1. Limits

1.1 Definition of a Limit

- **The *limit* is the central object of calculus.**
- **It is a tool from which other fundamental definitions develop.**
- **The key difference between calculus and everything before is this idea.**
- **We say things like:**

  a function $f(x)$ has a limit at a point y

$$\lim_{x \to y} f(x) = L \text{ if, for all } \epsilon > 0, \text{ there exists some } \delta > 0$$

$$\text{such that if } 0 < |x - y| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

- **In other words, if a point x is close to y, then the outpoint $f(x)$ is close to L.**

- **The limit definition *does not say* $f(x)$ *needs to exist!***

- **The special case when $f(x)$ exists and is equal to $\lim_{y \to x} f(y)$ is special, and will be discussed later.**

- **One can sometimes *visually check if a limit exists*, but the definition is very important too.**

- **It’s a tough one the first time, but is a thing of great beauty.**

1.2 Computing Basic Limits

- **Computing limits can be easy or hard.**

- **A limit captures what the function looks like *around a certain point*, rather than *at a certain point.***

- To compute limits, you need to ignore the function’s value, and only analyze what happens nearby.
- This is what the $\epsilon - \delta$ definition attempts to characterize.

Compute $\lim_{x \to 0} (x+1)^2$

Compute $\displaystyle\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}$

Compute $\lim\limits_{x \to 1} \dfrac{x^2 + 2x + 1}{x + 1}$

Compute $\lim\limits_{x \to 0} \dfrac{1}{x}$

Compute $\lim_{x \to 0} \left(\frac{\sqrt{x^4 + x^2}}{x} \right)$

1.3 Continuity

- **Sometimes, plugging into a function is the same as evaluating a limit. But not always!**

- **Continuity captures this property.**

f is continuous at x if

$$\lim_{y \to x} f(y) = f(x)$$

- **Intuitively, a function that is continuous at every point can be drawn *without lifting the pen.***

f is *continuous* if it is continuous at x for all x

Discuss the continuity of $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of $f(x) = \begin{cases} 2x+1 & \text{if } x \leq 1 \\ 3x^2 & \text{if } x > 1 \end{cases}$

- **Polynomials, exponential functions, and $\sin, \cos$ are continuous functions.**

- **Rational functions are continuous except at points where the denominator is 0.**

- **Logarithm is continuous, because its domain is only $(0, \infty)$.**

1.4 Squeeze Theorem

- **There are no one-size-fits-all methods for computing limits.**
- **One technique that is useful for certain problems is to *relate one limit to another.***
- **A foundational technique for this is based around the *Squeeze Theorem.***

Squeeze Theorem

Suppose $g(x) \leq f(x) \leq h(x)$ for some interval containing y.

$$\Rightarrow \lim_{x \to y} g(x) \leq \lim_{x \to y} f(x) \leq \lim_{x \to y} h(x)$$

- We will not prove this (or any, really) theorem.

- One classic application of the theorem is computing

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

- **Direct substitution (which one should be very wary of when computing limits) fails.**

- **Indeed, plugging in** $x = 0$ **yields**

$$\frac{\sin(0)}{0} = \frac{0}{0} = \text{DNE}$$

- **An instructive exercise is to show that, for**

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$$

$$\Rightarrow \lim_{x \to 0} \cos(x) \leq \lim_{x \to 0} \frac{\sin(x)}{x} \leq \lim_{x \to 0} 1$$

$$\Rightarrow 1 \leq \lim_{x \to 0} \frac{\sin(x)}{x} \leq 1$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

