Affine planes, ternary rings, and examples of non-Desarguesian planes

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Preface

The main goal of this paper is to present a detailed self-contained exposition of a part of the theory of affine planes leading to a construction of affine (or, equivalently, projective) planes not satisfying the Desarques axiom. Unfortunately, most expositions of the theory of affine and projective planes stop before such a construction. Perhaps, the reason is that this theory is usually presented as a part of combinatorics, while such constructions are in the spirit of the abstract algebra.

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This article is intended to be a complement to the introductory expositions of the theory of affine and projective planes, and as an easy reading for mathematicians with a taste for abstract algebra (and the geometry of points, lines, and planes).

We start with an axiomatic definition of affine planes and show that all of them admit a coordinate system over a so-called *ternary ring* (in fact, ternary rings are defined in such a way as to make this statement true). It is well known that the Desargues axiom for an affine plane is equivalent to the existence of a coordinate system over a skew-field. Since there are excellent presentations of this equivalence (see, for example, the elegant book [Ha] by R. Harstshorne), we simply define a Desarguesian plane as a plane which admits a coordinate system over a skew-field. Similarly, since the correspondence between affine and projective planes is very well presented in the literature, we do not consider the projective planes in this article.

A novelty of our exposition is the notation $(a, x, b) \longmapsto \langle ax + b \rangle$ for the ternary operation in a ternary ring, replacing the standard notation $(a, x, b) \longmapsto x \cdot a \circ b$. The author hopes that the much more suggestive notation $\langle ax + b \rangle$ will made this beautiful theory accessible to a more wide audience.

1. Introduction

Affine planes. An *affine plane* $\mathbb{A}$ is a set, the elements of which are called *points*, together with a collection of subsets, called *lines*, satisfying the following three axioms.

A1. *For every two different points there is a unique line containing them.*

A2. *For every line* l *and a point* P *not in* l, *there is a unique line containing* P *and disjoint from* l.

A3. *There are three points such that no line contains all three of them.*

Two lines are called *parallel* if they are either equal, or disjoint. Note that being parallel is an equivalence relation. Indeed, this relation is obviously reflexive and symmetric. If two lines l_1, l_2 are parallel to a line l, then the intersection of l_1 and l_2 is empty by the axiom **A2**, i.e. l_1 is parallel to l_2.

And *isomorphism* of an affine plane $\mathbb{A}$ with an affine plane $\mathbb{A}'$ is defined as a bijection $\mathbb{A} \to \mathbb{A}'$ taking lines to lines. Two affine planes $\mathbb{A}$, $\mathbb{A}'$ are called *isomorphic*, if there exist an isomorphism $\mathbb{A} \to \mathbb{A}'$.

Affine planes and skew-fields. A *skew-field* is defined in the same way as a field, except that the commutativity of the multiplication is not assumed. Skew-fields (in particular, fields) lead to the main examples of affine planes. Namely, for a skew-field K, let $\mathbb{A} = K^2$ and let (x, y) be the canonical coordinates in K^2. A *line* in $\mathbb{A}$ is defined as subset of $\mathbb{A}$ described by an equation having either the form $y = ax + b$ for some $a, b \in K$, or the form $x = c$ for some $c \in K$. An easy exercise shows that K^2 with lines defined in this way is indeed an affine plane. If an affine plane is isomorphic to K^2, then we say that it is *defined over* K.

The class of affine planes defined over skew-fields can be characterized in purely geometric terms, i.e. in terms involving only points and lines. Namely, an affine plane $\mathbb{A}$ is defined over a skew-field if and only if $\mathbb{A}$ satisfies the so-called *Minor* and *Major Desargues axioms*. Another way to look at this characterization involves *projective planes* (which are not used in this article and by this reason are not even defined). Every affine plane can be canonically embedded in a projective plane, called its *projective completion*, by adding a *line at the infinity* to it. In particular, one can construct a projective plane starting from a skew-field K. A projective plane constructed in this way is said to be *defined over* K. A projective plane is defined over a skew-field if and only if it satisfies the *Desargues axiom* for projective planes. Also, the projective completion of an affine plane $\mathbb{A}$ is defined over a skew-field K, i.e. can be obtained from some affine plane by adding the line at infinity (which may be different from the original line at infinity added to $\mathbb{A}$) if and only $\mathbb{A}$ is defined over the same skew-field K.

In the present paper these characterizations serve only as a justification of the following term. Namely, an affine plane is said to be *non-Desarguesian* if it is not defined over a skew-field. Our main goal is to construct examples of non-Desarguesian planes.

Prerequisites. The prerequisites for reading this article are rather modest. It is not even strictly necessary to be familiar beforehand with the notion of an affine plane. But the reader is expected to be familiar with the notions of rings and fields. At some places we speak about vector spaces over skew-fields; without much loss the reader may assume that these skew-fields are actually fields. In Section 8 we use one basic result from the Galois theory, but it can be well taken on faith. Mainly, only a taste for abstract algebra is expected, especially in Section 8.

The organization of the paper. In Section 2 we introduce the notion of coordinates in an affine plane, not necessarily defined over a skew-field. The coordinates of a point are taken from a set with a ternary operation, called a *ternary ring*. Conversely, every ternary ring defines an affine plane, as explained in Section 2. A source of difficulties is the fact that isomorphic affine planes can be coordinatized by non-isomorphic ternary rings. They are isomorphic under an obvious additional condition; this is discussed in Section 3. In Section 4 we discuss a weaker notion of an isomorphism for ternary rings (namely, the notion of an *isotopism*), which is better related to isomorphisms of affine planes. But, in fact, this notion is not needed for our main goal, namely, for construction of non-Desarguesian planes, and Section 4 may be skipped without loss of the continuity. In Section 5 we introduce the most tractable class of ternary rings, namely, the class of *Veblen-Wedderburn systems*, also called *quasi-fields*. Like the fields, they are sets with two binary operations, but satisfying only a fairly weak version of the axioms of a field. Sections 6 and 7 are devoted to two different ways to prove that an affine plane is not defined over a skew-field. Section 8 is devoted to a construction of quasi-fields not isomorphic to a skew-field. Finally, in Section 9 we combine the results of the previous sections in order to construct non-Desarguesian planes. The main ideas are contained in Sections 2, 5, and 8.

Further reading. There are several excellent books exploring deeper these topics, in particular, exploring the role of the Desargues axioms. For a systematic introduction to the theory of affine and projective planes the reader may turn to the classical unsurpassed books by E. Artin [Ar] (see [Ar], Chapter II), M. Hall [H1] (see [H1], Chapter 20), and R. Hartshorne [Ha]. The book by Hartshorne is the most elementary one of them. The reader is not assumed even to be familiar with the notions of a group and of a field; in fact, the affine and projective geometries are used to motivate these notions. The book by M. Hall (or, rather, its last chapter, which actually does not depend much on the previous ones) gives an in depth exposition directed to mature mathematicians of the theory of projective planes and its connections with algebra. E. Artin's elegant exposition is written on an intermediate level between books of R. Hartshorne and M. Hall. All these books present in details the characterization of planes defined over skew-fields in terms of Desargues axioms.

A lot of books in combinatorics discuss some elementary parts of the theory of affine and projective planes, but very rarely include a construction of a non-Desarguesian plane. An exception is another M. Hall's classics, namely [H2]. We followed [H2] in that we deal with affine planes and not with projective ones, and in the way we coordinatize affine planes in Section 2.

The book of D. Hughes and F. Piper [HP] is, probably, the most comprehensive exposition of the theory of projective planes (the study of which is essentially equivalent to the study of affine planes). The state of the art as of 2007 is discussed by Ch. Weibel [W].

