\ &= a + m_C [C] - m_D [D]. \end{aligned}$$

It follows that if $t \in \mathcal{J}(S)$, then $m_C = m_D = 0$. A similar argument using D, E instead of C, D shows that if $t \in \mathcal{J}(S)$, then also $m_D = m_E = 0$. It follows that $t \in \mathcal{J}(S)$ if and only if $m_C = m_D = m_E = 0$, i.e. if and only if $t = 1$.

4. Dehn multi-twists in Torelli groups

Homology equivalence. Two circles C, C' in S are said to be *homology equivalent* if they can be oriented in such way that their homology classes in $H_1(S)$ are equal,

$$[C] = [C'].$$

Changing the orientation of a circle C replaces $[C]$ by $-[C]$. It follows that the homology equivalence is indeed an equivalence relation. If C is a separating circle, then $[C] = 0$, and hence all separating circles are homology equivalent.

Necklaces. Let s be a one-dimensional closed submanifold of S . The homology equivalence induces an equivalence relation on the set $\pi_0(s)$ of components of s . A *necklace* of s is an equivalence class of this equivalence relation containing a non-separating circle. If a necklace consists of more than 1 circle, then the union of all circles in it is a *BP-necklace* in the sense of Appendix. See Corollary A. 2. The description of BP-necklaces in Appendix is the motivation behind the term *necklace*.

Graph associated with a submanifold. Let c be a one-dimensional closed submanifold of S . One can associate with the pair (S, c) a graph $\mathfrak{G}(S, c)$ as follows. Its set of vertices is the set $\pi_0(S//c)$ of components of the cut surface $S//c$, and its set of edges is the set $\pi_0(c)$ of components of c . Every component C of c is the image under the canonical map $p = p//c: S//c \rightarrow S$ of two components of $\partial S//c$. Let C_1, C_2 be these components, and let Q_1, Q_2 be the components of $S//c$ containing C_1, C_2 respectively (it may happen that $Q_1 = Q_2$). The component C considered as an edge of $\mathfrak{G}(S, c)$ connects the components Q_1, Q_2 considered as vertices. If $Q_1 = Q_2$, then the edge C is a loop, and if the intersection of the images $p(Q_1), p(Q_2)$ consists of several components of c , then the vertices Q_1, Q_2 are connected by several edges.

Separating edges. Let a, b be two vertices of a connected graph \mathfrak{G} . A set T of edges of \mathfrak{G} is *separating edge set* for a and b if every path in \mathfrak{G} contains at least one edge from T . The vertices a, b are said to be τ -edge separated if τ is the minimal number of elements in a separating edge set for a and b .

4.1. Theorem. *If two vertices a, b of a connected graph \mathfrak{G} are τ -edge separated, then there exists τ simple paths connecting a with b and having pair-wise disjoint sets of edges.*

Proof. See, for example, Theorem 12.3.1 in the classical book of O. Ore [o]. ■

4.2. Corollary. *If two vertices a, b of a connected graph \mathfrak{G} remain connected by a path after removing any edge of \mathfrak{G} , then there exists at least two 2 simple paths connecting a with b and having disjoint sets of edges.*

Remark. We will use only Corollary 4.2. But Theorem 4.1 is so beautiful that the author could not resist including it. It is an edge-separation version of vertex-separation results of K. Menger [me] and H. Whitney [w]. It is worth to note that K. Menger's paper [me] is a paper in topology, and H. Whitney is better known not for his seminal contributions to graph theory, but as one of the creators of differential topology.

The idea to apply Corollary 4.2 to Dehn multi-twists in Torelli groups is due to W. Vautaw [v1]. Unfortunately, the application of this result is hidden deep inside of a technical argument in [v1], and no references related to this result are provided.

For a textbook exposition the vertex-separation version of Corollary 4.2, known as Whitney's theorem, see [BM], Theorem 3.2, or [BR], Theorem 3.3.7. The Corollary 4.2 itself is the Exercise 3.9 in [BM].

4.3. Lemma. *Suppose that c is a one-dimensional submanifold of S such that all components of c are non-separating and no two components are homology equivalent. Let D be a*

component of c . Then there exists two circles A, B on S intersecting each component of c transversely in no more than one point and such that D intersects both circles A, B and no other component of c does.

Proof. Let us consider the graph $\mathcal{G}(S, c)$. Let Q_1, Q_2 be the components of S/c connected by the edge D of $\mathcal{G}(S, c)$. If $Q_1 = Q_2$, then there is a circle in S intersecting D transversely in one point and disjoint from all other components of c . In this case we can take this circle as both A and B .

Suppose now that $Q_1 \neq Q_2$. Let \mathcal{G} be the result of removing the edge D from the graph $\mathcal{G}(S, c)$. Since the circle D is non-separating, the graph \mathcal{G} is connected. If after removing an edge C of \mathcal{G} the vertices Q_1, Q_2 are not connected, then C and D together separate S , and hence C and D together bound a subsurface of S . In this case the circles C and D are homology equivalent, contrary to the assumption. Therefore the vertices Q_1, Q_2 of \mathcal{G} remain connected after removing any edge of \mathcal{G} .

Let us choose some points x_1, x_2 in the interior of surfaces Q_1, Q_2 respectively. Let J be an arc in S connecting x_1 with x_2 , intersecting D transversely at one point, and disjoint from all other components of c .

A path in \mathcal{G} connecting Q_1 with Q_2 can be “realized” by an arc in S connecting x_1 with x_2 , contained in the union

$$\bigcup_Q p(Q),$$

where Q runs over the set of vertices of this path, intersecting only those components of c which are the edges of this path, and intersecting each of these components transversely at one point.

By Corollary 4.2, there exist two paths in \mathcal{G} connecting Q_1 with Q_2 and having disjoint sets of edges. Therefore, there exist two arcs in S connecting x_1 with x_2 , disjoint from D , and such that no other component of c intersects both of these arcs. By taking the unions of these two arcs with the arc J we get circles A, B with the required properties. ■

Difference maps. The *difference map* of a diffeomorphism $G : S \rightarrow S$ is the map

$$\Delta_G : H_1(S) \rightarrow H_1(S)$$

defined by $\Delta_G = G_*(a) - a$. Clearly, Δ_G depends only on the isotopy class g of G and hence may be denoted by Δ_g . The isotopy class g belongs to $\mathcal{J}(S)$ if and only if the difference map Δ_g is equal to 0 .

Let c be a one-dimensional closed submanifold of S , and let

$$u = \prod_C t_C^{n_C} \in \mathcal{J}(S),$$

where C runs over all components of c and n_C are integers, t_C be a Dehn multi-twist about c . The formula (3.1) allows to compute the difference map Δ_u . Namely,

$$(4.1) \quad \Delta_u(a) = \sum_C n_C \langle a, [C] \rangle [C],$$

where C runs over all components of c .

4.4. Theorem. *Let s be a one-dimensional closed submanifold of S , and let t be a Dehn multi-twist about s , i.e.*

$$t = \prod_O t_O^{m_O},$$

where O runs over all components of s , and $m_O \in \mathbf{Z}$. Then $t \in \mathcal{J}(S)$ if and only if

$$\sum_{O \in \mathcal{N}} m_O = 0$$

for all necklaces \mathcal{N} of s .

