# 2. Theory of the Derivative

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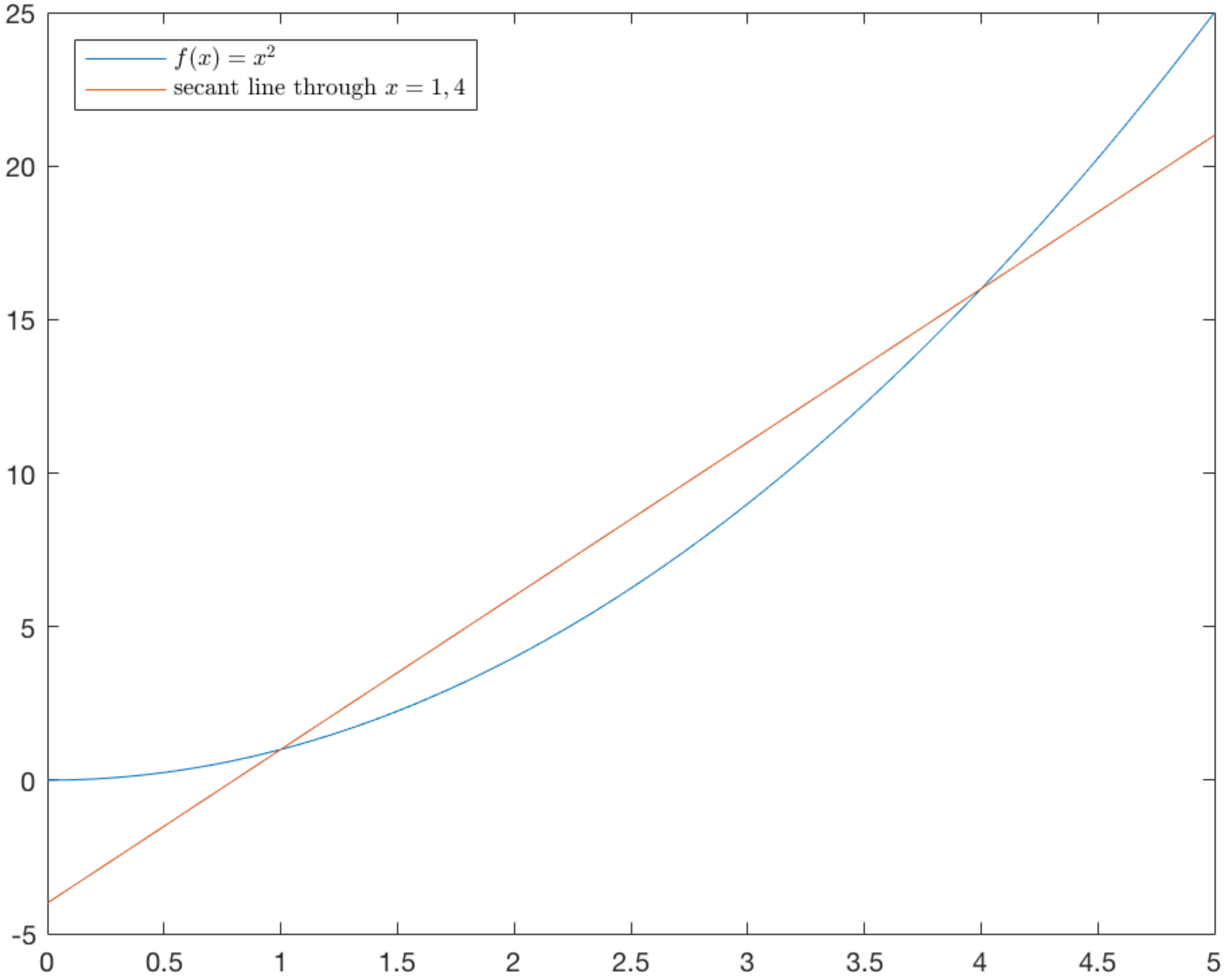
# 2.1 Tangent Lines

