Isometries of Teichmüller spaces from the point of view of Mostow rigidity

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To the memory of V. A. Rokhlin

Abstract. The paper is devoted to a new proof of a famous theorem of Royden and Earle–Kra to the effect that (in the most cases) the group of isometries of a Teichmüller space is equal to the corresponding Teichmüller modular group (also known as the mapping class group). This new proof has a global and geometric nature as opposed to the local and analytic proofs of Royden and Earle–Kra. It follows the same general outline as Mostow's proof of the rigidity theorem for symmetric spaces of rank at least 2. The analogy between the theorem of Royden and Earle–Kra and the Mostow rigidity theorem is quite unexpected and was not anticipated before.

Introduction

Let S be a closed orientable surface of genus g with b distinguished points, considered as punctures. We are interested in conformal structures on S minus punctures which can be extended across the punctures, i.e. in the conformal structures of finite type on S minus punctures. Let T_S be the Teichmüller space of such conformal structures. Two conformal structures on S define the same point of T_S if one can be transformed to the other by a homeomorphism $S \to S$ fixing the distinguished points and isotopic to the identity in the class of such homeomorphisms. The Teichmüller space T_S

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is equipped with a natural metric, namely with its Teichmüller metric. Let Mod_S be the modular group (also known as the mapping class group) of S, i.e. the group of isotopy classes of diffeomorphisms $S \to S$; both diffeomorphisms and isotopies are assumed to preserve the set of punctures. Obviously, Mod_S acts on T_S. Moreover, Mod_S acts by isometries of the Teichmüller metric.

Theorem A. *If S is not a sphere with ≤ 4 punctures, nor a torus with ≤ 2 punctures, then all isometries of T_S with its Teichmüller metric belong to* Mod_S.

This famous theorem is due to Royden [Ro] in the case of closed surfaces S and isometries preserving the natural complex structure on T_S, to Royden and to Earle and Kra [EK1] in the case of surfaces with punctures and isometries preserving the natural complex structure on T_S, and to Earle and Kra [EK2] in the general case. If S is a sphere with ≤ 3 punctures, then T_S is a point and there are no nontrivial isometries. If S is a sphere with 4 punctures or a torus with ≤ 1 punctures, then, as is well known, T_S is isometric to the hyperbolic plane and the conclusion of the theorem is obviously not true. Earle and Kra [EK2] observed that the conclusion of the theorem is not true also for a torus with 2 punctures. The reason is more subtle: if $S_{1,2}$ is a torus with 2 punctures and $S_{0,5}$ is a sphere with 5 punctures, then $T_{S_{1,2}}$ is isometric to $T_{S_{0,5}}$, but $\mathrm{Mod}_{S_{1,2}}$ is not isomorphic to $\mathrm{Mod}_{S_{0,5}}$. The proofs of Royden [Ro] and Earle and Kra [EK1], [EK2] have a local and analytic nature. The Teichmüller metric is fairly nonsmooth (first of all, it does not comes from a riemannian metric), and their proofs are based on a detailed investigation of its nonsmoothness.

The goal of this paper is to present a completely new proof of Theorem A. This new proof has a global and geometric nature. It follows the same general outline as Mostow's proof [Mo] of the rigidity theorem for symmetric spaces of rank at least 2. The analogy between Theorem A and the Mostow rigidity theorem is quite unexpected and was not anticipated before. Some remarks of Kra (see [Kr], p.268, footnote 2) suggest that this new proof may be in some sense the right one.

A key role in the proof of the Mostow rigidity theorem is played by a theorem of Tits about automorphisms of buildings. An analogue of the notion of building was introduced in the theory of Teichmüller spaces by Harvey

[H1], [H2]. It is the notion of the complex of curves of a surface S (with punctures). The *complex of curves* $C(S)$ of S is a simplicial complex in the sense given to this word, for example, in [S], Chapter 3. Thus it consists of a set of *vertices* and a set of *simplexes,* which are nonempty sets of vertices. The vertices of $C(S)$ are the isotopy classes of simple closed curves (also called *circles*) on S minus punctures (curves are not allowed to pass through the punctures during isotopies), which are *nontrivial,* i.e. are not contractible in S minus punctures and are not isotopic to a small circle going once round a puncture. A set of vertices is declared to be a simplex if and only if these vertices can be represented by (pairwise) disjoint circles.

Every diffeomorphism $S \to S$ preserving the set of punctures takes nontrivial circles to nontrivial circles and obviously preserves the disjointness of circles. Thus it defines an automorphism $C(S) \to C(S)$. Clearly, this automorphism depends only on the isotopy class of our diffeomorphism $S \to S$. Hence, we have a natural action of Mod_S on $C(S)$. Now we can state an analogue of the theorem of Tits about automorphisms of buildings.

Theorem B. *If S is not a sphere with ≤ 4 punctures, nor a torus with ≤ 2 punctures, then all automorphisms of $C(S)$ are given by the elements of Mod_S.*

This theorem is due to the author [I3] in the case of surfaces of genus at least 2 (see also [I1] for a preliminary version) and to Korkmaz [Ko] in the remaining cases. A different, but largely based on similar ideas, proof was suggested by Luo [Lu]. If S is a sphere with ≤ 3 punctures, then $C(S)$ is empty and there are no nontrivial automorphisms. If S is a sphere with 4 punctures or a torus with ≤ 1 punctures, then $C(S)$ is an infinite set of vertices without any simplexes of dimension ≥ 1. Obviously, the conclusion of the theorem is not true in this case. Luo [Lu] observed that the conclusion of the theorem is also not true for a torus with 2 punctures, by the following reason: if $S_{1,2}$ is a torus with 2 punctures and $S_{0,5}$ is a sphere with 5 punctures, then $C(S_{1,2})$ is isomorphic to $C(S_{0,5})$, but, as we noticed above, $\mathrm{Mod}_{S_{1,2}}$ is not isomorphic to $\mathrm{Mod}_{S_{0,5}}$. Thus, the exceptional cases of Theorems A and B naturally correspond one to the other.

Theorem B is a purely topological theorem, and the ideas involved in its proof are completely different from the ideas involved in the proof of Theorem

A assuming Theorem B. This situation is similar to the one encountered in the proof of Mostow rigidity. The present paper is devoted to the proof of Theorem A assuming Theorem B. This proof is based on the theory of Teichmüller spaces and the theory of measured foliations. A fairly detailed sketch of the proof of Theorem B can be found in [I3].

The reader is expected to be familiar with the theory of measured foliations as presented in [FLP], and with the part of the theory of Teichmüller spaces outlined by Kerckhoff [K], §§ 1,3. Further results of Kerckhoff [K] play a crucial role in our proofs, as do also the ideas of Masur [M1]. See Sections 2 and 3 respectively. The influence of Mostow's ideas on this paper should be obvious to everyone at least superficially familiar with the rigidity theory.

The rest of the paper is organized as follows. In Section 1 we recall some material from the theory of the Teichmüller spaces and, in particular, state some results of Kerckhoff [K]. Also, in Section 1 we introduce the notions of divergent and parallel rays in Teichmüller spaces, which play a central role in our approach to Theorem A. In Section 2 we prove a sufficient condition for two rays to be divergent, and in Section 3 we prove a sufficient condition for two rays to be parallel; see Corollaries 2.2 and 3.3 respectively. We deduce them from Theorems 2.1 and 3.2 respectively. These theorems seem to be of some independent interest and are proved in their natural generality (and not only in a form needed to deduce Corollaries 2.2 and 3.3). Section 4 is devoted to a characterization of measured foliations corresponding to circles; this result (Theorem 4.1) is crucial to establish a connection between the Teichmüller metric on T_S and the complex of curves $C(S)$. Finally, in Section 5 we prove Theorem A (assuming Theorem B). The geometric arguments of this section are inspired by Mostow's rigidity theory.

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1. Preliminaries

1.1. Geodesic rays and measured foliations. First of all, note that in the context of punctured surfaces we should include in the consideration foliations with one-prong singularities at punctures and we should always treat all punctures as singularities (sometimes they are two-prong singularities).

Consider some point $m \in T_S$ and represent it by a conformal structure on S. The punctured surface S together with this conformal structure forms a Riemann surface M. Our main tool for understanding the geometry of the Teichmüller space will be the set of all geodesic rays with respect to the Teichmüller metric starting at m (for various points m). We will denote the set of rays starting at m by R_m. There is a natural one-to-one correspondence between this set of Teichmüller rays and the set of straight rays starting at 0 in the space Q_M of quadratic differentials on M (which may have first order poles at punctures).