Proof. Since all Dehn twists about separating circles belong to $\mathcal{J}(S)$, we may assume that s has no separating components. Let us select a circle from every necklace of s . Let c be the union of all selected circles. For every selected circle C let

$$n_C = \sum_O m_O,$$

where the sum is taken over all circles in the necklace containing C . Since Dehn twists about homology equivalent circles induce the same automorphism of $H_1(S)$, the Dehn multi-twist t belongs to $\mathcal{J}(S)$ if and only if

$$u = \prod_C t_C^{n_C} \in \mathcal{J}(S),$$

where C runs over all components of c .

We need to prove that $u \in \mathcal{J}(S)$ if and only if $n_C = 0$ for all components C of c . If all $n_C = 0$, then $u = \mathbf{1} \in \mathcal{J}(S)$. This proves the “if” part.

Suppose now that $u \in \mathcal{J}(S)$. By the choice of c , all components of c are non-separating and no two of them are homology equivalent. Let D be a component of c , and let A, B be two circles provided by Lemma 4.3. Let us orient circles A, B, D

and consider their homology classes $a = [A]$, $b = [B]$, $[D]$. We may assume that $\langle a, [D] \rangle = \langle a, [D] \rangle = 1$. The formula (4.1) implies that

$$\langle b, \Delta_u(a) \rangle = \sum_C n_C \langle a, [C] \rangle \langle b, [C] \rangle,$$

where C runs over all components of c . By the choice of circles A, B , for all components $C \neq D$ either $\langle a, [C] \rangle = 0$, or $\langle b, [C] \rangle = 0$. It follows that

$$\langle b, \Delta_u(a) \rangle = n_D.$$

On the other hand, $\Delta_u(a) = 0$ because $u \in \mathcal{J}(S)$. It follows that $n_D = 0$. Since the component D of c was arbitrary, this proves the “only if” part of the theorem. ■

5. The rank of multi-twist subgroups of Torelli groups

The rest of the paper is focused on the estimates of the rank of various abelian subgroups of $\mathcal{J}(S)$. We denote the rank of an abelian group \mathfrak{A} by $\text{rank } \mathfrak{A}$ its rank.

Dehn multi-twist subgroups. Let s be a one-dimensional closed submanifold of S . By $\mathcal{T}(s)$ we will denote the subgroup of $\text{Mod}(S)$ generated by Dehn twists about components of s , i.e. the group of Dehn multi-twists about s . If s is a system of circles, then $\mathcal{T}(s)$ is well known to be a free abelian group having (say, left) Dehn twists about components of s as free generators.

5.1. Theorem. *Let s be a system of circles on S . Then*

$$\text{rank } \mathcal{T}(s) \cap \mathcal{J}(S) = N - n,$$

where N is the number of components of s , and n is the number of necklaces of s .

Proof. Since $\mathcal{T}(s)$ is a free abelian group of rank N freely generated by Dehn twists about components of s , this theorem immediately follows from Theorem 4.4. ■

Two examples. The system of circles s pictured on Fig. 1 consists of $2g - 3$ separating circles. Therefore, for this system of circles $N = 2g - 3$, $n = 0$, and hence $\text{rank } \mathcal{T}(s) \cap \mathcal{J}(S) = 2g - 3$. The system of circles s' pictured on Fig. 2 consists of $g - 1$ separating circles and $g - 1$ non-separating circles, all of which are homology equivalent. Therefore, for this system of circles $N = 2g - 2$, $n = 1$, and hence $\text{rank } \mathcal{T}(s') \cap \mathcal{J}(S) = (2g - 2) - 1 = 2g - 3$.

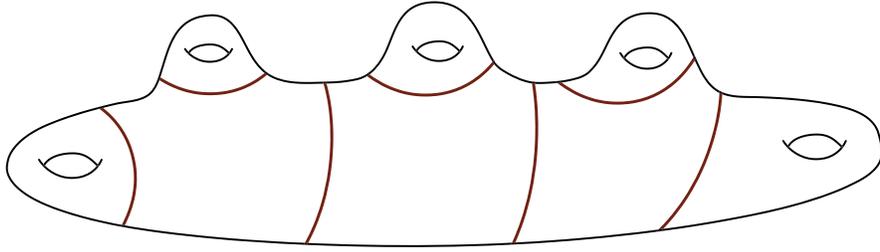


Figure 1: A system of $2g - 3$ separating circles.

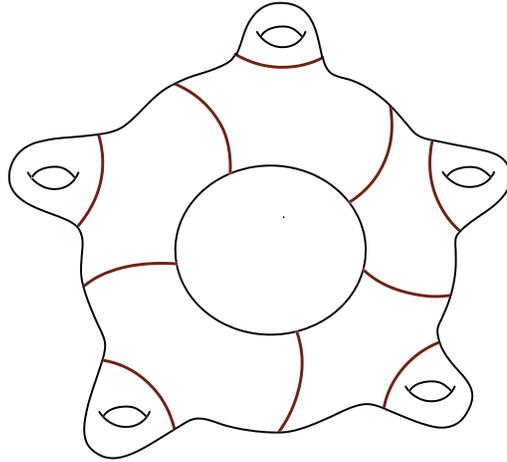


Figure 2: A system of $g - 1$ separating and $g - 1$ non-separating circles.

5.2. Lemma. *Let s be a one-dimensional closed submanifold of S partitioning S into parts of genus 0. Then s has at least $g(S)$ necklaces.*

Proof. Let argue by induction by the genus $g = g(S)$. The cases of $g = 0, 1$ are trivial. Let us consider a component C of s . Let us cut S along C and glue two discs to the two boundary components of the resulting surface and denote by R the resulting surface. The submanifold $s \setminus C$ partitions R into discs with holes.

If C is separating, then R consists of two components. Let us denote them by R_1 and R_2 , and let $g_1 = g(R_1)$ and $g_2 = g(R_2)$. Then $g = g_1 + g_2$ and $g_1, g_2 < g$. The submanifolds $s_1 = (s \setminus C) \cap R_1$ and $s_2 = (s \setminus C) \cap R_2$ partition R_1 and R_2 respectively into parts of genus 0. By the inductive assumption submanifolds s_1 and s_2 have at least g_1 and g_2 necklaces respectively. Each of these necklaces is also a necklace of s in S . It follows that s has at least g necklaces.

If C is non-separating, then R is connected and $g(R) = g - 1$. Since C is non-separating, C belongs to some necklace of s . Other components of this necklace are separating in R and hence do not belong to any necklace of $s \setminus C$ in R . On the other hand, any non-separating circle in R is non-separating in S , and if two such circles are not homology equivalent in R , then they are not homology equivalent in S . Hence the number of necklaces of $s \setminus C$ in R is smaller than the number of necklaces of s . By the inductive assumption there are at least $g - 1$ necklaces of $s \setminus C$ in R , and hence at least g necklaces of s in S . This completes the step of the induction. ■

5.3. Corollary. *Let s be a one-dimensional closed submanifold of S . Then*

$$\text{rank } \mathcal{T}(s) \cap \mathcal{J}(S) \leq 2g - 3.$$

Proof. Since Dehn twists about trivial circles are equal to 1 and Dehn twists about isotopic circles are equal, we may assume that s is a system of circles. Adding new components to s cannot decrease the rank of $\mathcal{T}(s) \cap \mathcal{J}(S)$. Therefore, we may assume that s partitions S into discs with two holes. Then the number of components of s is equal to $3g - 3$, and Corollary 5.1 together with Lemma 5.2 imply that the rank of $\mathcal{T}(s) \cap \mathcal{J}(S)$ is $\leq 3g - 3 - g = 2g - 3$. ■

An invariant of surfaces with boundary. Let Q be a compact orientable surface which is not an annulus and which has non-empty boundary. As usual, let $g = g(Q)$ be the genus of Q . Let $b = b(Q) \geq 1$ be the number of components of ∂Q . Let

$$d(Q) = 2g(Q) - 3 + b(Q) = 2g - 3 + b.$$

The geometric meaning of $d(Q)$ is the following. The maximal number of components of a system of circles c on Q is equal to $3g - 3 + b$, and any system of circles with the maximal number of components partitions Q into discs with two holes. One can construct system of circles with maximal number of components, in particular, as follows. Choose first g disjoint non-trivial circles in Q such that their union c_0 does separate Q . The surface Q cut along c_0 has genus 0. By adding $d(Q)$ circles to c_0 one can get a system of circles on Q with the maximal number of components.