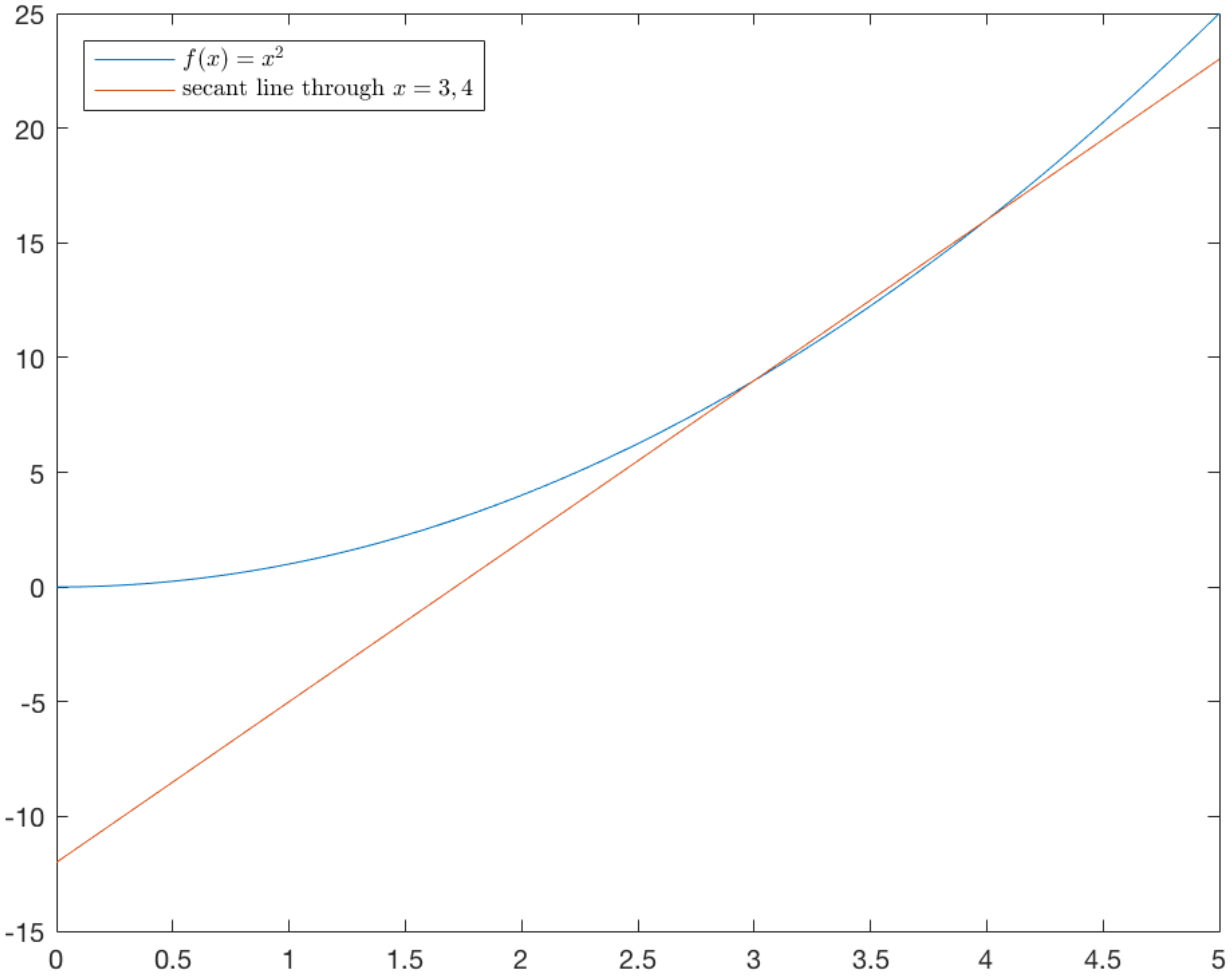
- Before we do any heavy lifting, let's get a mental picture.
- One of the classical ideas behind calculus is the notion of *tangent line to* a function.
- This will motivate the limit definition of a derivative in the next submodule.