2. Affine planes and ternary rings

Let $\mathbb{A}$ be an affine plane. We start by introducing some sort of cartesian coordinates in $\mathbb{A}$. We follow the approach of M. Hall (see [H2], Section 12.4). Then we use these coordinates in order to define a ternary operation on any line in $\mathbb{A}$. Next, we will turn the main properties of this operation into axioms. This leads to the notion of a *ternary ring*. Our main novelty here is the notation $(a, x, b) \mapsto \langle ax + b \rangle$ for the ternary operation in ternary rings, replacing the notation $(a, x, b) \mapsto x \cdot a \circ b$ used by M. Hall and other authors. The notation $\langle ax + b \rangle$ seems to be much more suggestive than $x \cdot a \circ b$ and makes the whole theory much more transparent.

The simplest form of cartesian coordinates on $\mathbb{A}$***.*** *Fix two non-parallel lines* l, m *in* $\mathbb{A}$.

The lines l, m allow us to identify $\mathbb{A}$ with the cartesian product $l \times m$ in the same way as one does this while introducing the cartesian coordinates in the usual Euclidean plane. Namely, given a point $p \in \mathbb{A}$ we assign to it the pair $(x, y) \in l \times m$, where x is the point of intersection with the line l of the line containing p and parallel to m (such a line cannot be parallel to l, because otherwise l would be parallel to m contrary to the assumption), and where y is defined in a similar manner. This leads to a map $\mathbb{A} \to l \times m$. One can also define a map $l \times m \to \mathbb{A}$ by assigning to (x, y) the intersection point of the line containing x and parallel to m and the line containing y and parallel to l. Clearly, these two maps are the inverses of each other. We will use them to identify $\mathbb{A}$ with $l \times m$. Such an identification is the simplest form of the *cartesian coordinates* on $\mathbb{A}$.

Let $\mathbf{o}$ be the point of intersection of l and m; the point $\mathbf{o}$ serves as the origin of our coordinate system.

Identifying l ***and*** m***.*** Next, we would like to identify the lines l and m. In order to do this, we need a line d passing through $\mathbf{o}$ and different from l, m. One can get such a line, for example, by taking any line d_0 intersecting both l and m in such a way that the points of intersection are different from $\mathbf{o}$, and then take as d the line parallel to d_0 and passing through $\mathbf{o}$.

From now on, we will assume that such a line d *is fixed.*

Using d, we can construct a natural bijection between l and m. Namely, given $x \in l$, let $z(x) \in d$ be the intersection point with d of the line containing x and parallel to m, and let $y(x) \in m$ be the intersection point with m of the line containing $z(x)$ and parallel to l. Clearly, $x \mapsto y(x)$ is a bijection $l \to m$.

Now, let K be any set endowed with a bijection $K \to l$. By composing it with the bijection from the previous paragraph, we get a bijection $K \to m$. Formally, we can simply set $K = l$, but we are going to treat K and l differently, and by this reason it is better to consider them as different objects. The set K is going to play a role similar to the role of $\mathbf{R}$ in Euclidean geometry.

The cartesian coordinates on $\mathbb{A}$**.** Our bijections allow us to identify $\mathbb{A}$ with K^2. We consider this identification as the *cartesian coordinates* on $\mathbb{A}$. This identification, obviously, turns the line d into the *diagonal* $\{(x, x) : x \in K\}$. Guided by the construction of the usual cartesian coordinates, we denote the element of K corresponding to the point $\mathbf{0} \in l$ by 0. Then $\mathbf{0} = (0, 0)$.

We would like also to have an analogue of the number 1. In fact, we can choose as 1 an arbitrary element of K different from 0. This freedom of choice of 1 corresponds to the freedom of choice of the unit of measurement in the Euclidean geometry.

From now on, we will assume that such an element $1 \in K$ *is fixed.*

The slopes of lines. Next, we define the *slope* of a line L in $\mathbb{A}$. If L is parallel to l, its slope is defined to be 0. Such lines are called *horizontal*. If L is parallel to m, its *slope* is defined to be ∞. Such lines are called *vertical*. If the line L is not vertical, consider the line L' parallel to L and passing through $(0, 0)$. Let $(1, a)$ be the intersection point of L' with the line $\{(1, z) : z \in K\}$ (i.e. with the the vertical line passing through $(1, 0)$). The *slope* of L is defined to be a. Note that it depends on the choice of 1.

By the definition, the parallel lines have the same slope. Since d contains the point $(1, 1)$, the slope of d is equal to 1. Clearly, two lines have the same slope if and only if they are parallel.

The ternary operation on K**.** Let us define a ternary operation $(a, x, b) \longmapsto \langle ax + b \rangle$ on K as follows. Let L be the unique line intersecting the line m at the point $(0, b)$ and having the slope $a \neq \infty$. Since L is a non-vertical line, it intersects the line m at a single point, say $(0, b)$. For every $x \in K$, the line L intersects the vertical line $\{(x, z) : z \in K\}$ at a unique point. Let (x, y) be this point and set

(1) $\quad \langle ax + b \rangle = y.$

We consider $(a, x, b) \longmapsto \langle ax + b \rangle$ as a ternary operation in K. In general we do not have separate multiplication and addition operations in K; the angle brackets are intended to stress this.

Clearly, every non-vertical line is the set of points $(x, y) \in K^2$ satisfying (1) for some $a, b \in K$. Every vertical line is the set of point $(x, y) \in K^2$ satisfying, for some fixed $c \in K$, the equation $x = c$.

The main properties of $(a, x, b) \longmapsto \langle ax + b \rangle$**.** They are the following.

T1. $\langle 1x + 0 \rangle = \langle x1 + 0 \rangle = x.$

T2. $\langle a0 + b\rangle = \langle 0a + b\rangle = b$.

T3. *If* $a, x, y \in K$, *then there is a unique* $b \in K$ *such that* $\langle ax + b\rangle = y$.

T4. *If* $a, a', b, b' \in K$ *and* $a \neq a'$, *then the equation* $\langle ax + b\rangle = \langle a'x + b'\rangle$ *has a unique solution* $x \in K$.

T5. *If* $x, y, x', y' \in K$ *and* $x \neq x'$, *then there is a unique pair* $a, b \in K$ *such that* $y = \langle ax + b\rangle$ *and* $y' = \langle ax' + b\rangle$.

Let us explain the geometric meaning of these properties. This explanation also proves that they indeed hold for $(a, x, b) \longmapsto \langle ax + b\rangle$.

T1: The equation $\langle 1x + 0\rangle = x$ means that $d = \{(x, x) : x \in K\}$ is a line with the slope 1. The equation $\langle x1 + 0\rangle = x$ means that the slope of the line passing through $(0, 0)$ and $(1, x)$ is equal to x (which is true by the definition of the slope).

T2: The equation $\langle a0 + b\rangle = b$ means that the line defined by the equation (1) intersects m at $(0, b)$ (which is true by the definition of $\langle ax + b\rangle$). The equation $\langle 0a + b\rangle = b$ means that the horizontal line passing through $(0, b)$ consists of points (a, b), $a \in K$.

T3: This means that for every slope $\neq \infty$ there is a unique line with this slope passing through (x, y).

T4: This means that two lines with different slopes $\neq \infty$ intersect at a unique point.

T5: This means that every two points not on the same vertical line (i.e. not on the same line with slope ∞) are contained in a unique line with slope $\neq \infty$.