Let us describe this correspondence. Let $q \in Q_M$. Away from its zeros (there are only finite number of them) the quadratic differential q locally has the form dz^2 for some local parameter z on M; $z = x + iy$. Therefore, the quadratic differential q defines in a neighborhood of any point of M which is not a zero of q local coordinates (x, y). Let $K \geq 1$. Replace the coordinate functions x, y by the new coordinate functions $K^{-1/2}x$, $K^{1/2}y$ (i.e., the point with the old coordinates (x, y) has $(K^{-1/2}x, K^{1/2}y)$ as its new coordinates). For a given K the new coordinate maps obtained in this way from q are, as is well known, conformally compatible and can serve as the charts of a new conformal structure on S minus the set of zeros of q. In fact, such a conformal structure uniquely extends across the zeros of q and, therefore, defines a new conformal structure on S. Alternatively, one may use x, Ky as the new coordinate functions (clearly, the chart (x, Ky) is conformally related to $(K^{-1/2}x, K^{1/2}y)$).

We denote this new conformal structure by $M_{q,K}$. Clearly, $M_{q,1} = M$ and $M_{aq,K} = M_{q,K}$ for any $a > 0$. In particular, $M_{q,K}$ depends only on the *ray* $\{\lambda q : \lambda \geq 0\}$ in Q_M defined by q. We denote the point of T_S corresponding to the conformal structure $M_{q,K}$ by $m_{q,K}$. The points $m_{q,K}, K \geq 1$ fill a geodesic ray in T_S; this geodesic ray corresponds to the ray in Q_M defined by q.

We denote this geodesic ray by r_q and consider it as a map $r_q : \mathbf{R}_{\geq 1} \to T_S$ defined by $r_q(K) = m_{q,K}$. If we will need to stress the dependence on the point m, we will denote r_q by $r_{m,q}$. Note that

$$d\,(m, m_{q,K}) = \frac{1}{2} \log K,$$

where $d\,(\cdot\,,\cdot)$ is the Teichmüller distance. In particular, our parametrization of rays is not the parametrization by length, but is obviously related to the latter.

A quadratic differential q gives rise to two measured foliations on S, the horizontal foliation $H(q)$ and the vertical foliation $V(q)$. If (x, y) are the local coordinates discussed above, then the leaves of $H(q)$ are given by the equation $y = \text{const}$ and the leaves of $V(q)$ are given by the equation $x = \text{const}$. The functions y and x respectively define the transverse measures. Note that $H(aq) = aH(q)$ and $V(aq) = aV(q)$ for any $a > 0$ (if $a < 0$, then $H(aq) = |a|V(q)$ and $V(aq) = |a|H(q)$).

According to a fundamental theorem of Hubbard and Masur [HuM] (see [K] for another proof; see also [G], Chapter 11) the map $h : Q_M \to MF_S$, where MF_S is the Thurston's space of measured foliations on the punctured surface S, assigning to q the equivalence class of the foliation $h(q)$, is a homeomorphism. We will denote by $q(\mu)$ the image of the foliation $\mu \in MF_S$ under the inverse homeomorphism h^{-1}. Clearly, $q(a\mu) = aq(\mu)$ for any $a \geq 0$. We will also denote the ray $r_{q(\mu)}$ simply by r_μ, or by $r_{m,\mu}$, if we need to stress the point m. Also, we will denote $M_{q(\mu),K}$ by $M_{\mu,K}$. We will always use Greek letters for points of MF_S and Latin letters for quadratic differentials, so this will cause no ambiguity.

Since the conformal structures $M_{q,K}$ depend only on the ray in Q_M defined by q, the ray $r_{m,q}$ also depends only on the ray defined by q. Since $h(aq) = ah(q)$ for every $a \geq 0$, the ray $r_{m,\mu} = r_{m,q(\mu)} = r_{m,h^{-1}(\mu)}$ depends only on the ray $\{\lambda\mu : \lambda \geq 0\}$ in MF_S defined by μ. In other words, the set R_m of Teichmüller rays at an arbitrary point $m \in T_S$ is in a natural 1-1 correspondence with the set PF_S of projective classes of measured foliations on S.

We will say that two geodesic rays r, r' starting at possibly different points m, m' are *divergent*, if

$$d\,(r(K), r'(K)) \longrightarrow \infty \;\text{ when }\; K \longrightarrow \infty,$$

where $d\,(\cdot\,,\cdot)$ is the Teichmüller distance. We will say that two geodesic rays

r, r' starting at m, m' are *parallel*, if the function

$$K \mapsto d\,(r(K), r'(K))$$

is bounded. These two relations of divergence and parallelness of rays will be the key ingredients of our investigation of the geometry of the Teichmüller space.

1.2. Extremal length and Teichmüller distance. In this section we remind some results of Kerckhoff [K] about extremal length and Teichmüller distance, which we will need in order to provide a sufficient condition for two rays r_μ, r_ν to be divergent. See Corollary 2.2.

Classically, extremal length was defined only for simple closed curves on a Riemann surface. Kerckhoff extended the definition of extremal length from simple closed curves to general measured foliations (as usual, we indentify a nontrivial simple closed curve with the corresponding foliation). So, for any Riemann surface M having S as the underlying topological surface, the extremal length is a continuous function $E_M : MF_S \setminus \{0\} \to \mathbf{R}$ (in fact, the extension to the whole MF_S by $E_M(0) = 0$ is also continuous). See [K], § 3. This function is homogenous of degree 2, i.e., $E_M(a\mu) = a^2 E_M(\mu)$ for any $a > 0$. Since the foliations of the form $a\gamma$, where γ is a simple closed curve and $a > 0$, are dense in MF_S, the function E_M is, in fact, uniquelly determined by its values on the set of simple closed curves.

According to a remark after Proposition 3 in [K], § 3, $E_M(\mu)$ is equal to the area of S in the flat metric defined by the quadratic differential $q(\mu)$ (if $q = dz^2$ in the local coordinate $z = x + iy$, then the corresponding metric is given by $dx^2 + dy^2$). Since this fact is of crucial importance for us, we give a proof. By continuity, it is sufficient to prove this only for points $a\gamma \in MF_S$, where $a > 0$ and γ is a simple closed curve. Any foliation representing $a\gamma$ has the property that the complement of critical leaves is a cylinder C foliated by circles isotopic to γ. If we represent $a\gamma$ the foliation $H(q)$ associated with $q = q(a\gamma)$, this cylinder acquires a conformal structure from M and ever a flat metric from q refining this conformal structure. Let l be the circumference and h be the height of this cylinder with respect to the flat metric coming from q. Then $E_M(a\gamma) = a^2 E_M(\gamma)$ and, by the classical geometric definition of extremal length of simple closed curves and a theorem of Jenkins [J] and Strebel [St] $E_M(\gamma) = 1/\text{mod}\,(C)$ (see also [K], Theorem 3.1), where mod $(C) = h/l$ is the modulus of the cylinder C. On the other hand, the area of S in the metric defined by q is equal hl. But, since $h(q) = a\gamma$, the height h is equal to a. Hence, $E_M(a\gamma) = a^2/\text{mod}\,(C) = a^2/(h/l) = h^2/(h/l) = hl$ and the area is also equal to hl. This completes the proof of Kerckhoff's remark.

We will need also the following estimate of the extremal length of a simple closed curve γ. Let q be a quadratic differential on M and let ρ be the corresponding flat metric. Let $L_\rho(\gamma)$ be the infinium of lengths of simple closed curves isotopic to γ measured with respect to ρ. Let $A(\rho)$ be the area of S with respect to ρ. Then

$$E_M(\gamma) \geq l_\rho^2(\gamma)/A(\rho).$$

In fact, if we allow ρ to range over all (possibly singular) metrics agreeing with the conformal structure M, then $E_M(\gamma)$ will be the supremum of the right hand of this inequality, by the classical analytic definition of the extremal length, whence the inequality. By the already cited theorem of Jenkins [J] and Strebel [St] (cf. also [K], Theorem 3.1), this supremum is in fact realized by the flat metric of an appropriate quadratic differential.

According to [K], Theorem 4, the Teichmüller distance can be expressed in terms of extremal lengths as follows. Let m', m'' be two points of T_S represented by Riemann surfaces M', M'' (with the same underlying topological surface S). Then

$$d\,(m', m'') = \frac{1}{2}\log(\sup_\gamma(E_{M'}(\gamma)/E_{M''}(\gamma))),$$

where γ ranges of all nontrivial simple closed curves on S. Because E_M is a continuous function on $MF_S \setminus \{0\}$, this implies that

$$d\,(m', m'') = \frac{1}{2}\log(\sup_\mu(E_{M'}(\mu)/E_{M''}(\mu))),$$

where μ ranges over all $MF_S \setminus 0$.

2. Divergent rays

The goal of this section is to provide a sufficient condition for two geodesic rays starting at the same point to be divergent. See Corollary 2.2. The basic tool will be the inequality of the following Theorem.

2.1. Theorem. *Let $\mu, \nu \in MF_S$, $m \in T_S$ and let $K_1, K_2 \geq 1$. If* $\mathrm{i}\,(\mu,\nu) \neq 0$*, then*

$$d\,(m_{\mu,K_1}, m_{\nu,K_2}) \geq d_1 + d_2 + \frac{1}{2}\log\frac{\mathrm{i}\,(\mu,\nu)^2}{E_M(\mu)E_M(\nu)},$$

where $d_1 = d\,(m, m_{\mu,K_1}) = \frac{1}{2}\log K_1$*, and* $d_2 = d\,(m, M_{\nu,K_2}) = \frac{1}{2}\log K_2$.