- A line is *tangent* to a function if it intersects it only once.
- This is somewhat of a simplification, in that the line is allowed to intersect multiple times outside of some small interval, but that is more advanced and theoretical than we will get into.

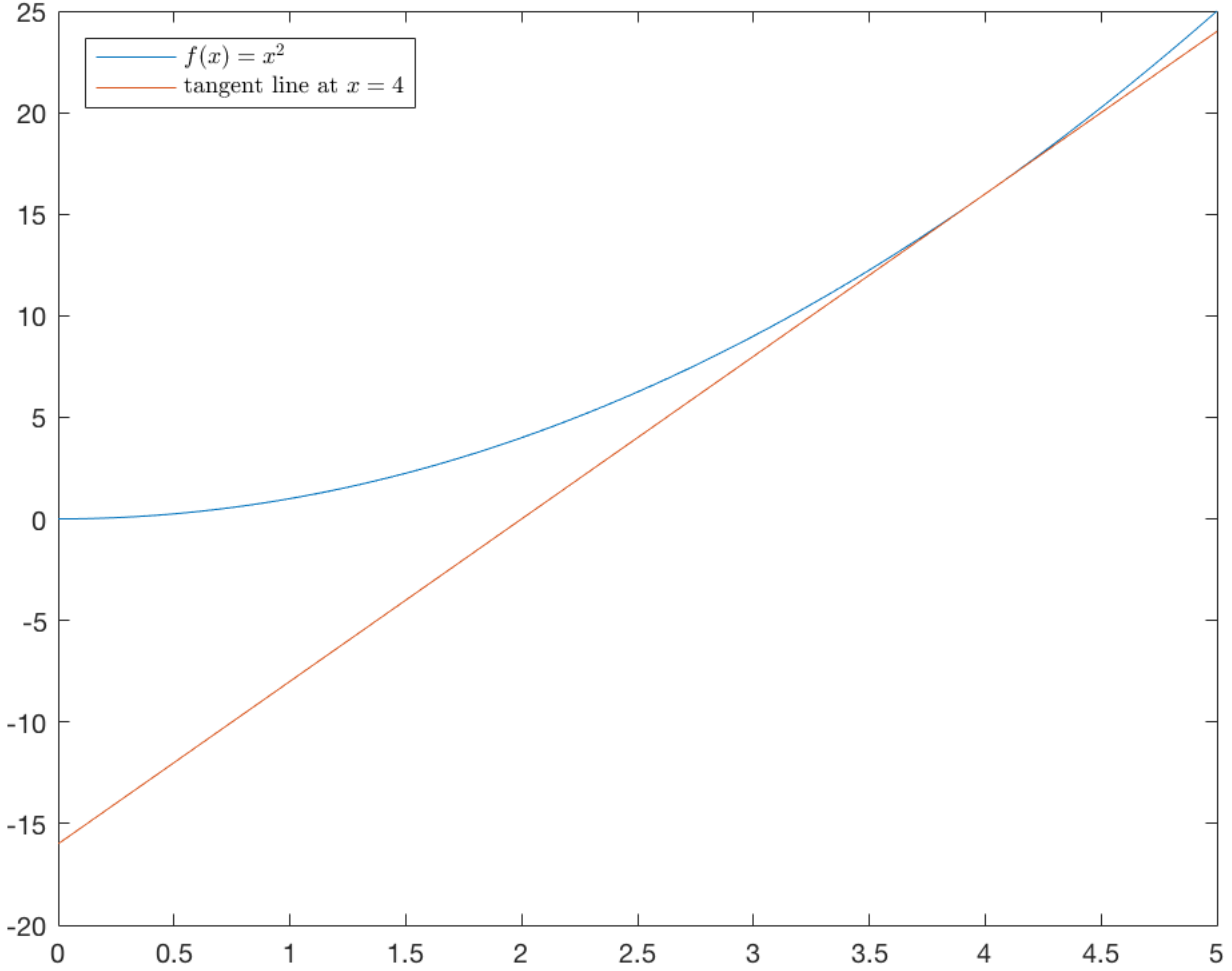
- **Tangent lines can be constructed as *limits of secant lines*, i.e. lines that intersect a function in exactly two points.**