Two invariants of systems of circles. Let s be a system of circles on S . Let

$$\mathcal{D}(s) = \sum_Q d(Q),$$

where Q runs over components of $S//s$. Since s is a system of circles, none of components Q of $S//s$ is an annulus, and hence $d(Q)$ is defined for all Q .

Let $\mathfrak{d}(s)$ be the number of components of $S//s$ which are neither a disc with two holes,

nor a torus with one hole. Then

$$\mathfrak{D}(s) \geq \mathfrak{d}(s)$$

because $\mathfrak{d}(Q) \geq 1$ if Q is neither a disc with two holes, nor a torus with one hole.

5.4. Corollary. *Let s be a system of circles on S . Then*

$$\text{rank } \mathcal{T}(s) \cap \mathcal{J}(S) \leq 2g - 3 - \mathfrak{D}(s) \leq 2g - 3 - \mathfrak{d}(s).$$

Proof. Each component Q of $S//s$ with $\mathfrak{g}(Q) \geq 1$ contains $\mathfrak{g}(Q)$ disjoint circles such that their union does separate Q . All these circles are pair-wise not homology equivalent, and not homology equivalent to any component of s . Let s_1 be the union of s and all these circles. Then s_1 is a system of circles, and every new circle in s_1 is the single element of a new necklace of s_1 . Therefore Theorem 4.4 implies that

$$\mathcal{T}(s_1) \cap \mathcal{J}(S) = \mathcal{T}(s) \cap \mathcal{J}(S).$$

Let N_1 is the number of components of s_1 , and n_1 is the number of necklaces of s_1 . By the construction, s_1 partitions S into subsurfaces of genus 0. Hence Lemma 5.2 implies that there are at least g necklaces of s_1 , i.e. $n_1 \geq g$. One can get from s_1 a system of circles in S with the maximal possible number of components by adding $\mathfrak{d}(Q)$ circles in Q for each component Q . Since the maximal possible number of components is $3g - 3$, it follows that $N_1 + \mathfrak{D}(s) \leq 3g - 3$, and hence $N_1 \leq 3g - 3 - \mathfrak{D}(s)$. By combining the last inequality with $n_1 \geq g$ we see that

$$N_1 - n_1 \leq 3g - 3 - \mathfrak{D}(s) - g = 2g - 3 - \mathfrak{D}(s).$$

But by Corollary 5.1 $\mathcal{T}(s_1) \cap \mathcal{J}(S)$ is a free abelian group of rank $N_1 - n_1$. ■

6. Pure diffeomorphisms and reduction systems

Pure diffeomorphisms and elements. Let F be a diffeomorphism of S . A system of circles c is said to be a *pure reduction system* for F if c is a reduction system for F and the following four conditions hold.

- (a) F is orientation-preserving.
- (b) Every component of $S//c$ is invariant under $F//c$.
- (c) F is equal to the identity in a neighborhood of $c \cup \partial S$.

- (d) For each component Q of $S//c$ the isotopy class of the restriction $F_Q: Q \rightarrow Q$ is either pseudo-Anosov, or contains id_Q .

A diffeomorphism F of S is said to be *pure* if F admits a pure reduction system. This definition is invariant under diffeomorphisms of S . More formally, if c is a pure reduction system for a diffeomorphism F of S , and if G is some other diffeomorphism of S , then $F(c)$ is a pure reduction system for $G \circ F \circ G^{-1}$.

An isotopy class $f \in \text{Mod}(S)$ is said to be *pure* if f contains a pure diffeomorphism. A system of circles c is said to be a *pure reduction system* for an element $f \in \text{Mod}(S)$ if c is a pure reduction system for some diffeomorphism F in the isotopy class f .

The *isotopy extension theorem* implies that the property of being a pure reduction system for f depends only on the isotopy class of the submanifold c in S . In addition, if c is a pure reduction system for f and g is the isotopy class of a diffeomorphism G of S , then $G(c)$ is a pure reduction system for gfg^{-1} .

6.1. Theorem. *If $m \geq 3$, then all elements of $\mathcal{J}_m(S)$ are pure.*

6.2. Lemma. *Suppose that c is a reduction system for a diffeomorphism G representing an element of $\mathcal{J}_m(S)$, where $m \geq 3$. Then G leaves every component of c invariant, and $G//c$ leaves every component of $S//c$ invariant.*

Proofs. See Theorem 1.7 and Theorem 1.2 of [I2] respectively. ■

6.3. Lemma. *Suppose that $f \in \text{Mod}(S)$ is a pure element. Let c be a pure reduction system for a diffeomorphism $F: S \rightarrow S$ representing f . If Q is a component of $S//c$ such that the restriction F_Q is pseudo-Anosov, then the image $p//c(\partial Q) \subset c$ is contained up to isotopy in any pure reduction system of f .*

Proof. This follows from the uniqueness of Thurston's normal form of f . ■

Minimal pure reduction systems. A system of circles c is said to be a *minimal pure reduction system* for an element $f \in \text{Mod}(S)$ if c is a pure reduction system for f , but no proper subsystem of c is. Any pure reduction system for f contains a minimal pure reduction system, and, in fact, it is unique.

Moreover, up to isotopy such a minimal pure reduction system depends only on f .

In fact, the set of the isotopy classes of components of a minimal pure reduction system for f is nothing else but the canonical reduction system of f in the sense of [I2]. This easily follows from the results of [I2], Chapter 7. See also [IM], Section 3. Since the

canonical reduction system of f is defined invariantly in terms of f , it depends only on f and hence the same is true for the minimal reduction systems for f .

Moreover, the notion of a minimal pure reduction system is invariant under diffeomorphisms of S in the same sense as the notion of a pure reduction system (see the first subsection of this section).

Reduction systems of subgroups. Let Γ be a subgroup of $\text{Mod}(S)$. A system of circles c on S is called a *reduction system* for Γ if c is a reduction system for every $f \in \Gamma$. The subgroup Γ is said to be *reducible* if there exists a non-empty reduction system for Γ , and is said to be *irreducible* otherwise.

A finite subgroup of $\text{Mod}(S)$ can be irreducible even if all its non-trivial elements are reducible. Such subgroups were constructed and classified by J. Gilman [G]. But an infinite irreducible subgroup always contains an irreducible element of infinite order, i.e. a pseudo-Anosov element. See [I2], Corollary 7.14.

Pure reduction systems of subgroups. Suppose that Γ is a subgroup of $\text{Mod}(S)$ consisting of pure elements. A system of circles c is said to be a *pure reduction system* for Γ if c is a pure reduction system for every element of Γ .