Proof. We will estimate $d\,(m_{\mu,K_1}, m_{\nu,K_2})$ using Kerckhoff's formula for the Teichmüller distance. This formula implies that

$$d\,(m_{\mu,K_1}, m_{\nu,K_2}) \geq \frac{1}{2}\log(E_{M_{\nu,K_1}}(\mu)/E_{M_{\mu,K_2}}(\mu))$$

and this inequality will eventually lead to the estimate of the theorem.

Let $K = K_1$. Consider the extremal length $E_{M_{\mu,K}}(\mu)$. We will denote it simply by E_K. As we explained above, we will be able to compute E_K if we will find a quadratic differential q_K on $M_{\mu,K}$ such that $h(q_K) = \mu$ (the extremal length E_K is equal to the area of S in the metric defined by q_K). If $z = x + iy$ is a local parameter on M such that $q(\mu) = dz^2$, then $K^{-1/2}x + iK^{1/2}y$ is a local parameter on $M_{\mu,K} = M_{q(\mu),K}$. We can define a quadratic differential q'_K on $M_{\mu,K}$ as the one having expression $d(K^{-1/2}x + iK^{1/2}y)^2$ in terms of such local parameters on $M_{\mu,K}$. Clearly, $h(q'_K) = K^{1/2}\mu$ and hence $h(K^{-1/2}q'_K) = \mu$. This means that we can take $q_K = K^{-1/2}q'_K$. Since the map $(x,y) \mapsto (K^{-1/2}x, K^{1/2}y)$ is area-preserving (with respect to the usual area on $\mathbf{R}^2$), the area of the metric the corresponding to q'_K is the same as the area of the metric corresponding to $q(\mu)$. The last area is, of course, equal to $E_M(\mu)$. This implies that the area of the metric defined by q_K is equal to $E_M(\mu)/K$, i.e. $E_K = E_M(\mu)/K = E_M(\mu)/K_1 = E_M(\mu)/e^{2d_1}$.

Now, let $K = K_2$. Consider the extremal length $E_{M_{\nu,K}}(\mu)$. We will denote it by E^K. Let q''_K be the quadratic differential on $M_{\nu,K}$ constructed from $q(\nu)$ and K in exactly the same way as we constructed q'_K from $q(\mu)$ and K in the previous paragraph. Let ρ_K be the flat metric defined by q''_K. Let us choose a sequence $\{\gamma_i\}$ of simple closed curves and a sequence $\{c_i\}$ of positive real numbers such that $c_i\gamma_i \to \mu$ in MF_S when $i \to \infty$. Then we have

$$E_{M_{\nu,K}}(\mu) = \lim_{i\to\infty} E_{M_{\nu,K}}(c_i\gamma_j).$$

Note that if ρ is the flat metric defined by an arbitrary quadratic differential q and γ is a simple closed curve, then, clearly, $l_\rho(\gamma) \geq \mathrm{i}\,(h(q), \gamma)$. This implies

$$E_{M_{\nu,K}}(c_i\gamma_i) = c_i^2 E_{M_{\nu,K}}(\gamma_i) \geq \frac{c_i^2 l_{\rho_K}^2(\gamma_i)}{A(\rho_K)} \geq$$

$$\geq \frac{c_i^2\,\mathrm{i}\,(h(q_K''), \gamma_i)^2}{A(\rho_K)} = \frac{\mathrm{i}\,(h(q_K''), c_i\gamma_i)^2}{A(\rho_K)} =$$

$$= \frac{\mathrm{i}\,(K^{1/2}\nu, c_i\gamma_i)^2}{A(\rho_K)} = \frac{K\mathrm{i}\,(\nu, c_i\gamma_i)^2}{A(\rho_K)}.$$

Passing to the limit, we get

$$E_{M_{\nu,K}}(\mu) \geq \frac{K\mathrm{i}\,(\nu, \mu)^2}{A(\rho_K)} = \frac{K\mathrm{i}\,(\mu, \nu)^2}{A(\rho_K)}.$$

Exactly as in our discussion of $E_{M_{\mu,K}}(\mu)$, the area $A(\rho_K)$ of the flat metric defined by the differential q_K'' is, in fact, independent of K and is equal to $E_M(\nu)$. So, finally, we get $E^K \geq K\mathrm{i}\,(\mu, \nu)^2/E_M(\nu)$, i.e. $E_{M_{\nu,K_2}} \geq K_2\mathrm{i}\,(\mu, \nu)^2/E_M(\nu) = e^{2d_2}\mathrm{i}\,(\mu, \nu)^2/E_M(\nu)$. Collecting our results together, we see that

$$d(m_{\mu,K_1}, m_{\gamma,K_2}) \geq \frac{1}{2}\log\frac{E_{M_{\nu,K_2}}(\mu)}{E_{M_{\mu,K_1}}(\mu)} \geq$$

$$\geq \frac{1}{2}\log\frac{e^{2d_2}\,\mathrm{i}\,(\mu, \nu)^2/E_M(\nu)}{E_M(\mu)/e^{2d_1}} =$$

$$= \frac{1}{2}\log\frac{e^{2d_1+2d_2}\,\mathrm{i}\,(\mu, \nu)^2}{E_M(\nu)E_M(\mu)} =$$

$$= d_1 + d_2 + \frac{1}{2}\log\frac{\mathrm{i}\,(\mu, \nu)^2}{E_M(\nu)E_M(\mu)}.$$

This completes the proof. □

2.2. Corollary. *If* $\mathrm{i}\,(\mu, \nu) \neq 0$, *then the rays* $r_{m,\mu}, r_{m,\nu}$ *are divergent.*

Proof. Take $K_1 = K_2 = K \to \infty$ in the theorem. □

2.3. Corollary. *For any two foliations $\nu, \mu \in MF_S$ we have*

$$\mathrm{i}\,(\nu, \mu)^2 \leq E_M(\nu) E_M(\mu).$$

Proof. Let $K \geq 1$ and let $d = \frac{1}{2} \log K$. Since $d\,(m_{\mu,K}, m) = d\,(m_{\nu,K}, m) = d$, we have $d(m_{\mu,K}, m_{\nu,K}) \leq 2d$. Together with the theorem this implies that

$$\frac{1}{2} \log \frac{\mathrm{i}\,(\mu, \nu)^2}{E_M(\mu) E_M(\nu)} \leq 0,$$

i.e.

$$\frac{\mathrm{i}\,(\mu, \nu)^2}{E_M(\mu) E_M(\nu)} \leq 1.$$

This completes the proof. □

2.4. Remark. This corollary is know; see, for example, [Mi]. We will not use it. It is similar to an estimate of the intersection number in terms of hyperbolic lengths proved in [FLP]; cf. [FLP] Expose 4, § I, Lemma 2, but more precise in the sense that an unknown constant is involved in the estimate from [FLP].

3. Parallel rays

The goal of this section is to prove a sufficient condition for two geodesic rays to be parallel. See Theorem 3.2. A special case (see Corollary 3.3) will play a crucial role in our proof of Theorem A.

3.1. Preliminaires. Recall that two measured foliations $\mathcal{F}$, $\mathcal{F}'$ are called *topologically equivalent* if they are Whitehead equivalent to measured foliations $\mathcal{F}_1$, $\mathcal{F}_1'$ respectively such that the underlying singular foliations of $\mathcal{F}_1$, $\mathcal{F}_1'$ (i.e., $\mathcal{F}_1$, $\mathcal{F}_1'$ considered *without* their transverse measures) can be transformed one to the other by a homeomorphism (not necessarily smooth) isotopic to the identity (in the class of homeomorphisms fixing the punctures). Two points $\mu, \mu' \in MF_S$ are called *topologically equivalent* if some (and then any) measured foliations $\mathcal{F}$, $\mathcal{F}'$ representing μ, μ' respectively are topologically equivalent.

Let us describe, in a slightly different form, a construction from [FLP], Exp. 9, § I. Given a measured foliation $\mathcal{F}$ on S, let us blow up all singularities of $\mathcal{F}$ in such a way that the singularities of the resulting measured foliation $\mathcal{F}_0$ on S minus several open discs (one for each singularity of $\mathcal{F}$) are all three-prong singularities (and all are boundary singularities). Next, let us split $\mathcal{F}_0$ along all leaves connecting two singularities. The result is a measured foliation $U\mathcal{F}$ on a subsurface of S such that all its singularities are three-prong boundary singularities. Note that this subsurface does not contain punctures, because all of them are considered as singularities and are blown up in the first step.