$f(x) = x^2$
secant line through $x = 1, 4$
25
20
15
10
5
0
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



$f(x) = x^2$
secant line through $x = 3, 4$
25
20
15
10
5
0
-5
-10
-15
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



$f(x) = x^2$
tangent line at $x = 4$
25
20
15
10
5
0
-5
-10
-15
-20
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

- **The slopes of the secant lines are computed using the classical slope formula.**

- **If a line passes through:**

$$(a, f(a)), (b, f(b)),$$

**then the slope of the line is**

$$m = \frac{f(b) - f(a)}{b - a}$$

- **What is the slope of the tangent line? We need limits! This gives us the formal definition of the derivative!!!**

# 2.2 Definition of Derivative

- The derivative is one of the two central objects in calculus.

- It measures *rate of change of a function.*

- In module 2, we will discuss methods for computing it, and discuss its geometric role.

- In module 3, we will use it as a tool to solve real-world problems.

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- **What is the slope of the tangent line? We need limits! This gives us the formal definition of the derivative!!!**

Let $f(x)$ be a function. The derivative of $f$ at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **So, the derivative is defined in terms of a *limit*.**

- **Notice that plugging in $h = 0$ yields 0/0, so we must be careful.**

- **In later submodules, we will develop some nice tricks and formulae.**

Let $f(x) = x$. Compute $f'(x)$.

Let $f(x) = x^2$. Compute $f'(x)$.

# 2.3 Rates of Change

- **Recall that for a general function $f(x)$, the slope of the secant line through $a, b$ may be interpreted as the *average rate of change of $f$ on $(a, b)$*.**

- **More precisely,**

$$\text{Average change of } f \text{ on } (a, b) = \frac{f(b) - f(a)}{b - a}$$

- **Let $b = a + h$. Then we can say that**

$$\text{Average change of } f \text{ on } (a, a+h) = \frac{f(a+h) - f(a)}{h}$$

- **This looks an *awful lot like the definition of the derivative!***

- **Simply take the limit as $h \to 0.$**

- **This shrinks the interval in question to $\approx a$ alone.**

- **We conclude that**

$$\text{Instantaneous change of } f \text{ at } a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

- **So, derivatives are equal to *instantaneous rates of changes.***

Let $f(x) = x^2$. Find the average rate of change on $(-2, 2)$.

Find the instantaneous rate of change at $x = 2$.

Let $f(x) = cx,$ for some fixed constant c.

Find the average rate of change on $(-1, 3)$.

Find the instantaneous rate of change at $x = 1$.

# 2.4 Derivative Rules

## 2.4.1 Fundamental Derivative Rules

## 2.4.2 Chain Rule

## 2.4.3 Derivatives of Exponential and Logarithmic Functions

## 2.4.4 Trigonometric Derivatives

## 2.4.5 Derivatives of Inverse Trigonometric Functions

## 2.4.1 Fundamental Derivative Rules

- **The limit definition of the derivative is not always very convenient.**

- **For practical purposes, it is nice to know exactly how this definition works for certain types of functions.**

- **The following results are not obvious, but we will not prove them in this course.**

## Derivative of a Constant

$$[a]' = 0$$

# Derivative of a Polynomial

$$[x^a]' = ax^{a-1}, \text{ if } a \neq 0$$

Let $f(x) = x^4$. Compute $f'(x)$.

# Derivative of a Sum

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

Let $f(x) = x^3 - 2x + 1$. Compute $f'(x)$.