Ternary rings. Motivated by these properties, suppose that we have a set K with two distinguished elements 0 and $1 \neq 0$, and a ternary operation $(a, x, b) \longmapsto \langle ax + b\rangle$ satisfying **T1**–**T5**. Such a K is called a *ternary ring*. Consider the set of points $\mathbb{A} = K^2$, and introduce the lines in the following manner: for every $x_0 \in K$ we have a line $\{(x_0, y) : y \in K\}$ (such lines are called *vertical*), and for every $a, b \in K$ we have a line $\{(x, y) : y = \langle ax + b\rangle\}$. Clearly, this defines a structure of an affine plane on $\mathbb{A} = K^2$ (notice that the instances of the axioms of an affine plane involving vertical lines hold trivially).

1. Proposition. *If* K *is a finite set, then the condition* **T5** *follows from* **T3** *and* **T4**.

Proof. Given $x, x' \in K$ such that $x \neq x'$, consider the map $f\colon K^2 \to K^2$ defined by

$$f(a,b) = (\langle ax+b\rangle, \langle ax'+b\rangle).$$

Suppose that f is not injective, i.e. that

(2) $\langle ax+b\rangle = \langle a'x+b'\rangle$,

(3) $\langle ax'+b\rangle = \langle a'x'+b'\rangle$

for some $(a,b) \neq (a',b')$. If $a = a'$, then (2) contradicts **T3**. If $a \neq a'$, then the two equalities (2), (3) together contradict **T4**. Therefore **T3** and **T4** imply that f is injective. Since f is a self-map of a finite set to itself, the injectivity of f implies its surjectivity. So, f is a bijection. **T5** follows. ■

3. Isomorphisms of ternary rings

The choices involved in the construction of a ternary ring by an affine plane. The construction of the ternary ring K associated to an affine plane $\mathbb{A}$ involves several choices. First, we selected two non-parallel lines l and m. Then we choose a set K together with a bijection $K \to l$, which we may consider as an identification. This choice is not essential at all. As we noted, we could simply take $K = l$ and the map $K \to l$ be the identity map.

Then we chose a third line d passing through the intersection point $\mathbf{0}$ of l and m, and a element $1 \in K$, $1 \neq 0$. The line d and the element $1 \in K$ define a point $z \in \mathbb{A}$: the intersection point with d of the line parallel to m and passing through the point of l corresponding to 1, This point corresponds to $(1,1)$ under our identification of $\mathbb{A}$ with K^2. Conversely, given a point $z \in \mathbb{A}$ not on l, m, we can define d as the line connecting $\mathbf{0}$ with z, and define 1 as the element of K corresponding to the intersection point with l of the line parallel to m and containing z. Therefore, the choice of d and 1 is equivalent to the choice of a point z not contained in the union $l \cup m$.

The effect of choices. Let $\mathbb{A}'$ be another affine plane with two lines and a point l', m', z' as above, and let K' be its coordinate ring. Clearly, there is an isomorphism $f\colon \mathbb{A} \to \mathbb{A}'$ such that $f(l) = l'$, $f(m) = m'$, $f(z) = z'$ if and only if there is bijection $F\colon K \to K'$ such that $F(0) = 0$, $F(1) = 1$, and

$$F(\langle ax+b\rangle) = \langle F(a)F(x)+F(b)\rangle$$

for all $a, x, b \in K$. Such a bijection is called an *isomorphism* $K \to K'$.

2. Proposition. *The following two conditions are equivalent.*

(i) *There is an isomorphism* $K^2 \to (K')^2$ *taking* $\mathbf{0}$ *to* $\mathbf{0}$, $K \times 0$ *to* $K' \times 0$, $0 \times K$ *to* $0 \times K'$, *and* $(1,1)$ *to* $(1,1)$.

(ii) *There is an isomorphism* $K \to K'$.

Proof. It is sufficient to apply the above observation to $\mathbb{A} = K^2$, $l = K \times 0$, $m = 0 \times K$, $z = (1,1)$, and $\mathbb{A} = (K')^2$, $l = K' \times 0$, $m' = 0 \times K'$, $z' = (1,1)$. ■

We see that up to an isomorphism K is determined by the plane $\mathbb{A}$ with a fixed choice of l, m, z. We call the ternary ring K a *coordinate ring* of the plane $\mathbb{A}$ with a triple (l, m, z) as above.

4. Isotopisms of ternary rings

The later sections do not depend on this one.

Isotopism. Ternary rings corresponding to the same affine plane $\mathbb{A}$ and the same choice of lines l, m, but to different choices of the point z, may lead to non-isomorphic ternary rings. Still, different choices of z lead to ternary rings which are *isotopic* in the following sense. A triple (F, G, H) of bijections $K \to K'$ is called an *isotopism*, if $H(0) = 0$ and

$$H(\langle ax + b \rangle) = \langle F(a)G(x) + H(b) \rangle$$

for all $a, x, b \in K$. Such a triple induces a map $\varphi\colon K^2 \to (K')^2$ by the rule $\varphi(x,y) = (G(x), H(y))$. Clearly, φ takes vertical lines to vertical lines. The equation $y = \langle ax + b \rangle$ implies $H(y) = \langle F(a)G(x) + H(b) \rangle$, which means that $(x', y') = \varphi(x, y)$ satisfies the equation $y' = \langle F(a)x' + H(b) \rangle$. It follows that φ takes the lines with slope $a \neq \infty$ to the lines with slope $F(a) \neq \infty$. We see that $\varphi\colon K^2 \to (K')^2$ is an isomorphism of affine planes.

3. Lemma. φ *takes horizontal lines to horizontal lines* (*i.e.* $F(0) = 0$), *and also takes* $\mathbf{0}$ *to* $\mathbf{0}$.

Proof. In order to prove the first statement, note that φ takes the line $\{(x, 0) : x \in K\}$ to the line $\{(G(x), H(0)) : x \in K\} = \{(x', 0) : x' \in K'\}$ (since $H(0) = 0$ and G is a bijection). Since both these lines have slope 0, we have $F(0) = 0$. Therefore, φ takes horizontal lines to horizontal lines. Let a, a' be two different slopes. Then the lines with equations $y = \langle ax + 0 \rangle$, $y = \langle ax + 0 \rangle$ intersect at $\mathbf{0} = (0, 0)$. Their images have the equations $y = \langle F(a)x + 0 \rangle$, $y = \langle F(a')x + 0 \rangle$ (recall that $H(0) = 0$). Since F is a bijection, $F(a) \neq F(a')$ and therefore these two lines intersect only at $\mathbf{0}$. It follows that $\varphi(\mathbf{0}) = \mathbf{0}$. ■

4. Corollary. *For an isotopism* (F, G, H) *we have* $F(0) = 0$ *and* $G(0) = 0$, *in addition to* $H(0) = 0$.

Proof. $F(0) = 0$ is already proved. $G(0) = 0$ follows from the following two facts: (i) $\varphi(\mathbf{0}) = \mathbf{0}$; (ii) φ takes the vertical line $x = 0$ to the vertical line $x = G(0)$. ■

5. Corollary. *The isomorphism of affine planes induced by an isotopism of ternary rings takes the horizontal (respectively, vertical) line containing* $\mathbf{0}$ *to the horizontal (respectively, vertical) line containing* $\mathbf{0}$. ■

6. Theorem. *Let* K *be a coordinate ring of the plane* $\mathbb{A}$ *with a choice of* l, m, z *as above, and let* K' *be the coordinate ring of the plane* $\mathbb{A}'$ *with a choice of* l', m', z'. *There is an isomorphism* $\mathbb{A} \to \mathbb{A}'$ *taking* l *to* l' *and* m *to* m' (*but not necessarily* z *to* z') *if and only if there is an isotopism* $K \to K'$.

Proof. The *"if"* direction is already proved. Let us prove the *"only if"* direction.

