1. Limits

1.1 Definition of a Limit

1.2 Computing Basic Limits

1.3 Continuity

1.4 Squeeze Theorem

1.1 Definition of a Limit

- **The *limit* is the central object of calculus.**
- **It is a tool from which other fundamental definitions develop.**
- **The key difference between calculus and everything before is this idea.**
- **We say things like:**

a function $f(x)$ has a limit at a point y

$$\lim_{x \to y} f(x) = L \text{ if, for all } \epsilon > 0, \text{ there exists some } \delta > 0$$

$$\text{such that if } 0 < |x - y| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

- **In other words, if a point x is close to y , then the outpoint $f(x)$ is close to L .**

- **The limit definition *does not say* $f(x)$ *needs to exist!***

- **The special case when $f(x)$ exists and is equal to $\lim_{y \to x} f(y)$ is special, and will be discussed later.**

- One can sometimes *visually check if a limit exists*, but the definition is very important too.

- It’s a tough one the first time, but is a thing of great beauty.

1.2 Computing Basic Limits

- **Computing limits can be easy or hard.**

- **A limit captures what the function looks like *around a certain point*, rather than *at a certain point.***

- **To compute limits, you need to ignore the function's value, and only analyze what happens nearby.**

- **This is what the** $\epsilon - \delta$ **definition attempts to characterize.**

Compute $\lim_{x \to 0} (x+1)^2$

Compute $\displaystyle\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}$

Compute $\displaystyle \lim_{x \to 1} \frac{x^2 + 2x + 1}{x + 1}$

Compute $\lim_{x \to 0} \frac{1}{x}$

Compute $\lim_{x \to 0} \left(\frac{\sqrt{x^4 + x^2}}{x} \right)$

1.3 Continuity

- **Sometimes, plugging into a function is the same as evaluating a limit. But not always!**

- **Continuity captures this property.**

f is continuous at x if

$$\lim_{y \to x} f(y) = f(x)$$

- **Intuitively, a function that is continuous at every point can be drawn *without lifting the pen.***

f is *continuous* if it is continuous at x for all x

Discuss the continuity of $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of $f(x) = \begin{cases} 2x+1 & \text{if } x \leq 1 \\ 3x^2 & \text{if } x > 1 \end{cases}$

- **Polynomials, exponential functions, and $\sin, \cos$ are continuous functions.**

- **Rational functions are continuous except at points where the denominator is 0.**

- **Logarithm is continuous, because its domain is only $(0, \infty)$.**

1.4 Squeeze Theorem

- **There are no one-size-fits-all methods for computing limits.**
- **One technique that is useful for certain problems is to *relate one limit to another.***
- **A foundational technique for this is based around the *Squeeze Theorem.***

Squeeze Theorem

Suppose $g(x) \leq f(x) \leq h(x)$ for some interval containing y.

$$\Rightarrow \lim_{x \to y} g(x) \leq \lim_{x \to y} f(x) \leq \lim_{x \to y} h(x)$$

- We will not prove this (or any, really) theorem.

- One classic application of the theorem is computing

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

- **Direct substitution (which one should be very wary of when computing limits) fails.**

- **Indeed, plugging in $x = 0$ yields**

$$\frac{\sin(0)}{0} = \frac{0}{0} = \mathrm{DNE}$$

- **An instructive exercise is to show that, for**

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$$

$$\Rightarrow \lim_{x \to 0} \cos(x) \leq \lim_{x \to 0} \frac{\sin(x)}{x} \leq \lim_{x \to 0} 1$$

$$\Rightarrow 1 \leq \lim_{x \to 0} \frac{\sin(x)}{x} \leq 1$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$