4. Theory of the Integral

4.1 Antidifferentiation

4.2 The Definite Integral

4.3 Riemann Sums

4.4 The Fundamental Theorem of Calculus

4.5 Fundamental Integration Rules

4.6 U-Substitutions

4.1 Antidifferentiation

- **We will begin our study of the *integral* by discussing antidifferentiation.**

- **As you might expect, this is the process of undoing a derivative.**

Let $f(x)$ be a function. A function $F(x)$ is an *antiderivative* of $f(x)$ if $F'(x) = f(x)$.

Let $f(x) = 1$. Find an antiderivative of $f(x)$.

Let $f(x) = \sin(x)$. Find an antiderivative of $f(x)$.

Let $f(x) = e^{2x}$. Find an antiderivative of $f(x)$.

- **Notice that I am asking to find *an* antiderivative, not *the* antiderivative.**

- **That is because antiderivatives are not unique!**

- **Indeed, if $F(x)$ is an antiderivative for $f(x)$, then $F(x) + C$ is also an antiderivative for any constant C.**

4.2 Definite Integral

- **We will relate the antiderivative to another important object: the *definite integral.***

- **This is a quantity that depends on two endpoint values, a, b, and a function, $f(x)$.**

- **It is written as** $\displaystyle\int_a^b f(x)dx.$

- **The definite integral has many important interpretations.**

- **The most significant for us is *area under the curve* $f(x)$ *from* a *to* b.**

- **It is not obvious how to compute the area under the curve of a general function— this is the power of calculus!**

- **Let's start with simple things.**

Compute $\int_0^2 3dx$.

Compute $\int_{-1}^{1} x dx$.

Compute $\int_0^5 2x dx$.

4.3 Riemann Sums

4.3.1 Riemman Sums Part I

4.3.2 Riemman Sums Part II

4.3.1 Riemann Sums Part I

- **We have seen how to compute definite integrals of functions with certain simple properties, by exploiting well-known area formulas from geometry.**

- **What can we do in general? Not much yet.**

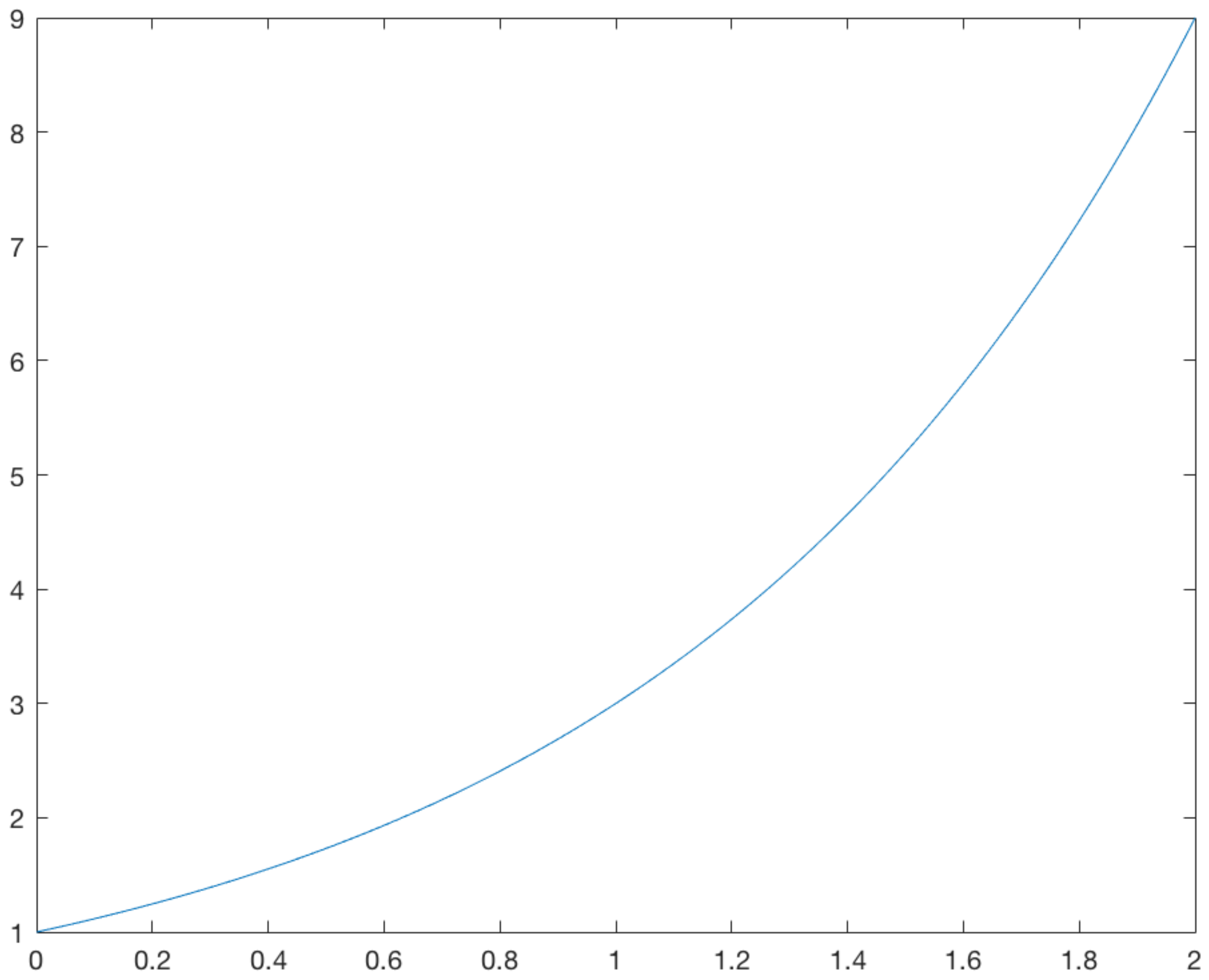
- **We can, however, approximate the area with *Riemann sums*.**