It is easy to see (cf. [FLP], Exp. 9, § I) that two measured foliations $\mathcal{F}$, $\mathcal{F}'$ are Whitehead equivalent if and only if $U\mathcal{F}$ can be transformed into $U\mathcal{F}'$ (respecting their transverse measures) by a diffeomorphism of S isotopic to the identity (in the class of diffeomorphisms fixing the punctures). Similarly, two measured foliations $\mathcal{F}$, $\mathcal{F}'$ are topologically equivalent if and only if the underlying singular foliations of $U\mathcal{F}$, $U\mathcal{F}'$ (i.e., $U\mathcal{F}$, $U\mathcal{F}'$ considered without their transverse measures) can be transformed one into the other by a homeomorphism isotopic to the identity.

Now, suppose that $\mathcal{F}$, $\mathcal{F}'$ are topologically equivalent and let $f : S \to S$ be a homeomorphism transforming the underlying foliation of $U\mathcal{F}$ into that of $U\mathcal{F}'$. Let λ, λ' be the transverse measures of $U\mathcal{F}$, $U\mathcal{F}'$ respectively. We will say that $\mathcal{F}$, $\mathcal{F}'$ are *comparable* if there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{\lambda'(f(I))}{\lambda(I)} \leq C$$

for every segment I transverse to $U\mathcal{F}$ (clearly, this condition does not depends on the choice of f).

We will say that two points $\mu, \mu' \in MF_S$ are *comparable* if some (and then any) measured foliations $\mathcal{F}$, $\mathcal{F}'$ representing μ, μ' are topologically equivalent and comparable.

We will need also the following variant of the construction of the "unglued" foliation $U\mathcal{F}$ associated to a measured foliation $\mathcal{F}$. Let $\mathcal{F}$ be a measured foliation. Let $C_0 = C_0(\mathcal{F})$ be the union of all singularities of $\mathcal{F}$ and all leaves of $\mathcal{F}$ connecting two singularities. For every leaf of $\mathcal{F}$ ending at some singularity let us add to C_0 a half leaf of $\mathcal{F}$ contained in this leaf and ending at the same singularity. Let us denote C_0 with these added half-leaves by $C = C(\mathcal{F})$. Now let us split $\mathcal{F}$ along C. The result of this splitting is a measured foliation $S\mathcal{F}$ on a subsurface of S having all its singularities on the boundary; in fact, all its singularities are three-prong singularitiees and correspond to the starting points of the added half-leaves. As in the case of $U\mathcal{F}$, this subsurface does not containd punctures, because they are considered as singularities.

As for the construction $\mathcal{F} \mapsto U\mathcal{F}$, it is easy to see that two measured foliations $\mathcal{F}$, $\mathcal{F}'$ are Whitehead equivalent if and only if $S\mathcal{F}$ can be transformed into $S\mathcal{F}'$ (respecting their transverse measures) by a diffeomorphism isotopic to the identity (in the class of diffeomorphisms fixing the punctures).

3.2. Theorem. *Let $m, m' \in T_S$. Suppose that two points $\mu, \mu' \in MF_S$ are topologically equivalent and comparable. Then the rays $r_{m,\mu}$, $r_{m',\mu'}$ are parallel.*

Proof. First of all, represent m by a structure M of a Riemann surface on S. Consider the quadratic differential $q = q(\mu)$ on M corresponding to μ, i.e. such that $\mu = h(q)$. Let $\mathcal{F} = H(q)$; this measured foliation represents the equivalence class μ.

Our first goal is to construct a structure M'' of Riemann surface on S and a quadratic differential q'' on M'' such that, in particular, the vertical foliations $V(q'')$, $V(q)$ are equal, but $h(q'') = \mu'$. We will denote by m'' the point of T_S corresponding to M''. After this we will prove that the rays $r_{m,\mu}$ and $r_{m'',q''}$ are parallel and that the rays $r_{m'',q''}$ and $r_{m',\mu'}$ are also parallel. Clearly, this will imply that the rays $r_{m,\mu}$ and $r_{m',\mu'}$ are parallel.

To begin with, choose a *good system of transversals* $I_1, \ldots, I_n$ in the sense [FLP], Exp. 9, § III for the measured foliation $\mathcal{F}$. This means that $I_1, \ldots, I_n$ are segments transversal to $\mathcal{F}$ and, moreover: the segments I_i may meet only at their endpoints; a segment I_j may contain a singularity only as one of its endpoints; if two segments meet at a singularity, then they are contained in two different sectors of this singularity. This system of transversals gives rise to a collection of immersed rectangles $R_1, \ldots R_N$ as in loc. cit., Lemma 4. In particular:

(i) the interiors of rectangles are disjoint;

(ii) two opposite sides of each rectangle are contained each in a single segment from our system of transversals;

(iii) two other opposite sides of each rectangle are contained each in a union of leaves and singularities of $\mathcal{F}$; moreover, every such side contains a singularity of $\mathcal{F}$ or an end of a segment I_i;

(iv) both sides of every segment I_i are covered by rectangles.

Thanks to the condition (iii), the collection of rectangles is canonically determined by the system of transversals $I_1, \ldots, I_n$ (see [FLP], loc. cit.). We may choose our system of transversals in such a way that every leaf of $\mathcal{F}$ intersects at least one of the segments I_i. Then, obviously, the union $R_1 \cup \ldots \cup R_N$ of our rectangles is equal to the whole surface S. In addition, we can choose our system of transversals in such a way that none of the segments I_i crosses in its interior a leaf joining two singularities (if the original system of transversals does not satisfy this condition, we need only to subdivide some of the I_i's). Then the operations leading from $\mathcal{F}$ to $U\mathcal{F}$ can be applied simultaneously to our transversals and rectangles, resulting in a good system of transversals $J_1, \ldots, J_n$ to $U\mathcal{F}$ and the corresponding collection of rectangles.

The construction of rectangles depends only on $\mathcal{F}$ and does not use any conformal structure on S. If we recall that S has a structure M of a Riemann surface and, moreover, that $\mathcal{F} = H(q)$ is the horizonatal foliation of a quadratic differential $q = q(\mu)$ on M, then it is natural to choose our segments I_i to be vertical with respect to q (i.e., to be contained in the leaves of $V(q)$). In this case q defines on each rectangle R_j a structure of a euclidean rectangle with sides parallel to the coordinate axes. The x-coordinate on a rectangle is given by the vertical foliation $V(q)$ and the y-coordinate is given by the horizontal foliation $\mathcal{F} = H(q)$; both coordinates are well defined up to a sign and an additive constant.

Up to now our considerations were restricted to only one of our two points $\mu, \mu' \in MF_S$; namely, to μ. Let us introduce μ' into the play. Represent μ' by a measured foliation $\mathcal{F}'$. Since μ is topologically equivalent to μ', there is a homeomorphism f of S transforming the underlying foliation of $U\mathcal{F}$ into that of $U\mathcal{F}'$. Let us equip our transversals J_i to $U\mathcal{F}$ with the measures induced by f from the transversal measure of $U\mathcal{F}'$. Collapsing of $U\mathcal{F}$ into $\mathcal{F}$ transforms these measures into measures on transvesals I_i; we will call these new measures the *induced* measures.

The induced measures can be used to define a new y-coordinate (the vertical coordinate) on every rectangle R_j. Together with the old x-coordinates (defined by $V(q)$) it defines a new structure of a euclidean rectangle with sides parallel to the coordinate axes on R_j. Since the induced measures are, in fact, induced from the transverse measure of $U\mathcal{F}'$, these structures of euclidean rectangles match together to define a euclidean structure with singularities on our surface S (the singularities are the same as the singularities of q). In other terms, we get another conformal structure on S together a quadratic differential q'': the new conformal coordinates are $z'' = x'' + iy''$, where x'' and y'' are the new horizontal and vertical coordinates (by the construction, $x'' = x$), and $q'' = (dz'')^2$. Let M'' be the surface S with this new conformal structure, and let m'' be the corresponding point of T_S.

The homeomorphism f transforming the underlying foliation of $U\mathcal{F}$ into that of $U\mathcal{F}'$ obviously transforms the measured foliation $UH(q'')$ into $U\mathcal{F}'$. It follows that $H(q'')$ and $\mathcal{F}'$ are Whitehead equivalent and, hence, $h(q'') = \mu'$.

