5. Applications of the Integral

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5.1 Area Under Curves

5.1.1 Area Under Curves
Part I

5.1.2 Area Under Curves
Part II

5.1.1 Area Under Curves Part I

- **One of the classic applications of the integral is to compute areas.**

- **We defined the integral to be the area under the curve:**

$$\int_a^b f(x)dx = \text{area under } f \text{ from } a \text{ to } b$$

Compute the area between x^2 and the x-axis from $x = 0$ to $x = 4$.

- **By convention, areas are positive. So if $f(x)$ is negative on $[a,b]$,**

$$-\int_a^b f(x)dx = \text{area under } f \text{ from } a \text{ to } b$$

- **Geometry also informs the following result:**

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \text{ if } a < c < b.$$

5.1.2 Area Under Curves Part II

- **One can also compute *the area between two curves* with the integral.**

- **Suppose** $f(x) \geq g(x) \text{ on } [a, b].$

- **The area between** $f(x), g(x)$ **on** $[a, b]$ **is**

$$\int_a^b (f(x) - g(x))dx.$$

Compute the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on $\left[0, \frac{\pi}{4}\right]$.

Compute the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on $\left[0, \frac{\pi}{2}\right]$.

Compute the area between $f(x) = x$ and $g(x) = x^2$ on $[0, 1]$.

5.2 Average Value

- The integral also has an interpretation as *the average of a function's value over an interval.*

- This makes sense if you recall that an integral $\int_a^b f(x)dx$ is approximated by Riemann sums, which are just rectangles whose heights are the function's values.

- The following statement is also worth considering for constant functions, which clearly have constant average.

- **The average value of $f(x)$ on the interval $[a, b]$ is**

$$\frac{1}{b-a}\int_a^b f(x)dx$$

- **So, we compute the integral, then divide by the length of the interval.**

- **Interpreting the integral as a sum, this bears resemblance to how the average of a finite set of numbers is computed.**

Compute the average value of $\ln(x)$ on $[1, 100]$.

Compute the average value of $\dfrac{1}{x^2+1}$ on $[-1, 1]$.

5.3 Growth and Decay Models

- **The integral allows us to solve certain basic *differential equations.***

- **Differential equations is a huge world of mathematics, and a subject with many problems without solutions.**

- It is a field of active research, including with computers.
- We will focus on an simple differential equation on the CLEP exam.

- **Consider the equation in terms of the unknown function $y(x)$:**

$$y' = ky, \quad \text{some constant } k.$$

- **To solve for $y(x)$, we do some algebra and recall the chain rule and formula for the derivative of $\ln(x)$.**

$$
\begin{aligned}
y' &= ky \\
\Leftrightarrow \frac{y'}{y} &= k \\
\Leftrightarrow \int \frac{y'}{y} dx &= \int k dx \\
\Leftrightarrow \ln(y) &= kx + C \\
\Leftrightarrow y(x) &= Ce^{kx}
\end{aligned}
$$

- If $k > 0,$ we have exponential growth.

- If $k < 0,$ we have exponential decay.

- The constant $C > 0$ is determined based on details in the problem, noting that $y(0) = C.$

Suppose $y' = 2y,\ y(0) = 100$. Find $y(5)$.

Suppose $y' = -5y, \ y(0) = 1000$. Find x such that $y(x) = 1$.

5.4 Return to Physics

- **Just as we used derivatives to understand *position, velocity, and acceleration* of a one-dimensional particle, so too can we use integrals.**

- **We simply follow the fundamental theory of calculus:**

$$\int_a^b f'(x)dx = f(b) - f(a).$$

- **Let $v(t)$ be the instantaneous velocity of a particle at time t.**

- **The position of the particle at time t is $p(t)$ and satisfies**

$$
\begin{aligned}
p'(t) &= v(t) \\
\Rightarrow \int_a^b p'(t)dt &= \int_a^b v(t)dt \\
\Rightarrow p(b) &= p(a) + \int_a^b v(t)dt.
\end{aligned}
$$

Suppose a particle has instantaneous velocity $v(t) = -t^2$ and initial position $p(0) = 10$. Find $p(5)$.

- **A similar game can be played with acceleration:**

$$
\begin{aligned}
v'(t) &= a(t) \\
\Rightarrow \int_{t_0}^{t_1} v'(t)dt &= \int_{t_0}^{t_1} a(t)dt \\
\Rightarrow v(t_1) &= v(t_0) + \int_{t_0}^{t_1} a(t)dt.
\end{aligned}
$$

- **With this formula for velocity, we can keep going and get a formula for position.**

Suppose a particle has instantaneous acceleration $a(t) = -10$, initial position $p(0) = 0,$ and initial velocity $v(0) = 0$. Find $p(5)$.