- A Riemann sum approximates an integral by covering the area beneath the curve with rectangles.

- The areas of the these rectangles are more easily computed.

- This is because the width of these rectangles is fixed, and the height is given by the value of the function at a given point.

- Programmers—try coding this! It's a classic.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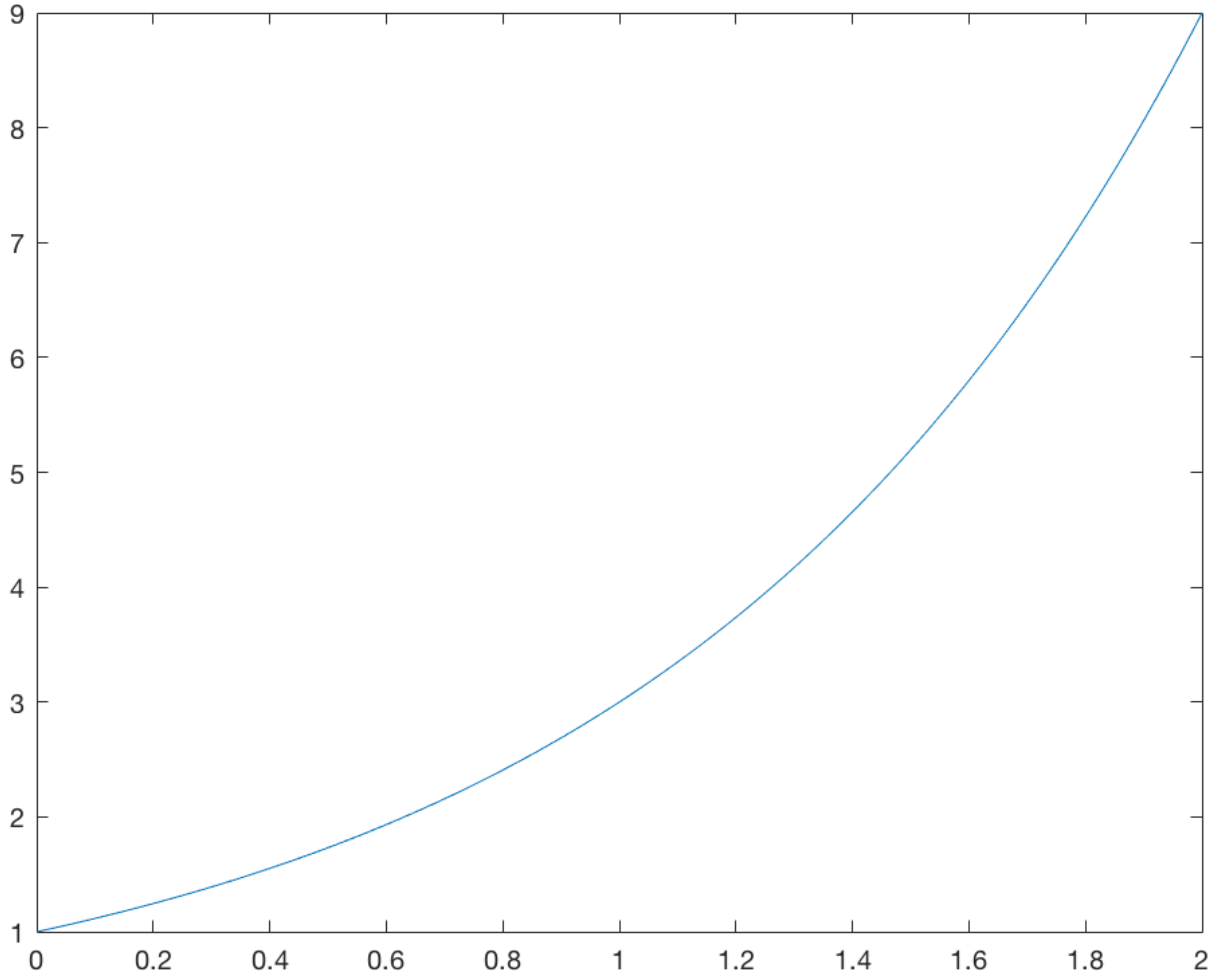
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Estimate $\int_0^4 x^2 dx$ with left and right Riemann sums of width 1.