Let us identify $\mathbb{A}$ with K^2 and $\mathbb{A}'$ with $(K')^2$. Let $G \colon K \to K'$ be the map corresponding to the map $K \times 0 \to K' \times 0$ induced by φ. Similarly, let $H \colon K \to K'$ be the map corresponding to the map $0 \times K \to 0 \times K'$ induced by φ. Using the fact that every point is the intersection of a unique vertical line with a unique horizontal line, and the fact that φ maps the vertical (respectively, horizontal) lines to the vertical (respectively, horizontal) lines, we see that φ is determined by the maps G, H, and, in fact, $\varphi(x, y) = (G(x), H(y))$. Clearly, $G(0) = 0$ and $H(0) = 0$.

In order to define F, consider for each $a \in K$ the line in $\mathbb{A}$ with the slope a passing through $\mathbf{0}$. The map φ takes it to a line in $\mathbb{A}'$ passing through $\mathbf{0}$. Let $F(a) \in K'$ be its slope.

Let us check that (F, G, H) is an isotopism. Since φ takes parallel lines to parallel lines, φ takes any line with the slope a to a line with the slope $F(a)$. So, it takes the line with the equation $y = \langle ax + b \rangle$ into a line with the equation of the form $y' = \langle F(a)x' + b' \rangle$. The first line contains the point $(0, b)$ (since $\langle a0 + b \rangle = b$ by **T2**). Therefore, the second line contains the point $\varphi(0, b) = (0, H(b))$. This implies that $H(b) = \langle F(a)0 + b' \rangle$. But $\langle F(a)0 + b' \rangle = b'$ by **T2**. Therefore, $b' = H(b)$.

We see that φ maps the line with the equation $y = \langle ax + b \rangle$ into the line with the equation $y' = \langle F(a)x' + H(b) \rangle$. Since $\varphi(x, y) = (G(x), H(y))$, we see that $y = \langle ax + b \rangle$ implies $H(y) = \langle F(a)G(x) + H(b) \rangle$. It follows that (F, G, H) is an isotopism. ■

Remark. If (F, G, H) is an isotopism, then F and G are determined by H and two elements $F^{-1}(1)$, $G^{-1}(1)$. Indeed,

$$F(a) = \langle F(a)1 + 0 \rangle = \langle F(a)G(G^{-1}(1)) + H(0) \rangle = H(\langle aG^{-1}(1) + 0 \rangle),$$

and

$$G(a) = \langle 1G(a) + 0 \rangle = \langle F(F^{-1}(1))G(a) + H(0) \rangle = H(\langle F^{-1}(1)a + 0 \rangle).$$

Historical note. For non-associative algebras, the notion of an equivalence weaker than an isomorphism was first introduced by A. A. Albert [Al]. He called two algebras A, A' *isotopic* if there is a triple of linear maps P, Q, $R\colon A \to A'$ such that

$$R(xy) = P(x)Q(y).$$

He called such a triple an *isotopy* of A and A'. Albert relates that

> *The concept of isotopy was suggested to the author by the work of N. Steenrod who, in his study of homotopy groups in topology, was led to study isotopy of division algebras.*

Albert noticed that if associativity of the multiplication is not assumed, the notion of isotopy is more suitable than the obvious notion of isomorphism, which leads to too many non-isomorphic (but isotopic) algebras.

It is only natural that the notion of an isomorphism is too narrow for the ternary rings also. The corresponding notion of an *isotopism* was introduced by M. V. D. Burmester [Bu], and, independently, by D. Knuth [Kn]. Both Burmester and Knuth proved Theorem 6 above. D. Knuth [Kn], moreover, found an affine plane $\mathbb{A}$ such that all ternary rings corresponding to different choices of z (but the same choice of l, m) are pairwise non-isomorphic. His plane is finite, and the corresponding ternary rings have 32 elements. See [Kn], Section 5. Unfortunately, his plane was found with the help of a computer, and, as Knuth writes, *"No way to construct this plane, except by trial and error, is known."* To the best knowledge of the author, this is still the case.

5. Veblen-Wedderburn systems

The left Veblen-Wedderburn systems. Let K be a set with two binary operations $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$, called the *addition* and the *multiplication*, respectively, and two distinguished elements 0, 1, $0 \neq 1$. If the following properties **VW1–VW5** hold, K is called a *left Veblen-Wedderburn system*, or, more recently, a *left quasi-field*.

VW1. K *is an abelian group with respect to the addition* $+$.

VW2. *Given* $a, b \neq 0$, *each of the equations* $ax = b$ *and* $xa = b$ *has a unique solution* x; *moreover, this solution is* $\neq 0$. *In addition, if* $a, b \neq 0$, *then* $ab \neq 0$.

VW3. $1x = x1 = x$, $0x = x0 = 0$, *and* $x + 0 = 0 + x = x$ *for all* x.

VW4. *Left distributivity:* $a(x + y) = ax + ay$ *for all* a, x, y.

VW5. *For* $a \neq a'$, *the equation* $ax = a'x + b$ *has a unique solution* x.

This notion was introduced by O. Veblen and J. Wedderburn [VW]. Notice that **VW5** is a weak version of the right distributivity. Clearly, under conditions **VW1**, **VW2** it follows from the right distributivity.

The right Veblen-Wedderburn systems. In order to define *right Veblen-Wedderburn system*, or *right quasi-fields*, we replace **VW4** and **VW5** by the following two conditions.

VW4-r. *Right distributivity:* $(x + y)a = xa + ya$ *for all* a, x, y.

VW5-r. *For* $a \neq a'$, *the equation* $xa = xa' + b$ *has a unique solution* x.

Clearly, K is a right quasi-field if and only if K with the same addition, 0, 1, and the opposite multiplication $a \cdot b = ba$, is a left quasi-field.

Weak quasi-fields. If K satisfies only conditions **VW1–VW4**, it is called a *a weak left quasi-field*. Similarly, K is called a *weak right quasi-field*, if it satisfies conditions **VW1–VW3** and **VW4-r.**

From Veblen-Wedderburn systems to ternary rings. If K is a left or right quasi-field, then we can define a ternary operation $(a, x, b) \mapsto \langle ax + b \rangle$ by the obvious rule $\langle ax + b \rangle = ax + b$. We claim that K with this ternary operation and the distinguished elements 0 and 1 is a ternary ring. Let us check this first for left quasi-fields.

T1: This condition follows from **VW3**.

T2: This condition also follows from **VW3**.

T3: This condition follows from **VW1**.

T4: Let $a, a', b, b' \in K$ and $a \neq a'$. The equation $ax + b = a'x + b'$ for x is equivalent to $ax = a'x + (b' - b)$ by **VW1**. It has a unique solution by **VW5**.

T5: Let $x, y, x', y' \in K$ and $x \neq x'$. The equations $y = ax + b$ and $y' = ax' + b$ for a, b imply

$$y - y' = ax - ax'$$

by **VW1**, and hence imply

$$y - y' = a(x - x')$$

by **VW4**. If $y \neq y'$, this equation is uniquely solvable for a by **VW2**. If we know a, we can find b from either of the equations $y = ax + b$, $y' = ax' + b$. Therefore b is unique. This proves **T5** in the case $y \neq y'$. If $y = y'$, then a has to be equal to 0 by **VW2** (since $x - x' \neq 0$). Therefore $b = y = y'$. This proves **T5** in the case $y = y'$.

For a right quasi-field K the conditions **T1**–**T3** hold by the same reasons as for the left quasi-fields (they do not depend on the distributivity). Let us check **T4** and **T5**.

T4: Let $a, a', b, b' \in K$ and $a \neq a'$. The equation $ax + b = a'x + b'$ for x is equivalent to $(a - a')x = (b' - b)$ by **VW1** and **VW4-r** (the right distributivity). It has a unique solution by **VW2** and **VW3** (the latter is needed if $b' - b = 0$).