Now we will prove that the rays $r_{m,q}$ and $r_{m'',q''}$ are parallel. Let us consider the identity map of S as a map $f_K : M_{q,K} \to M''_{q'',K}$, where $K \geq 1$. On every rectangle R_j we have the coordinates (x, y) defined by q and the coordinates (x, y'') defined by q''. The coordinates (x, Ky) and (x, Ky'') can be used as conformal coordinates on $M_{q,K}$ and $M''_{q'',K}$ respectively. Let $(X, Y) = (x, Ky)$ and $(X, Y'') = (x, Ky'')$. Note that y'' is a function of y; say $y'' = g(y)$. The fact that μ and μ' are comparable implies that

$$C^{-1} \leq \frac{g(y_1) - g(y_2)}{y_1 - y_2} \leq C$$

for some constant $C > 0$ and every $y_1 \neq y_2$. In its turn, this implies that g is absolutely continuous and the derivative g' of g satisfies $C^{-1} \leq g' \leq C$ (at points where the derivative is defined). Now, in the coordinates (X, Y), (X, Y'') the map f_K obviously has the form

$$f_K(X, Y) = (X, Kg(Y/K)).$$

Clearly, the function g_K given by the formula $g_K(Y) = Kg(Y/K)$ is absolutely continuous and satisfies $C^{-1} \leq g'_K \leq C$ (at points where the derivative is defined). Now it is clear that in the conformal coordinates (X, Y), (X, Y'') the map f_K is absolutely continuous on the lines and the conformal dilatation of its differential

$$\begin{pmatrix} 1 & 0 \\ 0 & g'_K \end{pmatrix}$$

is bounded independently of K. Since this holds for every rectangle R_j, the map $f_K : M_{q,K} \to M''_{q'',K}$ is quasiconformal with the distortion bounded independently of K. This implies that the Teichmüller distance $d\left(m_{q,K}, m''_{q'',K}\right)$

is bounded and hence the rays $r_{m,q}$, $r_{m'',q''}$ are parallel, or, what is the same, the rays $r_{m,\mu}$ and $r_{m'',q''}$ are parallel.

It remains to prove that the rays $r_{m'',q''}$ and $r_{m',\mu'}$ are parallel.

Let us represent m' by a structure M' of a Riemann surface on S. Consider the quadratic differential $q' = q(\mu)$ on M' corresponding to μ', i.e. such that $\mu' = h(q')$. While before it was sufficient to use an arbitrary representative $\mathcal{F}'$ of μ', now we will take $\mathcal{F}' = H(q')$, i.e. we will take the horizontal foliation $H(q')$ as our representative of μ'. As any representative of μ', it is Whitehead equivalent to $\mathcal{F}'' = H(q'')$ (because $h(q'') = \mu'$).

Let $C' = C(\mathcal{F}')$, $C'' = C(\mathcal{F}'')$ (see 3.1 for the notations). Let $\varepsilon > 0$. Consider all segments vertical with respect to q' (i.e., parts of the leaves of $V(q')$) having one of the endpoints in C' and the length ε. If ε is sufficiently small, as we will assume, then different segments may intersect only at a common endpoint in C'. Let $C'(\varepsilon)$ be the union of all these segments. Clearly, $C'(\varepsilon)$ is a subsurface of S (usually with corners) bounded by several vertical segments of length 2ε and several horizontal segements and, possibly, some horizontal circles. Let us construct $C''(\varepsilon)$ similarly, using C'' and q''.

The splitting of $\mathcal{F}'$ leading to $S\mathcal{F}'$ can be applied simultaneously to the measured foliation $V(q')$. In this way we will get a measured foliation V' defined on the same subsurface as $S\mathcal{F}'$ and transversal to $S\mathcal{F}'$ and to the boundary. Also, we may split $C'(\varepsilon)$; the resulting surface $SC'(\varepsilon)$ consists of several rectangles and annuli of height ε (different rectangles may have common corners). Let us construct V'' and $SC''(\varepsilon)$ in a similar manner.

Now, since $\mathcal{F}'$ is Whitehead equivalent to $\mathcal{F}''$, there is a diffeomorphism g of S mapping $S\mathcal{F}'$ into $S\mathcal{F}''$ and, in particular, respecting their transverse measures (see 3.1). Let us fix some sufficiently small, as described above, $\varepsilon > 0$. Obviously, we can choose g in such a way that $g(SC'(\varepsilon)) = SC''(\varepsilon)$ and g maps the rectangles and annuli of $SC'(\varepsilon)$ into the corresponding rectangles and annuli of $SC''(\varepsilon)$. Moreover, we can choose g to be (affine) linear on the parts of leaves of $S\mathcal{F}'$ contained in $SC'(\varepsilon)$ with respect to the coordinates on leaves of $S\mathcal{F}'$ and $S\mathcal{F}''$ given by the transverse measures of V' and V'' respectively (it is sufficient to arrange this on different rectangles and annuli separately). Such a map g uniformly stretches every rectangle and annulus from $SC'(\varepsilon)$ in the horizontal direction (i.e., in the direction of leaves of $S\mathcal{F}'$ and $S\mathcal{F}''$).

By collapsing back $S\mathcal{F}'$ to $\mathcal{F}'$ and $S\mathcal{F}''$ to $\mathcal{F}''$ we can get from g a map $G : S \setminus C' \to S \setminus C''$. In general, it cannot be extended to C'.

Given $K \geq 1$, we will construct now a quasiconformal map $f_K : M'_{q',K} \to M''_{q'',K}$. Recall that the underlying topological surface is the same for both $M'_{q',K}$ and $M''_{q'',K}$, namely S. We will define f_K differently on two parts $C'(\varepsilon/K)$ and $S \setminus \operatorname{int} C'(\varepsilon/K)$ of S. On $S \setminus \operatorname{int} C'(\varepsilon/K)$ we define f_K to be equal to G. Notice that, obviously, $G(S \setminus \operatorname{int} C'(\varepsilon/K)) = S \setminus \operatorname{int} C''(\varepsilon/K)$. In order to define f_K on $C'(\varepsilon/K)$ notice, first of all, that $C'(\varepsilon/K)$ is just a shrunken K times in the vertical direction toward C' copy of $C'(\varepsilon)$. More precisely, there is a canonical map $e'_K : C'(\varepsilon/K) \to C'(\varepsilon)$ fixed on C' and linearly mapping every vertical segment from $C'(\varepsilon/K)$ onto the corresponding segment of $C'(\varepsilon)$. Similarly, there is a canonical map $e''_K : C''(\varepsilon/K) \to C''(\varepsilon)$.

Recall now that $\mathcal{F}'$ is Whitehead equavalent to $\mathcal{F}'$. Following the Whitehead operations and isotopies transforming $\mathcal{F}'$ to $\mathcal{F}''$, we can construct a diffeomorphism $F : C'(\varepsilon) \to C''(\varepsilon)$ (even an ambient isotopy transforming $C'(\varepsilon)$ into $C''(\varepsilon)$). Obviously, we can adjust F to be linear on the vertical and horizontal sides of the boundary of $C'(\varepsilon)$. Consider the map $F_K = (e''_K)^{-1} \circ F \circ e'_K : C'(\varepsilon/K) \to C''(\varepsilon/K)$. We define f_K to be equal to F_K on $C'(\varepsilon/K)$. Since g is linear on parts of horizontal leaves (leaves of $S\mathcal{F}'$) contained in $SC'(\varepsilon)$, the map G is linear, in particular, on the horizontal part of the boundary of $C'(\varepsilon/K)$. It follows that F_K and G do agree on the horizontal part of the boundary of $C'(\varepsilon/K)$. Since g preserves the transverse measure, the map G is linear on the vertical part of the boundary of $C'(\varepsilon/K)$. It follows that F_K and G do agree on the vertical part of the boundary of $C'(\varepsilon/K)$ also. Hence, the map f_K is well defined.

Now we will show that the conformal dilatation of F_K is bounded independently of K. Let us consider F_K on $C'(\varepsilon/K)$ first. Recall that $C'(\varepsilon/K)$ is a shrunken K times in the vertical direction copy of $C'(\varepsilon)$. When we change the conformal structure of M' to that of $M'_{q',K}$ we dilate $C'(\varepsilon/K)$ in the vertical direction K times and the result is a conformal copy of $C'(\varepsilon)$. More formally, the map $e'_K : C'(\varepsilon/K) \to C'(\varepsilon)$ is a conformal equivalence if $C'(\varepsilon/K)$ is considered with the conformal structure of $M'_{q',K}$ and $C'(\varepsilon)$ with the conformal structure of M'. A similar remark applies to e''_K. It follows that $F_K = (e''_K)^{-1} \circ F \circ e'_K$ considered as map from the conformal structure of $M'_{q',K}$ to that of $M''_{q'',K}$ has the same distortion as a map $F : C'(\varepsilon) \to C''(\varepsilon)$ considered as a map from the conformal structure of M' to that of M''. Hence, the conformal dilatation of F_K (i.e., of f_K on $C'(\varepsilon/K)$) is bounded independently of K.

Consider now f_K on $S \setminus \operatorname{int} C'(\varepsilon_K)$, where it is equal to G (here the map remains unchanged, but conformal structures do change with K). Let (x', y') and (x'', y'') be some local coordinates on M' and M'' associated to q' and q'' respectively. Then the conformal structures on $M'_{q',K}$ and $M''_{q'',K}$ are given by the coordinates $(X', Y') = (x', Ky')$ and $(X'', Y'') = (x'', Ky'')$ respectively. Recall that G maps horizontal leaves to horizontal leaves and preserves the (vertical) distance between them. Hence, it has the form $G(x', y') = (G^1(x', y'), \pm y' + \text{const})$. By changing y'', if necessary, we may assume that it has the form $G(x', y') = (G^1(x', y'), y')$. In the new coordinates (X', Y') and (X'', Y'') it takes the form $G(X', Y') = (G^1(X', Y'/K), Y')$. The differential of G in the new coordinates is equal to

$$\begin{pmatrix} C^1_{x'}(X', Y'/K) & G^1_{y'}(X', Y'/K)/K \\ 0 & 1 \end{pmatrix},$$

where $G^1_{x'}$, $G^1_{y'}$ are the partial derivatives of G^1 in (x', y')-coordinates. Now, since G results from the diffeomorphism g defined on the surface supporting $S\mathcal{F}'$, and this surface is compact (while the domain of definition of G is not compact), the partial derivative $G^1_{x'}$ is uniformly bounded and bounded away from zero. Similarly, the partial derivative $G^1_{y'}$ is uniformly bounded. It follows that even as $K \to \infty$, the above diffferential remains is a compact subset of $GL_2(\mathbf{R})$, and hence its conformal dilatation is bounded independently of K.