4.3.2 Riemann Sums Part II

Estimate $\int_{-1}^{2}(1-x)dx$ with left and right Riemann sums of width 1.

4.4 The Fundamental Theorem of Calculus

- The *fundamental theorem of calculus* is a classic result.

- It links the derivative and the integral.

- We will not prove it, though we will use it extensively to compute areas under curves.

- Intuitively, definite integrals can be computed by evaluating an antiderivative at the endpoints of integration.

Suppose f has antiderivative $F(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Compute $\displaystyle\int_0^2 x^2 dx.$

Compute $\int_0^{2\pi} \cos(x) dx$.

- **When no particular endpoints are specified, the FTC suggests that we write** $$\int f(x) = F(x) + C$$

- **Here,** C **is an arbitrary constant.**

Compute $\displaystyle\int e^{3x}\,dx.$

Compute $\displaystyle\int \frac{2}{x}\,dx.$

- **Another way to interpret the FTC is as stating that the derivative and integral *undo each other.***

- **More precisely,**

$$\frac{d}{dx}\int f(x)dx = f(x)$$

- **This is valid for *all* $f(x)$ likely to appear on the CLEP exam.**

4.5 Basic Integral Rules

4.5.1 Basic Integral Rules I

4.5.2 Basic Integral Rules II

4.5.1 Basic Integral Rules I

- **Using the FTC, we see that all the basic *derivative rules* apply, in an inverted way, to *integrals.***

- **This means that to know the basic rules for integrals, it suffices to know the basic rules for derivatives.**

For constants a, b, $\int (af(x) + bg(x))dx = a \int f(x)dx + b \int g(x)dx$

$$\text{If } n \neq -1, \int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\text{If } n = -1, \int x^n dx = \ln(x) + C$$

Compute $\int (x^3 + 2x - 3)dx$

Compute $\int (x^{-1} + 1) dx$

$$\int e^x dx = e^x + C$$

Compute $\int \left(\frac{-4}{x} + 2e^x \right) dx$

4.5.2 Basic Integral Rules II

Compute $\int (\sin(x) + x^2)dx$

$$\int \sin(x)dx = -\cos(x) + C$$

$$\int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\tan(x) + \sec(x)| + C$$

Compute $\displaystyle\int (\tan(\theta) - \cos(\theta))d\theta$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C$$

$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}(x) + C$$

Compute $\displaystyle\int \frac{-3dx}{\sqrt{4-4x^2}}$

Compute $\displaystyle\int \frac{dy}{2|y|\sqrt{y^2-1}}$

4.6 U-Substitutions

- There are many more sophisticated types of integration methods.
- These include those based on the product rule (integration by parts), special properties of trigonometric functions (trig. substitutions), and those based on tedious algebra (partial fraction decomposition).

- **We focus on a method based on the *chain rule.***

- **Recall that to compute the derivative of a composition of functions, we use the chain rule:**

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

- **According to the FTC,**

$$\int \frac{d}{dx} f(g(x)) = f(g(x)) + C.$$

- **Hence,**

$$\int f'(g(x)) g'(x) dx = f(g(x)) + C$$

Compute $\displaystyle\int xe^{x^2}\,dx$

Compute $\displaystyle\int \cos(4x + 1)dx$

Compute $\int x^3 \sqrt{x^4 + 1} dx$

Compute $\int \tan(x) dx$