T5: Let $x, y, x', y' \in K$ and $x \neq x'$. The equations $y = ax + b$ and $y' = ax' + b$ for a, b imply

$$ax = ax' + (y - y').$$

Since $x \neq x'$, this equation has a unique solution a by **VW5-r**. As above, if we know a, we can find b from either of the equations $y = ax + b$, $y' = ax' + b$. Therefore b is unique. This proves **T5**.

Notice that going from left to right quasi-fields switches the roles of **VW4** and **VW5**.

Reconstructing quasi-field from the corresponding ternary ring. A left quasi-field can be restored from the corresponding ternary ring in an obvious manner: it has the same 0 and 1; the addition and the multiplication are defined by $a+b = \langle 1a+b\rangle$ and $ab = \langle ab+0\rangle$. Indeed, $1(ax+0)+b = ax+0+b = ax+b$. Therefore, we may consider quasi-fields as a special class of ternary rings. In particular, a quasi-field K defines an affine plane. Of course, this plane can be described directly: its set of points is K^2, and its lines are given by the equations of the form $x = a$ and of the form $y = ax+b$, where $(x, y) \in K^2$ and a, b are fixed elements of K.

7. Proposition. *If K is weak left quasi-field and is finite, then K is a left quasi-field (i.e.* **VW5** *follows from* **VW1**–**VW4** *if K is finite).*

Proof. For $a \neq a'$, let $f(x) = ax - a'x$. Suppose that f is not injective, i.e. $ax - a'x = ay - a'y$ for some $x \neq y$. Then $a(x-y) = a'(x-y)$ by **VW1** and **VW4** (the left distributivity). Since $a \neq a'$, this contradicts **VW2**. Therefore, f is injective. Being a self-map of a finite set to itself, it is bijective (cf. the proof of Proposition 1). Therefore, for every b there is a unique x such that $ax - a'x = b$. Hence, **VW5** holds. ■

Finiteness. Proposition 1 shows that in the finite case we can drop **T5** form the axioms of a ternary ring. By Proposition 7 we can also drop **VW5** form the axioms of a quasi-field for finite K. While checking **T4** for the ternary ring associated to a quasi-field above, we referred to **VW5**. If the quasi-field is finite and we drop the axiom **VW5**, we have to use the Proposition 7, and the role of **VW5** is passed to the left distributivity.

In some situations the finiteness can be replaced by the finite dimensionality over an appropriate skew-field.

8. Proposition. *Suppose that a weak left quasi-field K contains a subset F which is a skew-field with respect to the same operations and with the same 0 and 1. Suppose that, in addition,*

(4) $\quad (xy)a = x(ya),$

(5) $\quad (x+y)a = xa+ya,$

for all $a \in F$ and $x, y \in K$. Then K is a right vector space over F. If this vector space is finitely dimensional, then K is a left quasi-field (i.e. the condition **VW5** *holds).*

Proof. The first statement is clear.

Let us prove the second one. For $a \in K$, let $L_a \colon K \to K$ be the left multiplication by a, i.e. $L_a(x) = ax$. By **VW4** we have $L_a(x+y) = L_a(x) + L_a(y)$ for all $x, y \in$

K. Moreover, if $b \in F$, then $L_a(xb) = a(xb) = (ax)b = L_a(x)b$. It follows that L_a is (right) linear map of the vector space K to itself.

We need to check that for $a \neq a'$ the equation $L_a(x) = L_{a'}(x) + b$ has a unique solution x. Let $L = L_a - L_{a'}$. It is sufficient to show that the equation $L(x) = b$ has a unique solution x. Clearly, L is a linear map. If $L(y) = 0$, then $ay - a'y = 0$ and $ay = a'y$. Since $a \neq a'$, the condition **VW2** implies that this is possible only if $y = 0$. We see that L is linear self-map of K with trivial kernel. Since K is assumed to be finitely dimensional, L is an isomorphism. This implies that $L(x) = b$ has a unique solution. This proves the second statement of the proposition. ■

Our proof of Proposition 8 follows the proof of Theorem 7.3 in [HP].

6. Near-fields, skew-fields, and isomorphisms

In general, if affine planes K^2 and $(K')^2$ are isomorphic, the ternary rings K and K' do not need to be isomorphic. The goal of this section is to prove that they will be isomorphic if K' is a skew-field. See Corollary 10 below. A part of the proof works in a greater generality, namely for near-fields, which we will define in a moment.

A *left near-field* is a left quasi-field with associative multiplication. Non-zero elements of a left near-field form a group with respect to the multiplication. The *right near-fields* are defined in an obvious manner. Clearly, being a skew-field is equivalent to be being a left and right near-field simultaneously.

9. Lemma. *Let K' be a left near-field. Let $\mathbf{o} = (0,0) \in (K')^2$, and let l, m be, respectively, the horizontal and the vertical lines in $(K')^2$ passing through $\mathbf{o}$ (i.e. $l = K' \times 0$ and $m = 0 \times K'$). For every two points $z, z' \in (K')^2$ not in $l \cup m$, there is an automorphism of the affine plane $(K')^2$ preserving $\mathbf{o}$, l, and m, and taking z to z'.*

Proof. It is sufficient to consider the case when $z = (1,1)$. Let $z' = (u, v)$. Since (u, v) is not on l, m, both u and v are non-zero. Consider the map $f\colon (K')^2 \to (K')^2$ defined by $f(x, y) = (ux, vy)$. Clearly, $f(1,1) = (u, v)$, and f takes the vertical line $x = a$ to the vertical line $x = au$. If $y = ax + b$, then $vy = v(ax) + vb = (vau^{-1})ux + vb$ by the left distributivity and the associativity of the multiplication (here, as usual, u^{-1} is the unique solution of the equation $xu = 1$). It follows that f takes the line $y = ax + b$ to the line $y = (vau^{-1})x + vb$. Hence f is an automorphism of $(K')^2$. ■

10. Corollary. *Let* K *be a ternary ring, and let* K' *be a left near-field. Suppose that there is an isomorphism of planes* $f\colon K^2 \to (K')^2$ *taking* $\mathbf{o}$ *to* $\mathbf{o}$ *and taking the horizontal* (*respectively, vertical*) *line through* $\mathbf{o}$ *in* K^2 *to horizontal* (*respectively, vertical*) *line through* $\mathbf{o}$ *in* $(K')^2$*. Then* K *is isomorphic to* K' *as a ternary ring.*

Proof. Let $z = (1, 1) \in K^2$. By taking the composition of f with an appropriate automorphism $g\colon (K')^2 \to (K')^2$, if necessary, we can assume that $f(1, 1) = (1, 1)$ (the required g exists by the lemma). It remains to apply Proposition 2. ■

11. Lemma. *Let* K' *be a skew-field. Let* $\mathbf{o} = (0, 0) \in (K')^2$*, and let* l, m *be, respectively, the horizontal and the vertical lines in* $(K')^2$ *passing through* $\mathbf{o}$ (*i.e.* $l = K' \times 0$ *and* $m = 0 \times K'$). *Let* l', m' *be any two non-parallel lines in* $(K')^2$*. Then there is an automorphism of the affine plane* $(K')^2$ *taking* l *to* l' *and* m *to* m' (*and, in particular, taking* $\mathbf{o}$ *to the intersection point of* l' *and* m').

Proof. If K is a field, this is a fact well-known from the linear algebra. In general, one needs to check that there is no need to use commutativity of the multiplication. Let us first check that some natural maps are isomorphisms.