Combining the results of the last two paragraphs, we see that the distortion of f_K is bounded independently of K. This implies that the Teichmüller distance $d\,(m'_{q',K}, m''_{q'',K})$ is bounded and, hence, the rays $r_{m',q'}$, $r_{m'',q''}$ are parallel, or, what is the same, the rays $r_{m',\mu'}$ and $r_{m'',q''}$ are parallel. Since we already proved that the rays $r_{m'',q''}$ and $r_{m,\mu}$ are parallel, this implies that the rays $r_{m,\mu}$ and r'_m, μ' are parallel, and hence completes the proof of the theorem. □

Now, let us state the special case case of Theorem 3.2 which will use later. Let γ be the isotopy class of a circle C on S, and let $\mu \neq 0$ be some point of MF_Q, where Q is the result of cutting S along C. Using the more or less self-explanatory language of [I2], Section 2.4, we may consider the points of MF_S of the form $a\mu + b\gamma$, where $a, b \geq 0$.

3.3. Corollary. *Suppose that $a_1, b_1, a_2, b_2 > 0$. Let $\mu' = a_1\mu + b_1\gamma$, $\mu'' = a_2\mu + b_2\gamma$. Let $m', m'' \in T_S$. Then the rays $r_{m',\mu'}$, $r_{m'',\mu''}$ are parallel.*

Proof. Clearly, under the above assumptions the foliations μ' and μ'' are comparable. Hence, the corollalry immediately follows from the theorem. □

3.4. Remark. It is well known that the set of points $\mu' \in MF_S$ topologically equivalent to a point $\mu \in MF_S$ represented by a minimal foliation has a natural structure of a simplex. It is easy to see that if μ', μ'' belong to the interior of this simplex, then μ' and μ'' are comparable. Hence, the rays $r_{m',\mu'}$ and $r_{m'',\mu''}$ are parallel in this case. If $\mu \in MF_S$ is not minimal, then the set of points topologically equivalent to μ is still a simplex, but with some faces removed (for example, if μ corresponds to the union of several disjoint circles, then all faces are removed). Again, points belonging to the interior of this simplex are comparable and the corresponding rays are parallel.

3.5. Remark. The arguments of this section were inspired by the work of Masur [M1].

4. Foliations corresponding to simple closed curves

The goal of this section is to prove a characterization of measured foliations μ corresponding to (non-zero multiples of) simple closed curves in terms of the set $\mu^{\perp} = \{\nu : \mathrm{i}\,(\nu, \mu) = 0\}$. Such a characterization is provided by the next theorem. Later on this will allow us to give a geometric characterization of corresponding rays.

4.1. Theorem. *Suppose that S is not a sphere with ≤ 4 punctures or a torus with ≤ 1 punctures. Then the codimension of the set $\mu^{\perp} = \{\nu : \mathrm{i}\,(\nu, \mu) = 0\}$ in MF_S is equal to 1 if and only if $\mu = a\gamma$ for some $a > 0$ and some simple closed curve γ.*

Proof. Let us realize μ by a foliation $\mathcal{F}_0$ on S. Recall that $\mathcal{F}_0$ may have one-prong singularities at punctures and that in the context of punctured surfaces we should treat all punctures as singularities (sometimes they are two-prong singularities). Let us consider the *skeleton* K_0 of $\mathcal{F}_0$, i.e. the union of all singularities of $\mathcal{F}_0$ and leaves of $\mathcal{F}_0$ connecting two singularities.

Let us split $\mathcal{F}_0$ along all non-contractible components of K_0 following [FLP], Exposé 9, § I (but, in contrast to [FLP], we do not blow up contractible components). The result of this procedure is a foliation $\mathcal{F}$ on a possibly non-connected subsurface R of the surface S. Some components of R are annuli foliated by parallel circles (such components, of course, may be absent).

All other components Q have the property that the foliation $\mathcal{F}$ restricted to Q is rational in the sense of [FLP], Expose 9, § III (because we split all noncontractible components of the skeleton). Every half-leaf of $\mathcal{F}$ contained in such a component Q is dense in Q; see [FLP], Exposé 9, § III, Lemma 6. In particular, the restriction of $\mathcal{F}$ to any such component Q is minimal. We note also that no component of the boundary ∂R is trivial in S minus punctures (because we split only non-contactible components). Let ν be some other point of MF_S, represented by a foliation $\mathcal{F}'_0$. We can apply same procedure to $\mathcal{F}'_0$ and get a foliation $\mathcal{F}'$ defined on a subsurface R' of S. It has properties similar to that of $\mathcal{F}$. In particular, if a component Q' of R' is not an annulus, then $\mathcal{F}'$ is arational (and hence minimal) on Q'.

Suppose now that $\mathrm{i}\,(\nu, \mu) = 0$. We claim that then $\mathrm{i}\,(\gamma, \gamma') = 0$ for any component γ of ∂R and any component γ' of $\partial R'$. In fact, any such γ is isotopic to a component of the boundary of a regular neighbourhood of K_0. Therefore, Lemma 2.14 of [I2] implies that $\mathrm{i}\,(\alpha, \mu) \neq 0$ for any α such that $\mathrm{i}\,(\alpha, \gamma) \neq 0$. Similarly, $\mathrm{i}\,(\beta, \nu) \neq 0$ if $\mathrm{i}\,(\beta, \gamma') \neq 0$. Applying this to $\alpha = \gamma'$ and $\beta = \mu$ we see that $\mathrm{i}\,(\gamma, \gamma') \neq 0$ implies $\mathrm{i}\,(\gamma', \mu) \neq 0$ and $\mathrm{i}\,(\mu, \nu) \neq 0$. This proves the claim.

In view of the previous paragraph, we can move the subsurfaces R, R' by two ambient isotopies of S fixed at the punctures into a position such that every component of ∂R is either equal to a component of $\partial R'$ or is disjoint from $\partial R'$. Suppose now that a component γ of ∂R is contained in a component Q' of R'. If γ is nontrivial in Q', then $\mathrm{i}\,(\gamma, \nu) \neq 0$ since the foliation $\mathcal{F}'$ is arational; cf. [FLP], Exposé 5, Proposition II.6 (note that if Q' contains a nontrivial circle, it is not an annulus). But, as in the previous paragraph, this implies that $i(\mu, \nu) \neq 0$ and, so, contradicts to the our assumption. Hence, if a component γ of ∂R is contained in a component Q' of R', then γ is isotopic to a component of $\partial Q'$. The same fact is true with the roles of R and R' interchanged. Clearly, this implies that by some additional isotopies we can move R, R' into a position such that every component of R is either equal to a component of R' or is disjoint from R'.

Let Q be a common component of R and R' which is not an annulus. Let

$\mathcal{F}_Q$ (respectively $\mathcal{F}'_Q$) be the restriction of the foliation $\mathcal{F}$ (respectively $\mathcal{F}'$) on Q, and let μ_Q (respectively ν_Q) be the Whitehead equivalence class of $\mathcal{F}_Q$ (respectively $\mathcal{F}'_Q$); $\mu_Q, \nu_Q \in MF_Q$. Our assumption $\mathrm{i}\,(\mu, \nu) = 0$ implies that $\mathrm{i}\,(\mu_Q, \nu_Q) = 0$ (compare [I2], Section 2.7). Since $\mathcal{F}$ is arational and Q is not an annulus, every half-leaf of $\mathcal{F}_Q$ not ending in a singularity is dense in Q; cf. [FLP], Exposé 9, Section III, Lemma 6. This means that the foliation $\mathcal{F}_Q$ is minimal and, hence, $\mathrm{i}\,(\mu_Q, \nu_Q) = 0$ implies that ν_Q can be represented by a foliation topologically equivalent to $\mathcal{F}_Q$ (see [Re], Theorem 1.12 for a proof of this implication). In other words, ν_Q differs from μ_Q only by the transverse measure.

Note that all foliations of an annulus are topologically equivalent (they foliate the annulus by a family of parallel circles). Hence, if a common component Q of R and R' is an annulus, then the restrictions of foliations $\mathcal{F}$ and $\mathcal{F}'$ on Q are topologically equivalent and so differ only by the transverse measure. (Up to isotopy, the measure is determined by the height of the annulus.)

