

3. Applications of the Derivative

3.1 Plotting with Derivatives

3.2 Rate of Change Problems

3.3 Some Physics Problems

3.1 Plotting with Derivatives

3.1.1 Increasing and Decreasing Functions

3.1.2 Extrema

3.1.3 Concavity

3.1.1 Increasing and Decreasing Functions

- **Recall that the derivative of a function corresponds to *the rate of change of a function*.**
- **If the rate of change is positive, we say the function is increasing.**

- **If it is negative, we say it is decreasing.**
- **We can quantify this by discussing the sign of the derivative.**

- **Let $f(x)$ be a function.**
- **If $f'(x_0) > 0$, then $f(x)$ is increasing at x_0 .**
- **If $f'(x_0) < 0$, then $f(x)$ is decreasing at x_0 .**
- **If $f'(x_0) = 0$, no definitive conclusion can be made without further analysis.**

- **Note that a function may not even be differentiable and still be increasing/decreasing.**

Let $f(x) = \sin(x)$.

Is f increasing, decreasing at $x = 0, \frac{\pi}{4}, \pi$?

Let $f(x) = x - e^x$.

Is f increasing, decreasing at $x = -1, 0, 1$?

Let $f(x) = x^3 - 6x^2 + 3x - 2$.

Find where f is increasing and decreasing.

3.1.2 Extrema

- **We have seen that:**

$$f'(x) > 0 \Rightarrow f(x) \text{ increasing}$$

$$f'(x) < 0 \Rightarrow f(x) \text{ decreasing}$$

- **So, what about if $f'(x) = 0$?**
- **This is perhaps the most exciting aspect of differential calculus, and is a major reason it is studied by all kinds of people.**

- **Suppose** $f'(x) < 0, x < x_0$
 $f'(x_0) = 0$
 $f'(x) > 0, x > x_0$
- **Then f transitions from decreasing to increasing at $x = x_0$.**
- **This means $f(x)$ has a *local minimum at* x_0 .**

Show $f(x) = x^2$ has a local minimum at $x = 0$.

- **Suppose** $f'(x) > 0, x < x_0$
 $f'(x_0) = 0$
 $f'(x) < 0, x > x_0$
- **Then f transitions from increasing to decreasing at $x = x_0$.**
- **This means $f(x)$ has a local maximum at x_0 .**

Show $f(x) = \cos(x)$ has a local maximum at $x = 0$.

- A classic calculus problem is to find the *local extrema (minima and maxima) of a function*.
- To do so, set the derivative equal to 0 and check how the derivative changes sign.
- Not every place the derivative equals zero is a local extrema, however.

Find the local extrema of $f(x) = \sin(x)$.

Find the local extrema of $f(x) = x^3$.

3.1.3 Concavity

- We saw in the previous submodule that the properties of a function being *increasing*, *decreasing*, and its *local extrema* are governed by its first derivative, $f'(x)$.
- A more subtle notion, *concavity*, is governed by the second derivative, $f''(x)$.

- A loose metaphor is in order: when plotting a function, try pouring water on it.
- If the function holds the water, it is *concave up* there.
- If it doesn't hold water, it is *concave down* there.

- **A function $f(x)$ is *concave up* wherever $f''(x) > 0$.**
- **A function $f(x)$ is *concave down* wherever $f''(x) < 0$.**

Determine the concavity and sketch $f(x) = x^3 - 12x + 1$

- The second derivative can also be used to classify *critical points*, i.e. points where $f'(x) = 0$.

- ***Second Derivative Test:***

Suppose $f'(x_0) = 0$.

If $f''(x_0) > 0$, x_0 is a local maximum.

If $f''(x_0) < 0$, x_0 is a local minimum.

Use the second derivative test to determine the nature of the critical points of $f(x) = 2 \cos(4\pi x)$.

3.2 Rate of Change

- **A classic application of the derivative is to compute the *instantaneous rate of change* of a quantity.**
- **Recall that the instantaneous rate of change of $f(x)$ at $x = a$ is $f'(a)$.**
- **In contrast, the average rate of change of $f(x)$ on the interval $[a, b]$ is $\frac{f(b) - f(a)}{b - a}$**

Let $f(x) = x^4 - x^2 + 2$.

Find the average rate of change of f on $[0, 2]$.

Find the instantaneous rate of change of f at $x = 0, 2$.

Let the size of a population be given by $P(x) = 100 \cdot 2^{\frac{x}{100}}$.

Find the average rate of change of f on $[0, 200]$.

Find the instantaneous rate of change of f at $x = 0, 200$.

Let the value of an investment be $P(t) = 10 \cdot e^{\frac{x}{15}}$.

When will the instantaneous rate of growth of the investment first exceed 300?

3.3 Some Physics Problems

- **Another classic application of derivatives is related to the physical laws of motion.**
- **In this context, a one-dimensional particle's *position* is given by a function $p(t)$**
- **Related quantities, like its *velocity* $v(t)$ and its *acceleration* $a(t)$ may be understood as certain derivatives of the position.**

- **Let the position of a particle be given by $p(t)$.**
- **The velocity of the particle is given by $v(t) = p'(t)$.**
- **The acceleration of the particle is given by $a(t) = v'(t) = p''(t)$.**
- **So, velocity is the rate of change of position, and acceleration is the rate of change of velocity.**

Suppose a one-dimensional particle has position function $p(t) = 4 - 10t^2$.

When is the particle moving with velocity -10 ?

What is the acceleration of the particle?

Suppose a one-dimensional particle has position $p(t) = \ln(t^4 + t^2), t > 0$.

Show that the particle never changes direction.