In general, a subgroup of $\text{Mod}(S)$ consisting of pure elements does not admit any pure reduction system. For example, $\mathcal{J}(S)$ does not admit a pure reduction system. Indeed, $\mathcal{J}(S)$ contains both pseudo-Anosov elements and Dehn multi-twists. On the other hand, a reduction system of a pseudo-Anosov element should be empty, but a reduction system of a Dehn multi-twist cannot be empty.

6.4. Theorem. *If \mathcal{G} is an abelian subgroup of $\mathcal{J}_m(S)$, where $m \geq 3$, then there exists a pure reduction system for \mathcal{G} . In particular, if \mathcal{G} is an abelian subgroup of $\mathcal{J}(S)$, then there exists a pure reduction system for \mathcal{G} .*

Proof. See the description of abelian subgroups of $\text{Mod}(S)$ in [I2], Section 8.12. ■

Abelian subgroups of $\mathcal{J}(S)$. Let \mathcal{A} be an abelian subgroup of the group $\mathcal{J}_m(S)$, where $m \geq 3$. In particular, \mathcal{A} may be an abelian subgroup of $\mathcal{J}(S)$. Let c be a pure reduction system for \mathcal{A} . Then \mathcal{A} is contained in a free abelian subgroup \mathcal{G} of $\text{Mod}(S)$ constructed as follows.

Suppose that π is a set of components of $S//c$. Suppose that for each component $Q \in \pi$ a diffeomorphism $F^Q: Q \rightarrow Q$ fixed on ∂Q is given. Let $f^Q \in \text{Mod}(Q)$ be the isotopy class of F^Q . Suppose that each isotopy class f^Q is pseudo-Anosov. Let us extend these diffeomorphisms by the identity to diffeomorphisms of S , and let

\mathfrak{G} be the subgroup of $\text{Mod}(S)$ generated by $\mathcal{T}(s)$ and the isotopy classes of these extensions.

Then \mathfrak{G} is an abelian subgroup of $\text{Mod}(S)$, and $\mathfrak{A} \subset \mathfrak{G}$. Moreover, these extensions and the Dehn twists about components of c are free generators of \mathfrak{G} .

This is just a rephrased description of abelian subgroups from [I2], Section 8.12.

7. The rank of abelian subgroups of Torelli groups

7.1. Theorem. *Every abelian subgroup of $\mathcal{J}(S)$ is a free abelian group of rank $\leq 2g - 3$. If c is a pure reduction system for an abelian subgroup $\mathfrak{A} \subset \mathcal{J}(S)$, then*

$$\text{rank } \mathfrak{A} \leq \mathfrak{d}(c) + \text{rank } \mathcal{T}(c) \cap \mathcal{J}(S) \leq 2g - 3 - (\mathfrak{D}(c) - \mathfrak{d}(c)).$$

Proof. Let c be a pure reduction system for an abelian subgroup \mathfrak{A} of $\mathcal{J}(S)$. The subgroup \mathfrak{A} is contained in a free abelian subgroup \mathfrak{G} of $\text{Mod}(S)$ of the form described at the end of Section 6. In the rest of the proof we will use notations introduced in the construction of \mathfrak{G} in Section 6.

Restriction to the components of $S//c$ defines a canonical surjective homomorphism

$$\rho: \mathfrak{G} \longrightarrow \prod_Q \mathbb{Z}^Q,$$

where the product is taken over components $Q \in \pi$, and \mathbb{Z}^Q is the infinite cyclic subgroup of $\text{Mod}(Q)$ generated by f^Q . The image of ρ is a free abelian group of rank equal to the number of elements of π , and the kernel of ρ is equal to $\mathcal{T}(s)$. Let $\rho|_{\mathfrak{A}}$ be the restriction of ρ to \mathfrak{A} . Let us estimate the ranks of its image and kernel.

If $Q \in \pi$, then Q is not a disc with two holes because f^Q is a pseudo-Anosov class. If $Q \in \pi$ and Q is a torus with one hole, then F^Q acts non-trivially on $H_1(Q)$. In this case ∂Q is a separating circle in S , and hence $H_1(Q)$ is a direct summand of $H_1(S)$. It follows that F^Q cannot be the restriction to Q of a diffeomorphism $S \rightarrow S$ acting trivially on $H_1(S)$. In turn, this implies that the image of $\rho|_{\mathfrak{A}}$ is contained in the product of factors \mathbb{Z}^Q with Q not a torus with one hole.

It follows that the rank of the image of $\rho|_{\mathfrak{A}}$ is $\leq \mathfrak{d}(c)$. The kernel of $\rho|_{\mathfrak{A}}$ is contained in $\mathcal{T}(s) \cap \mathfrak{A} \subset \mathcal{T}(s) \cap \mathcal{J}(S)$. It follows that

$$\text{rank } \mathfrak{A} \leq \mathfrak{d}(c) + \text{rank } \mathcal{T}(c) \cap \mathcal{J}(S).$$

By Corollary 5.4 $\text{rank } \mathcal{T}(s) \cap \mathcal{J}(S) \leq 2g - 3 - \mathfrak{D}(c)$, and hence

$$\text{rank } \mathfrak{A} \leq \mathfrak{d}(c) + (2g - 3 - \mathfrak{D}(c)) = 2g - 3 - (\mathfrak{D}(c) - \mathfrak{d}(c)).$$

Since $\mathfrak{D}(c) \geq \mathfrak{d}(c)$, this implies that $\text{rank } \mathfrak{A} \leq 3g - 3$. ■

7.2. Lemma. *Let c be a system of circles partitioning S into surfaces of genus 0. If there is a component Q of $S//c$ such that $g(Q) = 0$, the canonical map $p//c$ embeds Q in S , and the complementary surface cQ is connected, then c has at least $g(S) + 1$ necklaces.*

Proof. Let us argue by induction by the number of components of ∂Q . Since c is a system of circles, Q is neither a disc, nor an annulus, and hence this number is ≥ 3 .

Suppose that ∂Q consists of 3 components, and let C, D, E be these components. Since cQ is connected, all of them are non-separating and no two of them are homology equivalent. Hence C, D, E belong to three different necklaces.

As in the proof of Lemma 5.2, let us cut S along C and glue two discs to the two boundary components of the resulting surface. Let R be the result of this glueing. The necklace containing C disappears in R (since the two circles in R resulting from C bound discs in R), and the two necklaces containing D and E respectively coalesce in R into one (since the union $D \cup E$ bounds in R an annulus). Hence the number of necklaces of c in S is bigger than the number of necklaces of $c \setminus C$ in R by at least 2. By Lemma 5.2 there are at least $g(R) = g(S) - 1$ necklaces of $c \setminus C$ in R , and hence there are at least $g(S) + 1$ necklaces of c in S .

Suppose now that ∂Q consists of ≥ 4 components, and let C be one of them. Again, let us cut S along C and glue two discs to the two boundary components of the resulting surface, and let R be the result of this glueing. One of the two glued discs is glued to Q . Let P be the result of this glueing. Let $c' = c \setminus C$. Then c' is a system of circles on R , and P is a component of $R//c'$. Moreover, $p//c'$ embeds P in R , and the complementary surface cP is connected (being the result of glueing a disc to a boundary component of cQ).

The necklace n_C containing C disappears in R (by the same reason as above). If a non-separating component of c does not belong to n_C , then it is non-separating in R also, and every non-separating in R component of c' is non-separating in S also. If two non-separating components of c do not belong to n_C and are homology equivalent in S , then they are homology equivalent in R also.