Using the more or less self-explanatory language of [I2], Section 2.4, we can represent μ as a sum

$$\mu = \Sigma_Q \mu_Q + \Sigma_\gamma a_\gamma \gamma,$$

where Q ranges over non-annular components of R and $\mu_Q \in MF_Q$ for such Q, and where γ ranges over some set of representatives of isotopy classes of components of ∂R (if some component of R or of $S \setminus \mathrm{int}\; R$ is an annulus, then there are isotopic components) and $a_\gamma \geq 0$. Note that an annulus component of R contributes $a_\gamma \gamma$ for some $a_\gamma > 0$ and γ to this expression.

Similarly, ν can be represented as a sum

$$\nu = \Sigma_Q \nu_Q + \Sigma_P \nu_P + \Sigma_\gamma b_\gamma \gamma,$$

where Q ranges over non-annular components of R and $\nu_Q \in MF_Q$ for such Q, where P ranges over non-annular components of $S \setminus \mathrm{int}\; R$ and $\nu_P \in MF_P$ for such P, and where γ ranges over some set of representatives of isotopy classes of components of ∂R (and $b_\gamma \geq 0$). Clearly, ν_Q can be non-zero only if Q is a common component of R and R', and in this case ν_Q is topologically equivalent to μ_Q in the view of the above discussion.

Conversely, if ν has the above form and ν_Q is topologically equivalent to μ_Q for all non-annular components Q of R, then $\mathrm{i}\,(\nu, \mu) = 0$. This follows from the fact that $\mathrm{i}(\nu_Q, \mu_Q) = 0$ for all such Q (since $\mathcal{F}_Q$ is minimal and ν_Q can be represented by a foliation topologically equivalent to $\mathcal{F}_Q$, the intersection number $\mathrm{i}(\nu_Q, \mu_Q) = 0$ by [Re], Theorem 1.12) and the ditributivity of the intersection number (cf. [I2], Section 2.7).

We see that the set $\mu^\perp = \{\nu : \mathrm{i}(\mu, \nu) = 0\}$ is a product of several factors corresponding to the summands in the above expression for ν. The factor corresponding to a component Q of R (which is not an annulus) is homeomorphic to the set of transverse measures on $\mathcal{F}_Q$ and by a theorem of Levitt (see [L], Chapter IV) has dimension $\leq 3g_Q - 3 + b_Q$, where g_Q is the genus and b_Q is the number of boundary components and punctures of Q. See [P], Section 2, for another proof of this theorem (in [P] this is proved only for closed surfaces, but this restriction does not seem to be essential for the proof). The factor corresponding to a component P of $S \setminus \mathrm{int}\ R$ is simply the space MF_P of measured foliations on P; it has the dimension $6g_P - 6 + 2b_P$, where g_P is the genus and b_P is the number of boundary components and punctures of P. Finally, the factor corresponding to a component γ of ∂R is simply a copy of $\mathbf{R}_{\geq 0}$. Summing all this together, we get

$$\dim \mu^\perp \leq \Sigma_Q(3g_Q - 3 + b_Q) + \Sigma_P(6g_P - 6 + 2b_P) + b',$$

where, as before, Q and P range over non-annular components of R and $S \setminus \mathrm{int}\ R$ respectively, and where b' is the number of the isotopy classes of components of ∂R.

Now we are ready to prove the *"only if"* statement of the theorem. The dimension of MF_S is equal to $6g - 6 + 2b$, where g is the genus and b is the number of punctures of S. We would like to know when our upper bound for $\dim \mu^\perp$ is equal to $6g - 7 + 2b$. The additivity of the Euler characteristic gives

$$\Sigma_Q(2g_Q - 2 + b_Q) + \Sigma_P(2g_P - 2 + b_P) = 2g - 2 + b,$$

since we can ignore annular components (they have Euler characteristic 0). This implies that

$$\Sigma_Q(6g_Q - 6 + 3b_Q) + \Sigma_P(6g_P - 6 + 3b_P) = 6g - 6 + 3b.$$

Counting of the boundary components gives

$$\Sigma_Q b_Q + \Sigma_P b_P = b + 2b'$$

(if we collapse all annular components into circles, this will be completely obvious). The last two equalities imply

$$\Sigma_Q(6g_Q - 6 + 2b_Q) + \Sigma_P(6g_P - 6 + 2b_P) = 6g - 6 + 2b - 2b'.$$

Hence,

$$\begin{aligned}\dim \mu^{\perp} &\leq \Sigma_Q(3g_Q - 3 + b_Q) + \Sigma_P(6g_P - 6 + 2b_P) + b' \leq \\ &\leq \Sigma_Q(6g_Q - 6 + 2b_Q) + \Sigma_P(6g_P - 6 + 2b_P) + b' = \\ &= 6g - 6 + 2b - b'.\end{aligned}$$

The last term of this chain of inequalities is $\geq 6g - 7 + 2b$ only if $b' \leq 1$. Moreover, if $3g_Q - 3 + b_Q \geq 1$ for some Q, then the second inequality is, in fact, strict, and if $3g_Q - 3 + b_Q \geq 2$ for some Q or there are at least two Q such that $3g_Q - 3 + b_Q = 1$, then the difference between $6g - 7 + 2b$ and the last term is at least 2. Therefore, if $\dim \mu^{\perp} \geq 6g - 7 + 2b$, then either $b' = 1$ and $3g_Q - 3 + b_Q = 0$ for all Q, or $b' = 0$ and $3g_Q - 3 + b_Q = 1$ for one Q and $3g_Q - 3 + b_Q = 0$ for all other Q. Note that if $3g_Q - 3 + b_Q = 0$ then Q is either a sphere with 3 boundary components and punctures, or is a torus without punctures. In fact, a sphere with 3 boundary components and punctures cannot occur as a component Q, because it carries no foliations (having singularities on all boundary components). And if Q is a torus without punctures, then $Q = R = S$ and S is a torus without punctures, which cannot be the case by the assumptions of the theorem. Similarly, if $b' = 0$ and $3g_Q - 3 + b_Q = 1$ for some Q, then this component Q is a sphere with 4 punctures or a torus with one puncture and $Q = S$. This also cannot be the case by the assumption of the theorem.

So, we are left with only one possibility: $b' = 1$ and there are no Q's at all. This means that R is, actually, an annulus and, hence, μ has the required form. (It is easy to see that the splitting of $\mathcal{F}_0$ cannot lead to several isotopic annuli. Alternatively, several isotopic annuli also result in a foliation of the form $a\gamma$.) This proves the *"only if"* statement.

It remains to prove the *"if"* statement of the theorem. If $\mu = a\gamma$, then there are no Q-terms in our expression for ν. This implies that there are no Q-terms in all further formulas and our inequalities are, in fact, equalities. Hence, $\dim \mu^{\perp} = 6g - 6 + 2b - b' = 6g - 7 + 2b$ in this case. Since the dimension of MF_S is equal to $6g - 6 + 2b$, this means that codim $\mu^{\perp} = 1$ in this case. This completes the proof of the *"if"* statement and, hence, of the theorem. □

5. Proof of Theorem A

Let us introduce a relation $\bowtie$ on $MF_S \setminus \{0\}$. Choose a base point $m \in T_S$. By the definition, $\mu \bowtie \nu$ if and only if there exist two sequences $\{\mu_i\}_{i=1,2\ldots}$, $\{\nu_i\}_{i=1,2\ldots}$ of points in MF_S such that $\lim_{i\to\infty} \mu_i = \mu$, $\lim_{i\to\infty} \nu_i = \nu$ and for every i the rays r_{m,μ_i}, r_{m,ν_i} are *not* divergent. Let m' be some other point in T_S. Suppose that the sequences $\{\mu_i\}_{i=1,2\ldots}$, $\{\nu_i\}_{i=1,2\ldots}$ have the above property. Theorem 3.2 implies that the rays r_{m,μ_i}, r_{m,ν_i} are parallel to r_{m',μ_i}, r_{m',ν_i} respectively. It follows that the sequences $\{\mu_i\}_{i=1,2\ldots}$, $\{\nu_i\}_{i=1,2\ldots}$ have the above property with m replaced by m'. Hence the relation $\bowtie$ does not depend on the choice of the base point $m \in T_S$.

Note that $\bowtie$ is not an equivalence relation. Since the intersection number $\mathrm{i}\,(\cdot\,,\cdot)$ is continuous, Corollary 2.2 implies that $\mathrm{i}\,(\mu,\nu) = 0$ if $\mu \bowtie \nu$. We set $\Delta(\mu) = \{\nu : \nu \bowtie \mu\}$. Clearly, $\Delta(\mu) \subset \mu^{\perp}$ (where $\mu^{\perp}$ is the set $\{\nu : \mathrm{i}\,(\nu,\mu) = 0\}$; see Section 4).

We will need a couple of lemmas about $\bowtie$ and $\Delta(\,\cdot\,)$.