# Derivative of a Product

$$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Let $f(x) = (x+1)\sqrt{x}$. Compute $f'(x)$.

# Derivative of a Quotient

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Let $f(x) = \dfrac{2x-3}{x^4+1}$. Compute $f'(x)$.

# 2.4.2 Chain Rule

- **The *chain rule* is arguably to most foundational property of derivatives.**

- **It tells how to compute the derivation of a *composition of functions, i.e. a function of the form***

  $$f(x) = g \circ h(x) = g(h(x))$$

$$[g \circ h(x)]' = [g' \circ h(x)] \cdot h'(x)$$

i.e. $[g(h(x))]' = [g'(h(x))] \cdot h'(x)$

Compute the derivative of $f(x) = (3x + 2)^{-2}$

Compute the derivative of $f(x) = (x^2 + 2)^3\sqrt{4x + 1}$

- What if we are considering just plain old $f(x)$ that does not appear to have the form of a composition?

- Well, we may always write:

$$f(x) = f(g(x)), g(x) = x$$

- Taking derivatives and applying the chain rule yields:

$$
\begin{aligned}
f'(x) &= f'(g(x)) \cdot g'(x) \\
&= f'(x) \cdot 1 \\
&= f'(x)
\end{aligned}
$$

- **This emphasizes that we are always *implicitly using the chain rule*, even when it might appear there is no composition.**

- **It may be necessary to apply the chain rule *iteratively:***

$$[f(g(h(x)))]' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Compute the derivative of $f(x) = (\sqrt{x^2 - 1} - 2)^{-1}$

# 2.4.3 Derivatives of Exponential and Logarithmic Functions

- **The exponential function with base $e$ is rather simple from the calculus standpoint.**

$$[e^x]' = e^x$$

- **More general exponential functions have a slightly more delicate formula:**

$$[a^x]' = a^x \cdot \ln(a)$$

Compute $\dfrac{d}{dx}\left[e^{2x}\right]$

Compute $\dfrac{d}{dz}\left[e^{z^2} + 4z\right]$

Compute $\dfrac{d}{dx}\left[xe^{x^3}\right]$

- **By contrast, logarithms are somewhat trickier. Derivatives of logarithms do not stay as logarithms:**

$$[\ln(x)]' = \frac{1}{x}$$

$$[\log_a(x)]' = \frac{1}{\ln(a)x}$$

Compute $\dfrac{d}{dx}\left[\ln(x^2)\right]$

Compute $\dfrac{d}{dy}\left[\ln(y+y^4)\right]$

Compute $\dfrac{d}{dx}\left[\ln(e^{2x+1})\right]$

# 2.4.4 Trigonometric Derivatives

- **The trigonometric functions all have derivatives that related to other trigonometric functions.**

- **The foundational ones are:**

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

Compute $\dfrac{d}{dx}[\cos(x^2+1)]$

- **We can use decompose into $\sin(x), \cos(x)$ and then use the quotient rule to compute the derivatives of the remaining trigonometric functions.**

- **We will prove that**

$$\frac{d}{dx}[\tan(x)] = \sec(x)^2$$

- **Proving the rest of the trigonometric derivatives in a similar way is an *excellent* exercise.**

$$\frac{d}{dx}[\cot(x)] = -\csc(x)^2$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

Compute $[\tan(\theta + 1)]'$

Let $f(x) = \csc(x^2)$. Compute $f'(x)$.

# 2.4.5 Derivatives of Inverse Trigonometric Functions

- **The inverse trigonometric functions also have derivatives that ought to be committed to memory for the CLEP exam.**

- We will see in a later submodule how to *prove these formulae* starting from a general principle for derivatives of inverse functions.

- Until then, we will take the basic rules for granted.

$$\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}$$

Compute the derivative of $f(x) = \sin^{-1}(x^3 + 1)$.

Compute the derivative of $f(x) = \sec^{-1}(e^x)$.

Compute the derivative of $f(x) = \arctan(\sin(x))$.