(i) The map $D(x, y) = (y, x)$ is an isomorphism. Indeed, it takes the line $x = a$ to the line $y = 0x + a$, and the line $y = 0x + b$ to the line $x = b$. If $a \neq 0$, it takes the line $y = ax + b$, i.e. the line $x = a^{-1}y - a^{-1}b$ (where a^{-1} is the unique solution of the equation $xa = 1$) to the line $y = a^{-1}x - a^{-1}b$. Here we used the left distributivity and the associativity of the multiplication.

(ii) For any $c, d \in K'$ the map $f(x, y) = (x + c, y + d)$ is an isomorphism. Indeed, it takes the line $x = a$ to the line $x = a + c$, and the line $y = ax + b$, to the line $y = ax - ac + b + d$. Here we used the left distributivity.

(iii) For any $c \in K'$ the map $f(x, y) = (x, y - cx)$ is an isomorphism. Indeed, it takes every line $x = a$ to itself, and it takes the line $y = ax + b$ to the line $y = (a - c)x + b$. Here we used the right distributivity.

(iv) For any $c \in K'$ the map $g(x, y) = (x - cy, y)$ is an isomorphism. Indeed, $g = D \circ f \circ D$, where $D(x, y) = (y, x)$ and $f(x, y) = (x, y - cx)$.

By using an isomorphism of type (ii) if necessary, we can assume that l', m' intersect at $\mathbf{o}$. By using the isomorphism D from (i) if necessary, we can assume that l' is not equal to $m = 0 \times K$. Then l' has the form $y = cx$. The map $f(x, y) = (x, y - cx)$ is of type (iii) and takes l' to l. Therefore we can assume that $l' = l$ and m' intersects $l' = l$ at $\mathbf{o}$. Then m' has a equation of the form $x = cy$ and an isomorphism of type (iv) takes m' to m. Since any automorphisms of type (iv) takes l to l, this completes the proof. ■

12. Theorem. *Let* K *be a ternary ring, and let* K' *be a skew-field. Suppose that there is an isomorphism of planes* $f\colon K^2 \to (K')^2$*. Then* K *is isomorphic to* K' *as a ternary ring.*

Proof. This follows from Lemma 11 and Corollary 10. ■

It follows that in order to construct an affine plane not coming from a skew-field, it is sufficient to construct a quasi-field which is not a skew-field (a quasi-field which is isomorphic to a skew-field is a skew-field itself). In particular, it is sufficient to construct a left quasi-field in which the right distributivity does not hold or a right quasi-field in which the left distributivity does not hold. Alternatively, it is sufficient to construct a (left or right) quasi-field in which the associativity of multiplication does not hold. We will present a construction of such quasi-field in Section 8.

7. Translations

The previous section provided us with a method of constructing affine planes not isomorphic to any affine plane defined by a skew-field. In this section we will present another method, based on an investigation of special automorphisms of affine planes called *translations*. It allows to show that some planes are not isomorphic even to any plane defined by a left quasi-field (see Theorem 17 below).

Let $\mathbb{A}$ be an affine plane. An automorphism $f\colon \mathbb{A} \to \mathbb{A}$ is called a *translation* if $f(l)$ is parallel to l for every line l (equal lines are considered to be parallel), and if f preserves every line from a class of parallel lines. Clearly, for a non-trivial (i.e., not equal to the identity) translation there is exactly one such class of parallel lines. Every line from this class is called a *trace* of f. If $\mathbb{A}$ is realized as K^2 for a ternary ring K, then a translation is called *horizontal* if it preserves all horizontal lines, i.e. if the class of horizontal lines is its trace.

The next two propositions are not used in the rest of the paper.

13. Proposition. *A non-trivial translation has no fixed points.*

Proof. Let f be a translation fixing a point z. Let l be trace of f.

Let m be a line containing z and not parallel to l. Since $f(m)$ is parallel to m and contains z, we have $f(m) = m$. Every point of m is the unique intersection point of m and a line parallel to l. The map f leaves both of these lines invariant. Therefore f fixes all points of m.

We see that f fixes all points except, possibly, the points of the line l_z passing through z and parallel to l. By applying the same argument to any point not on l_z in the role of z, we conclude that f fixes the points of l_z also, i.e. that $f = \mathrm{id}$. ■

14. Proposition. *Let z, z' be two different points, and let l be the line passing through z, z'. There is no more than one translation taking z to z', and if such a translation exists, it leaves invariant every line parallel to l.*

Proof. If f_1, f_2 are two different translations taking z to z', then $f_1^{-1} \circ f_2$ is a non-trivial translation fixing z, contradicting to Proposition 13.

Now, let f be a translation such that $f(z) = z'$, and let m be a trace of f. Let m_z be the line passing through z and parallel to m. Then m_z is also a trace of f. Clearly, we have $z \in m_z$ and $z' = f(z) \in m_z$. It follows that $l = m_z$ and, hence, l is a trace of f. Therefore, f leaves invariant every line parallel to l. This completes the proof. ■

15. Lemma. *Let K be a left quasi-field. For every two points (c_1, d_1), (c_2, d_2) of the plane K^2, there is a translation of K^2 taking (c_1, d_1) to (c_2, d_2).*

Proof. In this case there are obvious maps expected to be translations, namely the maps of the form $f(x, y) = (x + c, y + d)$, where $c, d \in K$. Clearly, if $c = c_2 - c_1$, $d = d_2 - d_1$, then $f(c_1, d_1) = (c_2, d_2)$. Let us check that these maps are indeed translations.

The map $f(x, y) = (x + c, y + d)$ takes the line $x = a$ to the line $x = a + c$, and the line $y = ax + b$ to the line $y = ax - ac + b + d$. (Cf. (ii) in the proof of Lemma 11 in the previous section.) In particular, it takes vertical lines to vertical lines, and the lines with the slope a to the lines with the slope a. If $c = 0$, then f preserves all vertical lines, and therefore is a translation. If $c \neq 0$, then $d = ec$ for some e by **VW2**. Since f takes the line $y = ex + b$ to the line $y = ex - ec + b + d$ and $ex - ec + b + d = ex - d + b + d = ex + b$, we see that f leaves invariant every line with the slope e. It follows that f is a translation in this case also. ■

16. Lemma. *Let K be a right quasi-field. Suppose that for every $v \in K$, $v \neq 0$, the plane K^2 admits a translation taking $\mathbf{0}$ to $(v, 0)$. Then the left distributivity law holds in K.*

Proof. Let f be a translation such that $f(0, 0) = (v, 0)$. Since the line passing through $(0, 0)$ and $(v, 0)$ is the horizontal line $K \times 0$, the translation f is a horizontal translation. Let us show that f has the expected form $f(a, b) = (a + v, b)$. This follows from the following four observations.

1. The line $y = x$ with the slope 1 passing through $(0, 0)$ is mapped to the line with the slope 1 passing through $(v, 0)$, i.e. to the line $y = x - v$.

2. For every $a \in K$, the map f preserves the line $y = a$. It follows that f takes the point of intersection of the lines $y = a$ and $y = x$ to the point of intersection of the lines $y = a$ and $y = x - v$. This means that $f(a, a) = (a + v, a)$.

3. The vertical line $x = a$ containing (a, a) is mapped to the vertical line containing $(a + v, a)$, i.e. to the line $x = a + v$.

4. The point of intersection of the lines $y = b$ and $x = a$ is mapped to the point of intersection of the lines $y = b$ and $x = a + v$. In other terms, $f(a, b) = (a + v, b)$.

Now, f takes the line $y = ax$ containing $(0, 0)$ to another line with the slope a, i.e. to a line of the form $y = ax - c$. Since it contains $(v, 0)$, we have $c = av$. So, the line $y = ax$ is mapped to the line $y = ax - av$. For every $u \in K$ the point (u, au) belongs to the line $y = ax$ and is mapped to the point $(u + v, au)$. Therefore, $(u + v, au)$ belongs to the line $y = ax - av$, i.e. $au = a(u + v) - av$, or $a(u + v) = au + av$.