5.1. Lemma. *If $\mu = c\gamma$, where γ is the isotopy class of some circle and $c > 0$, then $\mathrm{i}\,(\mu,\nu) = 0$, $\nu \neq 0$ implies $\mu \bowtie \nu$. In particular, in this case $\Delta(\mu) = \mu^{\perp}$.*

Proof. Represent γ by a circle C. Let Q be the result of cutting S along C. Then every ν such that $\mathrm{i}\,(\mu,\nu) = 0$ or, what is the same, $\mathrm{i}\,(\gamma,\nu) = 0$ has the form $\nu = \nu' + b\gamma$ for some $\nu' \in MF_Q$ and $b \geq 0$ (we again using the language of [I2], Section 2.4). Compare the proof of Theorem 4.1. If $\nu' = 0$, then $\mu = c\gamma$ and $\nu = b\gamma$ define the same ray (note that $b \neq 0$ because $\nu \neq 0$), and, hence, $\mu \bowtie \nu$. Assume now that $\nu' \neq 0$. If $b \neq 0$, then the sequences $\{\mu_i = (1/i)\nu' + c\gamma\}$ and $\{\nu_i = \nu' + b\gamma\}$ converge to μ and ν respectively (the second sequence is, of course, constant). By Corollary 3.3 the rays r_{m,μ_i} and r_{m,ν_i} are parallel and, in particular, are not divergent. It follows that $\mu \bowtie \nu$. If $b = 0$, then the sequences $\{\mu_i = (1/i)\nu' + c\gamma\}$ and $\{\nu_i = \nu' + (1/i)\gamma\}$ converge to μ and ν respectively. Again, by Corollary 3.3 the rays r_{m,μ_i} and r_{m,ν_i} are not divergent. It follows that $\mu \bowtie \nu$ in this case also. This completes the proof of the lemma. $\square$

5.2. Lemma. *A point $\mu \in MF_S$ has the form $\mu = c\gamma$, where γ is the isotopy class of some circle and $c > 0$, if and only if* $\operatorname{codim} \Delta(\mu) = 1$. *(The codimension is understood, naturally, to be the codimension in the space MF_S.)*

Proof. Suppose that $\mu = c\gamma$. Then $\Delta(\mu) = \mu^{\perp}$ by Lemma 5.1 and $\operatorname{codim} \mu^{\perp} = 1$ by Theorem 4.1. Hence, $\operatorname{codim} \Delta(\mu) = 1$.

Suppose now that $\operatorname{codim} \Delta(\mu) = 1$. Then $\operatorname{codim} \mu^{\perp} \leq 1$ because $\Delta(\mu) \subset \mu^{\perp}$, and hence $\operatorname{codim} \mu^{\perp} = 1$ (clearly, $\operatorname{codim} \mu^{\perp}$ cannot be 0). Hence, μ has the required form by Theorem 4.1; $\mu = c\gamma$. This proves the lemma. $\square$

Consider now an isometry $F : T_S \to T_S$. Take an arbitrary point $m \in T_S$. The isometry F maps the set R_m of rays in T_S starting at m into the set $R_{F(m)}$ of rays in T_S starting at $F(m)$. As it was explained in 1.1, both these sets are in a natural bijective correspondence with the projectivized space of measured foliations PF_S. Hence, our map $R_m \to R_{F(m)}$ leads to a map $F_* : PF_S \to PF_S$. Obviously, F_* is a homeomorphism.

Let $l : MF_S \to \mathbf{R}_{\geq 0}$ be an arbitrary *length function* on MF_S in the sense of [I2], Section 2.11. This means that $l(a\mu) = al(\mu)$ for every $\mu \in MF_S$, $a \geq 0$ and $l(\mu) = 0$ if and only if $\mu = 0$. It is well known that such functions do exist (see [I2], loc. cit.). Obviously, we can identify PF_S with the level set $l^{-1}(1)$. Then we can extend $F_* : PF_S \to PF_S$ to a map $MF_S \to MF_S$, still denoted by F_*, by the formula $F_*(a\mu) = aF_*(\mu)$ for $\mu \in PF_S$, $a \geq 0$. Obviuosly, $F_* : MF_S \to MF_S$ is also a homeomorphism.

Since $\Delta(\,\cdot\,)$ is defined in terms of the geometry of rays (and a ray r_μ depends only on the projective class of μ), we have $F_*(\Delta(\mu)) = \Delta(F_*(\mu))$. Since $F_* : MF_S \to MF_S$ is a homeomorphism, it takes the sets of codimension 1 into the sets of codimension 1. By combining these remarks with Lemma 5.2, we see that $F_* : MF_S \to MF_S$ preserves the set of points of the form $c\gamma$, where γ is the isotopy class of some circle and $c > 0$. It follows that $F_* : PF_S \to PF_S$ preserves the set $V(S) \subset PF_S$ of projective classes of foliations defined by circles. Hence, F_* gives rise to a map $F_{**} : V(S) \to V(S)$.

Now notice that $V(S)$ is essentially the set of vertices of $C(S)$. The projective classes of two isotopy classes of circles γ, γ' are connected by an edge in $C(S)$ if and only if $\mathrm{i}\,(\gamma, \gamma') = 0$, and by Lemma 5.1 this condition is equivalent to $\gamma \bowtie \gamma'$. The last condition $\gamma \bowtie \gamma'$ is defined in terms of the geometry of rays and hence preserved by F_{**}. In other words, $F_{**} : V(S) \to V(S)$ takes pairs of vertices connected by an edge to pairs of vertices connected by an edge. Hence, F_{**} is an automorphismof the complex of curves $C(S)$ (it is well-known that a set of vertices is a simplex of $C(S)$ if and only if every two vertices from this set are connected by an edge). Now Theorem B implies that F_{**} acts on $C(S)$ as an element f of the modular group Mod_S. Replacing F by $f^{-1} \circ F$, we can assume that $F_{**} = \mathrm{id}$. It remains to prove that in this case $F = \mathrm{id}$.

Let γ_1, γ_2 be two isotopy classes of circles on S such that the pair $\{\gamma_1, \gamma_2\}$ *fills* S in the sense that there is no nontrivial isotopy class of circles γ on S such that $\mathrm{i}\,(\gamma, \gamma_1) = \mathrm{i}\,(\gamma, \gamma_2) = 0$. A pair of circles representing γ_1, γ_2 can be thickened to a pair $\mathcal{F}_1$, $\mathcal{F}_2$ of transverse measured foliations as explained, for example, in [FLP], Exp. 13. By the construction γ_1, γ_2 (or, rather, the corresponding measured foliatons) are the equivalence classes of $\mathcal{F}_1$, $\mathcal{F}_2$. Two foliations $\mathcal{F}_1$, $\mathcal{F}_2$ together define a structure of a Riemann surface M on S and a quadratic differential q on M such that $H(q) = \mathcal{F}_1$, $V(q) = \mathcal{F}_2$. In its turn, q defines a geodesic g in T_S passing through the point m corresponding to M. The point m divides g into two rays and by the construction these two rays correspond to γ_1, γ_2 (i.e., they are the rays r_{m,γ_1}, r_{m,γ_2}). Since, as we now assume, $F_{**} = \mathrm{id}$, the isometry F takes g to a geodesic $F(g)$ such that $F(m)$ fivides $F(g)$ into two rays also corresponding to γ_1, γ_2. Such a geodesic is necessarily equal to g. This follows from [GM], Theorem 3.1 and the discussion of it in the Introduction. (See also the desciption of the geodesic flow on T_S in [M2].)

Let us consider one more isotopy class of circles γ_2' filling S together with γ_1. One can choose γ_2' in such a way that $\mathrm{i}\,(\gamma_2', \gamma_2) \neq 0$. In addition to g, let us consider the geodesic g' defined by γ_1, γ_2' in the same manner as g defined by γ_1, γ_2. By the previous paragraph, $F(g) = g$ and $F(g') = g'$. Clearly, F acts on each of these two geodesics as a translation. We claim that the translation distances are both equal to 0. Let $m \in g$ and $m' \in g'$. Then m divides g into two rays r_{m,γ_1}, r_{m,γ_2} and m' divides g' into two rays r_{m,γ_1}, $r_{m,\gamma_2'}$. By Theorem 3.2 the rays r_{m,γ_1}, r_{m',γ_1} are parallel. In addition, again by Theorem 3.2, the rays $r_{m',\gamma_2'}$, $r_{m,\gamma_2'}$ are parallel. By Corollary 2.2 the rays $r_{m,\gamma_2'}$ and r_{m,γ_2} are divergent. It follows that the rays $r_{m',\gamma_2'}$ and r_{m,γ_2} are divergent. In other words, the geodesics g, g' are parallel in one direction and divergent in the other direction. Clearly, this implies that the translation distances of F along both these geodesics are equal to 0 (note that if the translation distance is non-zero, we can get an arbitarily large translation distance by iterating F).

Since the set of all such geodesics is dense in the space of all geodesics (because the set of projective classes of circles is dense in PF_S), it follows that $F = \mathrm{id}$. This completes the proof of Theorem A.

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