# 2.5 Higher Order Derivatives

- **It is possible to *differentiate a function multiple times.***

- **The result of iterated differentiation is called a *higher order derivative.***

- **First derivative:** $f'(x)$
- **Second derivative:** $f''(x)$
- **Third derivative:** $f^{(3)}(x)$
- $n^{th}$ **derivative:** $f^{(n)}(x)$

Let $f(x) = x^3 - 4x + 1$. Compute $f', f'', f^{(3)}$

Let $f(x) = e^{x^2}$. Compute $f', f'', f^{(3)}$

Let $f(x) = \sin(2x)$. Find all values $x$ for which $f'' = 1$.

Let $f(x) = \ln(g(x))$. Compute $f''(x)$ in terms of $g(x)$.

# 2.6 Implicit Differentiation

- **All of our work has so far focused on differentiating a function where there was only one variable:**

$$f(x) = \text{ something depending on } x$$

- **We may at times come across an expression involving both $x$ and $y$**

- **In this case, $y$ is *implicitly* a function of $x$ .**

- **We differentiate in this case by noting that:**

$$\frac{d}{dx}[y] = y',$$
$$\frac{d}{dx}[x] = 1.$$

- **This allows us to differentiate both sides of an expression, and solve for the resulting $y'$.**

Solve for $y'$ : $2xy + y^2 = 1$

Solve for $y'$ : $\sqrt{y+1} + x^2 = y$

Solve for $y'$ : $e^{xy-1} = x^2$

# 2.7 L’Hôpital’s Rule

- **Recall that certain quantities are not well-defined:**

$$\frac{0}{0}, \frac{\infty}{\infty}$$

- **These *indeterminate forms* sometimes arise when taking limits of rational functions, i.e. computing limits of the form**

$$\lim_{x \to y} \frac{f(x)}{g(x)}$$

- **In these special indeterminate cases, one can apply manipulations to $\dfrac{f(x)}{g(x)}$ in order to compute the limit.**

- **Another, slicker, trick is to use *L'Hôpital's rule,* which we state loosely as**

$$\text{If } \lim_{x \to y} f(x) = \lim_{x \to y} g(x) = 0 \text{ or } \pm\infty,$$

$$\text{then } \lim_{x \to y} \frac{f(x)}{g(x)} = \lim_{x \to y} \frac{f'(x)}{g'(x)}, \text{ provided the second limit exists.}$$

Compute $\displaystyle\lim_{x \to \infty} \frac{x+1}{3x-1}$

Compute $\lim_{x \to 0} \frac{e^x - 1}{x}$

Compute $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

Compute $\lim_{x \to 0} \frac{\sin(x)}{x}$

Compute $\lim_{x \to 0} \frac{\cos(x)}{x}$

# 2.8 Some Classic Theoretical Results

- **This is not a course in theory, but certain results are important for the CLEP.**

- **Proving these would be an *excellent* learning experience, but is certainly not necessary. A basic understanding would suffice for the CLEP exam.**

## Differentiability Implies Continuity

Suppose a function $f$ is differentiable at a point $x$.

Then $f$ is continuous at $x$.

# Rolle's Theorem

Suppose a function $f$ is differentiable on an interval $(a, b)$.

If $f(a) = f(b)$, then there is a point $c, a < c < b$ such that $f'(c) = 0$.

# 2.9 Derivatives of Inverse Functions

- **We have seen already some special examples of derivatives of inverse functions: inverse trigonometric functions.**

- **Recall that the inverse function of $f(x)$ is a function $f^{-1}(x)$ satisfying**

$$f^{-1} \circ f(x) = f \circ f^{-1}(x) = x.$$

Suppose $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$. Then

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}.$$

Suppose $f(x) = x^3 + x - 1$. Compute the derivative of $f^{-1}$ at $x = 1$.

Suppose $f(x) = e^x + 2x + 3$. Compute the derivative of $f^{-1}$ at $x = 4$.