Since this is true for all $a, u, v \in K$, the left distributivity holds. ■

17. Theorem. *Let K be a right quasi-field for which the left distributivity does not hold, and let K' be a a left quasi-field. Then the planes K^2 and $(K')^2$ are not isomorphic.*

Proof. By Lemma 16 there is a point $(c, d) \in K^2$ such that no translation takes $(0, 0)$ to (c, d). Therefore, K^2 is not isomorphic to any plane constructed from a left quasi-field by Lemma 15. ■

8. André quasi-fields

The norm map. Let K be a field, and let G be a finite group of automorphisms of K. Let F be the subfield of K consisting of all elements fixed by G. By the Galois theory, the dimension of K as a vector space over F is equal to the order of G; in particular, it is finite. The *norm map* N is defined as follows:

$$N(x) = \prod_{g \in G} g(x).$$

Clearly, $N(x) \in F$ for all $x \in K$. Moreover, N defines a homomorphism $K^* \to F^*$ from the multiplicative group K^* to the multiplicative group F^* of the fields K, F respectively. Obviously, $N(g(a)) = N(a)$ for any $g \in G$.

Modifying the multiplication. Note that $N(1) = 1$, and, therefore, $1 \in N(K^*)$. Let $\varphi\colon N(K^*) \to G$ be a map subject to the only condition $\varphi(1) = 1$. In particular, φ does not need to be a homomorphism. Given such a map φ, we construct a new multiplication $\odot$ in K as follows. Of course, the new multiplication $\odot$ will depend on φ, but we will omit this dependence from the notations. Let α be equal to $\varphi \circ N$ on K^*, and let $\alpha(0) = 1 = \mathrm{id}_K$. So, α is a map $K \to G$. We will often denote $\alpha(x)$ by α_x; it is an automorphism $K \to K$ belonging to the group G. The multiplication $\odot$ is defined by the formula

$$x \odot y \;=\; x\,\alpha_x(y),$$

for all $x, y \in K$. Let K_φ be the set K endowed with the same addition and the same elements 0, 1 as K, and with the multiplication $\odot$.

18. Theorem. K_φ *is a left quasi-field.*

Proof. **VW1:** This property holds for K_φ because it holds for K.

VW3: Note that $\alpha_1 = \varphi(N(1)) = \varphi(1) = 1$. This implies $1 \odot x = x$ for all x. Also, since α_x is an automorphism of K, we have $\alpha_x(1) = 1$, $\alpha_x(0) = 0$, and, therefore, $x \odot 1 = x$, $x \odot 0 = 0$ for all $x \in K$. In addition, $0 \odot y = 0\alpha_0(y) = 0y = 0$. These observations imply the multiplicative part of **VW3** for K_φ; the additive part holds for K_φ because it holds for K.

VW4: Since α_x is an automorphism of K, we have $\alpha_x(y+z) = \alpha_x(y) + \alpha_x(z)$, and therefore $x \odot (y+z) = x \odot y + x \odot z$. So, the left distributivity law **VW4** holds for K_φ.

VW2: Suppose that $a, b \neq 0$. First of all, notice that $\alpha_a(b) \neq 0$ (because α_a is an automorphism of K), and, therefore, $a \odot b \neq 0$. Next, consider the equation $a \odot x = b$. It is equivalent to $a\,\alpha_a(x) = b$, which, in turn, is equivalent to $\alpha_a(x) = a^{-1}b$. It follows that $x = \alpha_a^{-1}(a^{-1}b)$ is the unique solution.

It remains to consider the equation $x \odot a = b$. Notice that

$$N(x \odot a) \;=\; N(x\alpha_x(a)) \;=\; N(x)N(\alpha_x(a)) \;= N(x)\,N(a),$$

since $N(g(a)) = N(a)$ for any $g \in G$. Therefore, $x \odot a = b$ implies that $N(x)N(a) = N(b)$. This, in turn, implies that

$$N(x) \;=\; N(a)^{-1}N(b) \;=\; N(a^{-1}b),$$

and $\alpha(x) = \varphi(N(x)) = \varphi(N(a^{-1}b)) = \alpha(a^{-1}b)$. In other terms, $\alpha_x = \alpha_{a^{-1}b}$.

Therefore, if $x\,\alpha_x(a) = x \odot a = b$, then $x\,\alpha_{a^{-1}b}(a) = b$ and

(6) $\quad x = b(\alpha_{a^{-1}b}(a))^{-1}.$

It follows that the equation $x \odot a = b$ has no more than one solution.

Let us check that (6) is, indeed, a solution. Let $g = \alpha_{a^{-1}b}$. If x is defined by (6), then

$$\alpha_x = \alpha(x) = \varphi(N(b(\alpha_{a^{-1}b}(a))^{-1}) = \varphi(N(b(g(a)^{-1})).$$

At the same time,

$$\begin{aligned} N(b(g(a)^{-1})) &= N(b)\,N(g(a)^{-1}) && \text{(because } N \text{ is a homomorphism)} \\ &= N(b)\,N(g(a^{-1})) && \text{(because } g \in G \text{ and hence } g(a)^{-1} = g(a^{-1})) \\ &= N(b)\,N(a^{-1}) && \text{(because } g \in G \text{ and hence } N(g(b)) = N(b)) \\ &= N(ba^{-1}) && \text{(because } N \text{ is a homomorphism)} \\ &= N(a^{-1}b) && \text{(because the multiplication in } K \text{ is commutative).} \end{aligned}$$

It follows that $\alpha_x = \varphi(N(b(g(a)^{-1})) = \varphi(N(a^{-1}b)) = \alpha_{a^{-1}b}$. Therefore

$$x \odot a = x\,\alpha_x(a) = x\,\alpha_{a^{-1}b}(a) = b\,(\alpha_{a^{-1}b}(a))^{-1}\,\alpha_{a^{-1}b}(a) = b$$

This proves that (6) is a solution of $x \odot a = b$ and completes our verification of **VW2**.

VW5: It remains to check **VW5**. To this end, we will apply Proposition 8 (see Section 5). We already established that K_φ is a weak left quasi-field. Notice that

(7) $\quad \alpha_x(a) = a \quad$ for any $x \in K$ and $a \in F$,

because F is fixed by all elements of G. Therefore,

(8) $\quad x \odot a = x\,a \quad$ for all $x \in K$, $a \in F$.

This immediately implies the condition (5) of Proposition 8. Now, let $x, y \in K$, and $a \in F$. The following calculation shows that the condition (4) of Proposition 8 also holds:

$$\begin{aligned} (x \odot y) \odot a &= (x \odot y)\,a && \text{(by (8))} \\ &= x\,\alpha_x(y)\,a && \text{(by the definition of } \odot) \\ &= x\,\alpha_x(y)\,\alpha_x(a) && \text{(by (7))} \\ &= x\,\alpha_x(ya) && \text{(because } \alpha_x \text{ is a homomorphism)} \\ &= x \odot (ya) && \text{(by the definition of } \odot) \\ &= x \odot (y \odot a) && \text{(by (8)).} \end{aligned}$$

Since K is finitely dimensional vector space over F, the Proposition 8 applies. It implies that **VW5** holds and hence K_φ is a left quasi-field. This completes the proof. ∎

The left quasi-fields K_φ are called *left André quasi-fields*. The *right André quasi-fields* are constructed in a similar manner, with the multiplication given by the formula $x \odot y = \alpha_y(x)y$. As the next two theorems show, in a left André quasi-field the multiplication is almost never associative (Theorem 19), and the right distributivity holds only if φ is the *trivial map*, i.e. the map taking every element to $1 \in G$ (Theorem 20). Of course, the corresponding results hold for the right André quasi-fields. We will call an André quasi-field *non-trivial* if φ is a non-trivial map.