It follows that there is a canonical surjective map from the set of different from n_C necklaces of c in S to the set of necklaces of c' in R . Hence the number of necklaces of c in S is bigger than the number of necklaces of c' in R by at least 1. By the

inductive assumption there are at least $g(R) + 1 = g(S)$ necklaces of c' . It follows that there are at least $g(S) + 1$ necklaces of c . This completes the induction step and hence the proof of the lemma. ■

7.3. Theorem. *Let c be a pure reduction system of an abelian subgroup $\mathfrak{A} \subset \mathcal{J}(S)$. Suppose that there is a component Q of $S//c$ such that $g(Q) = 0$, the canonical map $p//c$ embeds Q in S , and the complementary surface cQ is connected. Then $\text{rank } \mathfrak{A} \leq 2g - 4$.*

Proof. If a component Q of $S//c$ is neither a disc with two holes, nor a torus with one hole, then there is a circle contained in Q and non-trivial in Q . By adding these circles to c we will get a new system of circles c' . By the definition of $\mathfrak{d}(c)$, there are $\mathfrak{d}(c)$ such components, and hence there are $\mathfrak{d}(c)$ new circles in c' . On the other hand, c' consists of $\leq 3g - 3$ components because c' is a system of circles. It follows that the number of components of c is $\leq 3g - 3 - \mathfrak{d}(c)$. On the other hand, Lemma 7.2 implies that under our assumptions the number of necklaces of c is $\geq g + 1$.

By combining these estimates of the number of components and the number of necklaces of c with Theorem 5.1, we see that

$$\text{rank } \mathcal{J}(c) \cap \mathcal{J}(S) \leq 3g - 3 - \mathfrak{d}(c) - (g + 1) = 2g - 4 - \mathfrak{d}(c).$$

By Theorem 7.1 $\text{rank } \mathfrak{A} \leq \mathfrak{d}(c) + \text{rank } \mathcal{J}(c) \cap \mathcal{J}(S)$, and hence

$$\text{rank } \mathfrak{A} \leq \mathfrak{d}(c) + (2g - 4 - \mathfrak{d}(c)) = 2g - 4.$$

This completes the proof of the theorem. ■

7.4. Theorem. *Let c be a pure reduction system of an abelian subgroup $\mathfrak{A} \subset \mathcal{J}(S)$. Then $\text{rank } \mathfrak{A} \leq 2g - 4$ unless each component of $S//c$ is either a sphere with 3 or 4 holes, or a torus with 1 or 2 holes.*

Proof. By Theorem 7.1 $\text{rank } \mathfrak{A} \leq 2g - 3 - (\mathfrak{D}(c) - \mathfrak{d}(c))$. Since $\mathfrak{D}(c) \geq \mathfrak{d}(c)$,

$$\text{rank } \mathfrak{A} \leq 2g - 4$$

unless $\mathfrak{D}(c) = \mathfrak{d}(c)$. If the last equality holds, then $d(Q) = 1$ for every component Q of $S//c$ which is neither a disc with two holes (i.e. a sphere with 3 holes), nor a torus with 1 hole. But $d(Q) = 1$ if and only if Q is a sphere with 4 holes or a torus with 2 holes. The theorem follows. ■

8. Dehn and Dehn–Johnson twists in Torelli groups: I

Commutants and bicommutants. Let G be a group, and let $X \subset G$. The *commutant* X' of X is the set of all elements of G commuting with all elements of X . It is a subgroup of G . The *bicommutant* X'' of X is the commutant of X' .

The *commutant* g' of an element $g \in G$ is the set of all elements of G commuting with g . In other terms, it is the commutant of the one-element subset $\{g\}$. The *bicommutant* g'' of g is the commutant of g' .

In the rest of this section we will consider the commutants and bicommutants in $\mathcal{J}(S)$.

8.1. Theorem. *Suppose that $g(S) \geq 3$. Suppose that $f \in \mathcal{J}(S)$. If*

- *f belongs to an abelian subgroup of $\mathcal{J}(S)$ of rank $2g - 3$, and*
- *f'' does not contain abelian subgroups of rank 2 ,*

then f is a Dehn multi-twist.

Proof. Let \mathfrak{A} be an abelian subgroup of $\mathcal{J}(S)$ containing f . By Theorem 6.4 there exists a pure reduction system c for \mathfrak{A} . Then c is also a pure reduction system for f , and hence there is a diffeomorphism $F: S \rightarrow S$ representing f and such that c is a pure reduction system for F .

CLAIM. Let Q be a component of $S//c$ such that the isotopy class of F_Q is pseudo-Anosov, and let C be a component of $p//c(\partial Q)$. Then the Dehn twist t_C commutes with every element of the commutant f' of f in $\mathcal{J}(S)$.

Proof of the Claim. Let c_f be a minimal pure reduction system for f contained in c . Suppose that $g \in f'$, i.e. that $g \in \mathcal{J}(S)$ and g commutes with f . Let G be a diffeomorphism of S representing g . Then $G(c_f)$ is a minimal pure reduction system for $gfg^{-1} = f$, and hence $G(c_f)$ is isotopic to c_f . Replacing G by an isotopic diffeomorphism, if necessary, we may assume that $G(c_f) = c_f$.

Since the isotopy class of F_Q is pseudo-Anosov, Lemma 6.3 implies that $p//c(\partial Q)$ is contained in c_f . Hence Lemma 6.2 implies that G leaves every component of $p//c(\partial Q)$ invariant. In particular, $G(C) = C$, and hence

$$g t_C g^{-1} = t_C$$

(because G is orientation-preserving). In other terms, t_C commutes with g . Since g is an arbitrary element of f' , this proves the claim. \square

Suppose that f is not a Dehn multi-twist. Then there is a component Q of $S//c$ such that the isotopy class of F_Q is pseudo-Anosov. Theorem 7.4 implies that Q is either a sphere with 3 or 4 holes, or a torus with 1 or 2 holes.

Since there are no pseudo-Anosov isotopy classes on a sphere with 3 holes, Q cannot be a sphere with 3 holes. If Q is a torus with one hole, then every pseudo-Anosov isotopy class acts non-trivially on $H_1(Q)$. On the other hand, in this case the inclusion homomorphism $H_1(Q) \rightarrow H_1(S)$ is injective. Since the isotopy class of F_Q is pseudo-Anosov, in this case f acts non-trivially on $H_1(S)$, in contradiction with $f \in \mathcal{J}(S)$. Therefore Q cannot be a torus with 1 hole either. It follows that Q is either a sphere with 4 holes, or a torus with 2 holes.

Suppose that Q is a torus with 2 holes. If $p//c$ restricted to Q is not an embedding, then $p//c$ maps both components of ∂Q onto the same circle in S . In this case the image $p//c(Q)$ is a closed subsurface of S and hence is equal to S . It follows that in this case S is a surface of genus 2, contrary to the assumption. Therefore $p//c$ embeds Q into S and we can consider Q as a subsurface of S . Let C, D be the two boundary components of Q . Since the isotopy class of F_Q is pseudo-Anosov, the Dehn twists t_C and t_D commute with all elements of the commutant f' . While these Dehn twists themselves do not belong to $\mathcal{J}(S)$, the product

$$t = t_C \cdot t_D^{-1}$$

is a Dehn-Johnson twist and hence belongs to $\mathcal{J}(S)$. Therefore both f and t belong to the bicommutant f'' . Moreover, by the classification of abelian subgroups of $\text{Mod}(S)$ the elements f, t generate a free abelian group of rank 2. Since f'' cannot contain such a subgroup by the assumption, it follows that Q is not a torus with 2 holes.

The only possibility that remains is that Q is a sphere with 4 holes. Theorem 7.3 implies that in this case either $p//c$ restricted to Q is not an embedding, or cQ is not connected.

If $p//c$ restricted to Q is not an embedding, then $p//c$ maps two components of ∂Q onto the same circle in S . If $p//c$ also maps the two other components of ∂Q onto the same circle, then the image $p//c(Q)$ is a closed subsurface of S and hence is equal to S . It follows that in this case S is a surface of genus 2, contrary to the assumption. If $p//c$ maps two other components of ∂Q to two different circles C, D , then the image $p//c(Q)$ is a torus with 2 holes, and C, D are its boundary components. By arguing exactly as in the case of a torus with 2 holes, we conclude that in this case f'' contains a free abelian group of rank 2 having f and the Dehn-Johnson twist about the pair C, D as its free generators. This contradicts to the assumptions of the theorem, and hence $p//c$ actually embeds Q into S .

It remains to consider the case when Q is a sphere with 4 holes and Q is a sub-

surface of S . As we already noted, in this case Theorem 7.3 implies that cQ is not connected. Therefore cQ consists of 2, 3, or 4 components. In the first case the boundary of each of two components of cQ is a bounding pair in S . As in the case of the torus with 2 holes, this implies that f'' contains an abelian group of rank 2. In the other two cases there is a component of cQ with only 1 boundary component. Let C be this component. Then C is a bounding circle, and hence the Dehn twist t_C about C belongs to $\mathcal{J}(S)$. On the other hand, t_C belongs to the bicommutant f'' by the above Claim. Therefore, in this case f'' also contains a free abelian subgroup of rank 2, contrary to the assumption.

It follows there is no component Q of $S//c$ such that the isotopy class of F_Q is pseudo-Anosov. Hence all diffeomorphisms F_Q are isotopic to the identity, and hence f is a product of Dehn twist about components of c . ■

8.2. Theorem. *Under the assumption of Theorem 8.1 f is a non-zero power of either a Dehn twist about a separating circle, or a Dehn–Johnson twist about a bounding pair.*

Proof. By Theorem 8.1, f is a Dehn multi-twist. Therefore, f has the form (3.3) for some one-dimensional submanifold c of S and some integers m_O , where O runs over the components of c . Without any loss of generality we may assume that c is a system of circles and that all integers $m_O \neq 0$. Then c is a minimal pure reduction system for f .

Let $g \in \mathcal{J}(S)$ and let G be a diffeomorphism representing g . If $g \in f'$, then $gfg^{-1} = f$, and $G(c)$ is isotopic to c because c is a minimal reduction system for f (see Section 6). Replacing G by an isotopic diffeomorphism, if necessary, we may assume that $G(c) = c$. Then by Lemma 6.2 G leaves every component of c invariant. It follows that $gt_Cg^{-1} = t_C$ for every component C of c . It follows that g commutes with all elements of $\mathcal{T}(c)$. In particular, g commutes with all elements of $\mathcal{T}(c) \cap \mathcal{J}(S)$. Since g is an arbitrary element of f' , it follows that

$$\mathcal{T}(c) \cap \mathcal{J}(S) \subset f''.$$

But Theorem 5.1 implies that $\text{rank } \mathcal{T}(c) \cap \mathcal{J}(S) \geq 2$ unless c is either a separating circle, or the union of two circles forming a bounding pair. In the first case f is a power of the Dehn twist about this circle, and in the second case f is a power of a Dehn–Johnson twist about this bounding pair. ■

9. Dehn and Dehn–Johnson twists in Torelli groups: II

By Theorem 8.2 the two conditions of Theorem 8.1 are sufficient for an element f of $\mathcal{J}(S)$ to be a non-zero power of either a Dehn twist about a separating circle, or a Dehn–Johnson twist about a bounding pair. This section is devoted to a proof that these conditions are necessary. See Theorem 9.3. The ideas of this proof are essentially the same as the ideas of the original proof of B. Farb and the author, announced in [FI]. Naturally, it was adapted to the context of the present paper and differs from the original proof in details.

Theorems 8.2 and 9.3 together provide an algebraic characterization of non-zero powers of Dehn twists about separating circles and Dehn–Johnson twists about a bounding pair in terms of group structure of $\mathcal{J}(S)$. In order to distinguish between non-zero powers of Dehn twists about separating circles and Dehn–Johnson twists about a bounding pair, one needs to use an additional algebraic condition. See [FI], Proposition 9.

A theorem of Thurston. By a well known theorem of W. Thurston one can construct pseudo-Anosov isotopy classes of diffeomorphisms of a surface Q by taking the isotopy classes of various products of powers of twist diffeomorphisms of Q . If one uses only twist diffeomorphisms about circles bounding in Q subsurfaces with 1 boundary component, then these products diffeomorphisms act trivially on $H_1(Q)$. This is how Thurston constructed the first examples of pseudo-Anosov isotopy classes acting trivially on $H_1(Q)$, and, in particular, the first examples of pseudo-Anosov elements of Torelli groups (Thurston did not phrase his results in terms of Torelli groups).

In order for this construction to apply, it is sufficient for Q to contain circles bounding in Q subsurfaces with 1 boundary component. This excludes only surfaces of genus 0 and surfaces of genus 1 with 1 boundary component. When this construction applies, it leads to many examples of pseudo-Anosov isotopy classes. In particular, it leads to examples of non-commuting pseudo-Anosov isotopy classes.

Suppose now that Q is a subsurface of S . One can modify Thurston's examples by replacing the twist diffeomorphisms of Q by the twist diffeomorphisms of S about the same circles. Then instead of a diffeomorphism of Q we will get diffeomorphisms of S equal to the identity on cQ and such that the induced diffeomorphisms of Q are pseudo-Anosov. If we use only twist diffeomorphisms about circles bounding in Q subsurfaces with 1 boundary component, then these diffeomorphisms of S will act trivially on $H_1(S)$. In other terms, their isotopy classes will belong to $\mathcal{J}(S)$.

It follows that if a subsurface Q of S is neither a surface of genus 0 nor a surface of genus 1 with 1 boundary component, then there are diffeomorphisms of S equal to

the identity on cQ , such that their isotopy classes belong to $\mathcal{J}(S)$, and such that the induced diffeomorphisms of Q belong to pseudo-Anosov isotopy classes. Moreover, there are (pairs of) such diffeomorphisms with non-commuting isotopy classes.