19. Theorem. *The multiplication in K_φ is associative if and and only if φ is a homomorphism $N(K^*) \to G$, i.e. if and only if $\varphi(uv) = \varphi(u)\,\varphi(v)$ for all $u, v \in N(K^*)$.*

Proof. Let $x, y \in K^*$ and $g \in G$. Then

$$\begin{aligned}
\alpha(x\,g(y)) &= \varphi(N(x\,g(y)) && \text{(by the definition of } \alpha\text{)}\\
&= \varphi(N(x)\,N(g(y))) && \text{(because N is a homomorphism)}\\
&= \varphi(N(x)\,N(y)) && \text{(because } N(y) = N(g(y))\text{)}\\
&= \varphi(N(xy)) = \alpha(xy).
\end{aligned}$$

Therefore, $\alpha(x\,g(y)) = \alpha(xy)$, or, equivalently, $\alpha_{x\,g(y)} = \alpha_{xy}$ for all $x, y \in K$ and $g \in G$. It follows that

(9) $\quad \alpha_{x\,g(y)}(z) = \alpha_{xy}(z)$

for all $x, y \in K^*$, $z \in K$, and $g \in G$.

If $g = \alpha_x$, then $x\,g(y) = x\,\alpha_x(y) = x \odot y$ and hence (9) turns into

(10) $\quad \alpha_{x \odot y}(z) = \alpha_{xy}(z).$

By applying (10) we can compute $(x \odot y) \odot z$ as follows:

$$(x \odot y) \odot z = (x \odot y)\,\alpha_{x \odot y}(z) = (x \odot y)\,\alpha_{xy}(z) = x\,\alpha_x(y)\,\alpha_{xy}(z).$$

Next, let us compute $x \odot (y \odot z)$:

$$x \odot (y \odot z) = x \odot (y\,\alpha_y(z)) = x\,\alpha_x(y\,\alpha_y(z)) = x\,\alpha_x(y)\,\alpha_x(\alpha_y(z)).$$

By comparing the results of these computations, we see that for $x, y \in K^*$ the associativity law $(x \odot y) \odot z = x \odot (y \odot z)$ holds if and only if $\alpha_{xy}(z) = \alpha_x(\alpha_y(z))$. Since the associativity law trivially holds when $x = 0$ or $y = 0$, the associativity law for $\odot$ holds if and only if $\alpha_{xy}(z) = \alpha_x(\alpha_y(z))$ for all $x, y, z \in K$, or, equivalently, if and only if $\alpha_{xy} = \alpha_x \circ \alpha_y$ for all $x, y \in K$.

Recalling the definition of α, we see that $\alpha_{xy} = \alpha_x \circ \alpha_y$ is equivalent to

$$\varphi(N(x))\,\varphi(N(y)) = \varphi(N(xy)),$$

and since $N(xy) = N(x)N(y)$, is equivalent to

(11) $\varphi(N(x))\,\varphi(N(y)) = \varphi(N(x)N(y))$.

Clearly, (11) holds for all $x, y \in K$ if and only if $\varphi\colon N(K^*) \to G^*$ is homomorphism. It follows that the associativity is equivalent to φ being a homomorphism. ■

20. Theorem. *The right distributivity law holds for K_φ if and only if φ maps all elements of $N(K^*)$ to $1 \in G$ (and therefore $K_\varphi = K$).*

Proof. Clearly, if φ maps $N(K^*)$ to 1, then $K_\varphi = K$ and the right distributivity holds.

Suppose now that the right distributivity holds. Then for all $x, y, a \in K$ we have

$$x \odot a + y \odot a = (x+y) \odot a.$$

We can rewrite this as

(12) $xX(a) + yY(a) = (x+y)\,Z(a)$,

where $X = \alpha_x$, $Y = \alpha_y$, $Z = \alpha_{x+y}$. By using the fact that X, Y, Z are automorphisms of K and applying (12) to ab in the role of a, we get

(13) $xX(a)X(b) + yY(a)Y(b) = xX(ab) + yY(ab)$

$$= (x+y)\,Z(ab) = (x+y)\,Z(a)\,Z(b).$$

By combining (13) with (12), we get

$$xX(a)X(b) + yY(a)Y(b) = (xX(a) + yY(a))\,Z(b).$$

Let us multiply this identity by $x+y$, and then apply (12) to b in the role of a:

$$(x+y)(xX(a)X(b) + yY(a)Y(b))$$

$$= (x+y)(xX(a) + yY(a))\,Z(b)$$

$$= (x+y)\,Z(b)\,(xX(a) + yY(a)) = (xX(b) + yY(b))(xX(a) + yY(a)).$$

By opening the parentheses and canceling the equal terms, we get

$$yxX(a)X(b) + xyY(a)Y(b) \;=\; xyX(b)Y(a) + yxY(b)X(a).$$

Suppose that $x, y \neq 0$. Then we can divide the last equation by $xy \neq 0$ and get

$$Y(a)Y(b) + X(a)X(b) \;=\; X(b)Y(a) + Y(b)X(a).$$

The last identity is equivalent to

$$Y(a)Y(b) - Y(a)X(b) + X(a)X(b) - X(a)Y(b) \;=\; 0,$$

and, therefore, to

$$(14) \quad (Y(a) - X(a))(Y(b) - X(b)) \;=\; 0.$$

This holds for all $a, b \in K$. If $Y(a) - X(a) \neq 0$ for some a, then (14) implies that $Y(b) - X(b) = 0$ for all b, and, in particular, for $b = a$ in contradiction with $Y(a) - X(a) \neq 0$. It follows that $X(a) = Y(a)$ for all a. In other terms, $X = Y$. By recalling that $X = \alpha_x$, $Y = \alpha_y$, and that x, y are arbitrary non-zero elements of K, we conclude that all automorphisms α_x with $x \neq 0$ are equal, and, in particular, are equal to α_1. But the $\alpha_1 = \varphi(N(1)) = \varphi(1) = 1$ by the assumption. It follows that $\varphi(N(x)) = \alpha_x = 1$ for all $x \in K^*$, and hence $\varphi(z) = 1$ for all $z \in N(K^*)$. This completes the proof. ■

The Galois theory provides many explicit examples of field K with a finite group of automorphisms G. The freedom of choice of the map φ allows to construct left André quasi-field with non-associative multiplication (by using Theorem 19), and left André quasi-field in which the right distributivity does not hold (by using Theorem 20). One can also construct a left André quasi-field with associative multiplication in which the right distributivity does not hold. We leave this as an exercise for the readers moderately familiar with Galois theory.

Remark. In this section we to a big extent followed the exposition in [HP], Section IX.3.

9. Conclusion: non-Desarguesian planes

If K is a left quasi-field with non-associative multiplication (say, a left André quasi-field), then K is not isomorphic to any skew-field. By Theorem 12, K^2 is not isomorphic to any plane defined over a skew-field, and, therefore, is a non-Desarguesian plane.

If K is a left quasi-field in which the right distributivity does not hold, then, again, K is not isomorphic to any skew-field. By Theorem 12, K^2 is a non-Desarguesian plane.

Let K be a right quasi-field in which the left distributivity does not hold. For example, one can take as K any nontrivial right André quasi-field (we can take as K a left André quasi-field with the opposite multiplication, or use the right André quasi-field version of Theorem 20). Then, by Theorem 17, K^2 is not isomorphic to any plane defined over a left quasi-field. In particular, K^2 is not isomorphic to any plane defined over a skew-field, and, therefore, is a non-Desarguesian plane.

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