9.1. Lemma. *Let C be a circle on S separating S into two parts Q, R having C as their common boundary. Suppose that G is a diffeomorphism of S leaving each of these parts invariant and such that its isotopy class belongs to $\mathcal{J}(S)$. If R is a torus with 1 hole, then the restriction G_R is isotopic to the identity.*

Proof. Since ∂R is a separating circle, the homology group $H_1(R)$ is a direct summand of $H_1(S)$. By the assumption, G acts trivially on $H_1(S)$. It follows that G_R acts trivially on $H_1(R)$. By the classification of diffeomorphisms of a torus with 1 hole, this implies that G_R is isotopic to the identity. ■

9.2. Lemma. *Let c be a system of circles on S separating S into two parts having c as their common boundary. If both these parts have genus ≥ 1 , then there is a subset $X \subset \mathcal{J}(S)$ with the commutant X' equal to $\mathcal{T}(c) \cap \mathcal{J}(S)$.*

Proof. Let Q, R be the two parts into which c divides S . Let us consider diffeomorphisms of S equal to the identity on R and such that the isotopy class of the induced diffeomorphism of Q is pseudo-Anosov. Suppose that Q is not a torus with 1 hole. Then Thurston's construction leads, in particular, to two such diffeomorphisms F_1, F_2 having the additional property that their isotopy classes f_1, f_2 belong to $\mathcal{J}(S)$ and do not commute. If R is also not a torus with 1 hole, then there are also diffeomorphisms G_1, G_2 having the same properties, but with the roles of Q and R interchanged. Let g_1, g_2 be their isotopy classes, and let

$$X = \{f_1, f_2, g_1, g_2\}.$$

Then Dehn twists about components of c commute with all elements of X and hence $\mathcal{T}(c) \cap \mathcal{J}(S) \subset X'$. Let us prove the opposite inclusion $X' \subset \mathcal{T}(c) \cap \mathcal{J}(S)$.

Let $f \in X'$. Then the subgroup \mathfrak{A} of $\mathcal{J}(S)$ generated by f and, say, f_1, g_1 is abelian. By Theorem 6.4 there exists a pure reduction system c_0 for \mathfrak{A} . Then c_0 is also a pure reduction system for f_1 . By Lemma 6.3 c is contained up to isotopy in any pure reduction system for f_1 , and hence up to isotopy c is contained in c_0 . Therefore we may assume that $c_0 \supset c$. Then $c_0 = c$ because diffeomorphisms F_1, G_1 cannot leave invariant any system of circles in Q, R respectively.

Now the description of abelian subgroups of $\mathcal{J}(S)$ from Section 6 implies that \mathfrak{A} is

contained in the group generated by $\mathcal{T}(c) \cap \mathcal{J}(S)$ and f_1, g_1 . It follows that

$$(9.1) \quad f = t \cdot f_1^m \cdot g_1^n$$

for some $t \in \mathcal{T}(c) \cap \mathcal{J}(S)$ and $m, n \in \mathbf{Z}$. Since f_2 commutes with t, g_1 , and f , (9.1) implies that f_2 commutes with f_1^m . If $m \neq 0$, this implies that f_2 commutes with f_1 contrary to the assumption. Therefore $m = 0$. By a completely similar argument $n = 0$ and hence

$$f = t \in \mathcal{T}(c) \cap \mathcal{J}(S).$$

This proves the lemma in the case when neither Q , nor R is a torus with 1 hole.

Suppose now that one of the surfaces Q, R is a torus with 1 hole, but the other is not. We may assume that R is a torus with 1 hole and Q is not. Then Thurston's construction applies to Q and leads to diffeomorphisms F_1, F_2 with the same properties as above. Let f_1, f_2 be their isotopy classes, and let

$$X = \{f_1, f_2\}.$$

Then the Dehn twist about the circle $\partial Q = \partial R = c$ belongs to X' and generates $\mathcal{T}(c) = \mathcal{T}(c) \cap \mathcal{J}(S)$. Therefore $\mathcal{T}(c) \cap \mathcal{J}(S) \subset X'$.

In order to prove the opposite inclusion, consider an arbitrary $f \in X'$ and the group \mathfrak{A} generated by f and f_1 . Arguing as above, we see that there is a pure reduction system c_0 for \mathfrak{A} containing c . In view of Lemma 9.1, this implies that every element \mathfrak{A} can be represented by a diffeomorphism G leaving R invariant and such that the restriction G_R is isotopic to the identity. It follows that \mathfrak{A} is contained in the group generated by $\mathcal{T}(c)$ and f_1 and hence

$$(9.2) \quad f = t \cdot f_1^m$$

for some $t \in \mathcal{T}(c)$ and $m \in \mathbf{Z}$. Arguing as above, we see that $m = 0$. Hence

$$f = t \in \mathcal{T}(c) = \mathcal{T}(c) \cap \mathcal{J}(S).$$

This proves the lemma in the case when one of the parts Q, R is a torus with 1 hole.

If each part Q, R is a torus with 1 hole, then one can take $X = \{t_C\}$, where $C = \partial Q = \partial R$. We leave the details of this case to the reader. ■

9.3. Theorem. *If f is a non-zero power of either Dehn twist about a separating circle, or Dehn–Johnson twist about a bounding pair, then the f satisfies the conditions of Theorem 8.1.*

Proof. Suppose that f is a non-zero power of a Dehn twist about a separating circle c . Let Q_1, Q_2 be the parts into which C divides S , and let $g_1 = g(Q_1)$, $g_2 = g(Q_2)$. Then $g_1 + g_2 = g$, and the pair (S, C) is determined up to a diffeomorphism by the numbers g_1, g_2 .

The system of circles s illustrated on Fig. 1 contains a circle dividing S into two parts of genus g_1, g_2 for every pair g_1, g_2 such that $g_1 + g_2 = g$. We may assume that c is one of these circles. Then $f \in \mathcal{T}(s) \cap \mathcal{J}(S)$. The system of circles s consists of $2g - 3$ components and has no necklaces because all components of s are separating. Therefore, $\text{rank } \mathcal{T}(s) \cap \mathcal{J}(S) = 2g - 3$ by Theorem 5.1. It follows that f satisfies the first condition of Theorem 8.1.

By Lemma 9.2, there is a subset $X \subset \mathcal{J}(S)$ with $X' = \mathcal{T}(c)$. Since $f \in \mathcal{T}(s) \cap \mathcal{J}(S)$, the element f commutes with all elements of X . Therefore, $X \subset f'$ and hence $f'' \subset X' = \mathcal{T}(c) \cap \mathcal{J}(S) = \mathcal{T}(c)$. Since in this case $\mathcal{T}(c)$ is an infinite cyclic group, f'' does not contain free abelian groups of rank 2. It follows that f satisfies the second condition of Theorem 8.1.

Suppose now that f is a non-zero power of a Dehn–Johnson twist about a bounding pair C, D . Let $c = C \cup D$. Let Q_1, Q_2 be the parts into which c divides S , and let $g_1 = g(Q_1)$, $g_2 = g(Q_2)$. Then $g_1 + g_2 = g - 1$, and the pair (S, c) is determined up to a diffeomorphism by the numbers g_1, g_2 .

The system of circles s' illustrated on Fig. 2 contains a two circles forming a bounding pair and dividing S into two parts of genus g_1, g_2 for every pair g_1, g_2 such that $g_1 + g_2 = g - 1$. As above, we may assume that C, D is one of these bounding pairs. Then $f \in \mathcal{T}(s') \cap \mathcal{J}(S)$. The system of circles s' consists of $2g - 2$ components and 1 necklace. Therefore, $\text{rank } \mathcal{T}(s') \cap \mathcal{J}(S) = 2g - 3$ by Theorem 5.1. This proves that f satisfies the first condition of Theorem 8.1.

By using Lemma 9.2 in the same way as before, we see that $f'' \subset X' = \mathcal{T}(c) \cap \mathcal{J}(S)$. In this case $\mathcal{T}(c) \cap \mathcal{J}(S)$ is an infinite cyclic group generated by a Dehn–Johnson twist about the bounding pair C, D . It follows that f satisfies the second condition of Theorem 8.1. This completes the proof. ■

Appendix. BP-necklaces

BP-necklaces. A one-dimensional closed submanifold c of S is called a *BP-necklace* if the cut surface $S//c$ has at least 2 components and every component of $S//c$ has exactly 2 boundary components.

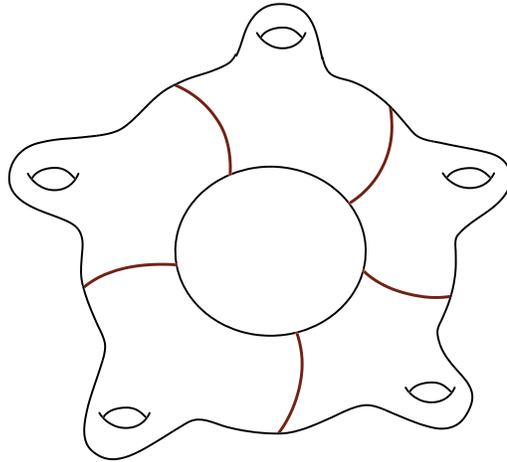


Figure 3: A BP-necklace.

An important example c of a BP-necklace on S is illustrated by Fig. 3. It is a union of $g - 1$ circles. Every component of $S//c$ is a torus with 2 holes. By adding to c for each component Q of $S//c$ a circle bounding a torus with 1 hole in Q , we get a system of circles s in S consisting of $2g - 2$ components and having exactly 1 necklace, namely the set of components of c . See Fig. 2. As we will see now, every BP-necklace looks like the one on Fig. 3, or, more generally, as the one on Fig. 4.

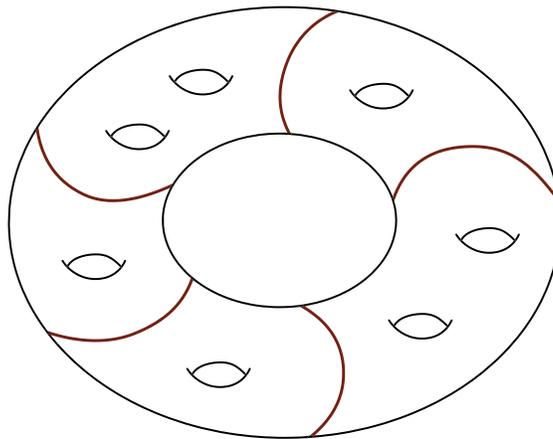


Figure 4: Another BP-necklace.

A description of BP-necklaces. Let c be a BP-necklace. If $p//c$ is not an embedding on a component Q of $S//c$, then $p//c$ maps both boundary circles of Q onto the

same circle in S . In this case $p//c(Q)$ is a closed surface and hence is equal to S . Therefore Q is the only component of $S//c$, contrary to the assumption. Hence $p//c$ embeds all components of $S//c$ and we may consider them as subsurfaces of S .

Every component of c is a component of the boundary of exactly two parts of S . Let Q_0 be one of the parts of S with respect to c , and let C_0 be one of components of the boundary ∂Q_0 . Let C_1 be the other component of the boundary ∂Q_0 , and let Q_1 be the second part of S having C_1 as a component of its boundary. By continuing in this way, one can consecutively number the components of c and the parts of S as

$$C_0, Q_0, C_1, Q_1, C_2, Q_2, \dots, C_n, Q_n$$

in such a way that $\partial Q_i = C_i \cup C_{i+1}$ if $0 \leq i \leq n-1$, and $\partial Q_n = C_n \cup C_0$. In particular, we see that all components of c are homology equivalent.

A. 1. Theorem. *If a closed one-dimensional submanifold c of S has ≥ 2 components and all components of c are non-separating and homology equivalent, then c is a BP-necklace.*

Proof. If two disjoint non-separating circles C, C' are homology equivalent, then C, C' is a bounding pair. Indeed, if the union $C \cup C'$ does not separate S , then the classification of surfaces implies that the homology classes $[C], [C']$ are linearly independent and hence C, C' cannot be homology equivalent. This observation implies the theorem in the case when c consists of 2 components. In order to prove the theorem in the general case, we will use the induction by the number of components of c . Suppose that the theorem is already proved for submanifolds with $\leq n-1$ components, and that c has n components. Let C be a component of c , and let $c_0 = c \setminus C$. By the inductive assumption, c_0 is a BP-necklace.

Since the circle C is disjoint from c_0 , it is contained in a component Q of $S//c_0$. If C is a non-separating in Q , then C is not homology equivalent to components of ∂Q , contrary to the assumption. Hence C separates Q into two parts. If both components of ∂Q are contained in the same part, then the other part is a subsurface of Q with boundary equal to C . This subsurface is also a subsurface of S . Therefore in this case C bounds a subsurface of S , contrary to the assumption that C is non-separating. Therefore the two components of ∂Q are contained in different parts of Q . Hence C divides Q into two subsurfaces, each of which has two boundary components. One of these two boundary components is C , and the other is a component of ∂Q . It follows that c divides S into components of $S//c_0$ different from Q and the two parts into which C divides Q . Therefore c is a BP-necklace. ■

A. 2. Corollary. *If s is a closed one-dimensional submanifold of S , then the union of circles in any necklace of s is a BP-necklace.* ■

References

- [BR] R. Balakrishnan, K. Ragnathan, *A textbook of graph theory*, Springer, 2012.
- [BM] J.A. Bondy, U.S.R. Mundy, *Graph Theory with Applications*, North-Holland, 1976.
- [FI] B. Farb, N.V. Ivanov, The Torelli geometry and its applications. Research announcement, *Mathematical Research Letters*, V. 12, Nos. 2–3 (2005), 293–301.
- [G] J. Gilman, Structures of elliptic irreducible subgroups of the modular group, *Proceedings of the London Mathematical Society*, V. 47, No. 1 (1983), 27–42.
- [I1] N.V. Ivanov, Automorphisms of Teichmüller modular groups, *Lecture Notes in Mathematics*, No. 1346, Springer, 1988, 199–270.
- [I2] N.V. Ivanov, *Subgroups of Teichmüller Modular Groups*, Translations of Mathematical Monographs, Vol. 115, AMS, 1992.
- [I3] N.V. Ivanov, Elements of Torelli topology: II. The extension problem, Preprint, 2016.
- [IM] N.V. Ivanov, J.D. McCarthy, On injective homomorphisms between Teichmüller modular groups I, *Inventiones mathematicae*, V. 135, F. 2 (1999), 425–486.
- [me] K. Menger, Zur allgemeinen Kurventheorie, *Fundamenta Mathematicae*, V. 10 (1927), 96–115.
- [O] O. Ore, *Theory of graphs*, Colloquium publications, Vol. XXXVIII, AMS, 1962.
- [T] W.P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, *Bulletin of the AMS*, V. 19, No. 2 (1988), 417–431.
- [v1] W. Vautaw, Abelian subgroups of the Torelli group, *Algebraic & Geometric Topology*, V. 2 (2002), 157–170.
- [v2] W. Vautaw, *Abelian subgroups and automorphisms of the Torelli group*, Ph. D. Thesis, Michigan State University, 2002.
- [w] H. Whitney, Congruent graphs and the connectivity of graphs, *American Journal of Mathematics*, V. 54, No. 1 (1932), 150–168.

June 24, 2016

<https://nikolaivivanov.com>