2. Theory of the Derivative

2.1 Tangent Lines

2.2 Definition of Derivative

2.3 Rates of Change

2.4 Derivative Rules

2.5 Higher Order Derivatives

2.6 Implicit Differentiation

2.7 L'Hôpital's Rule

2.8 Some Classic Theoretical Results

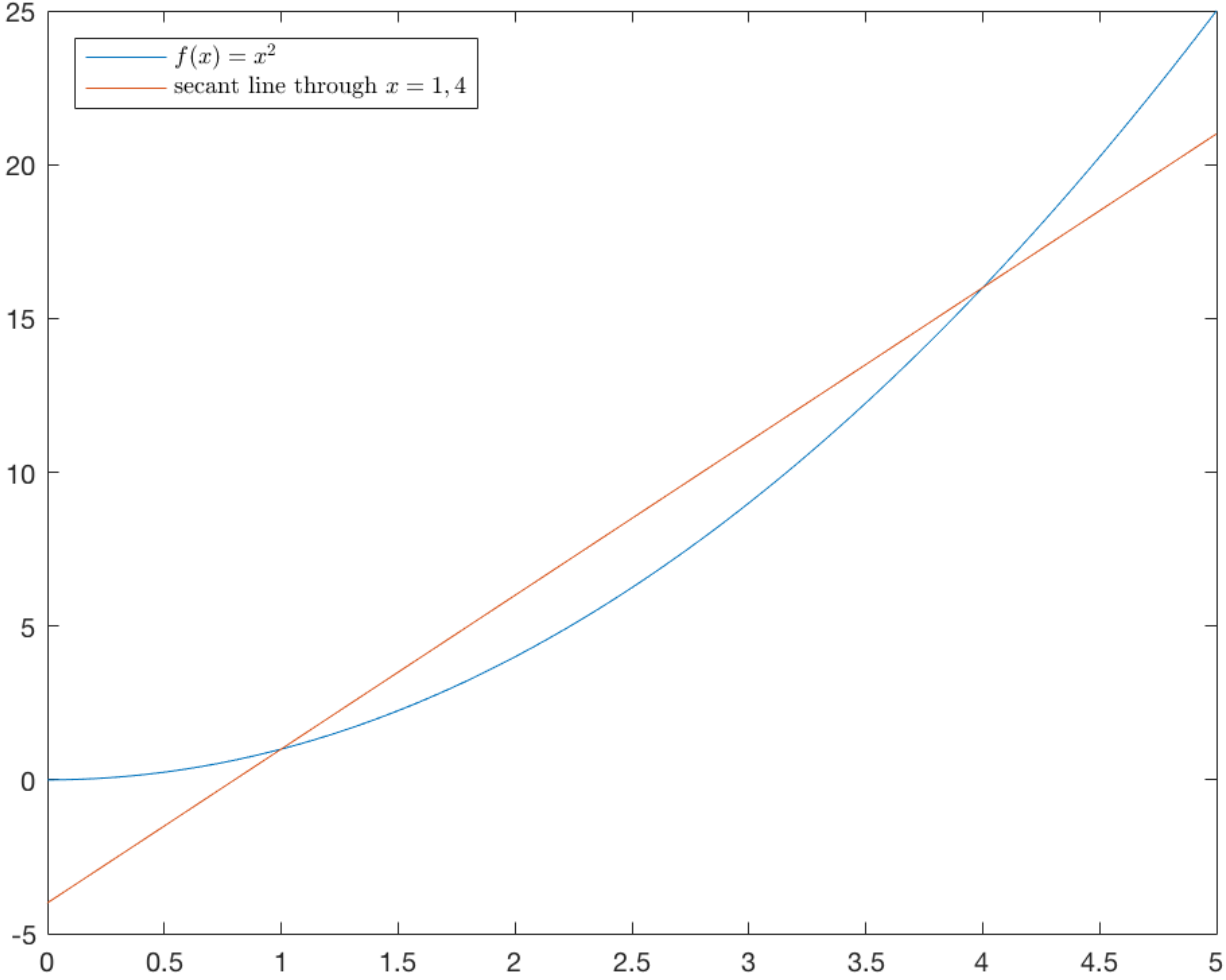
2.9 Derivatives of Inverse Functions

2.1 Tangent Lines

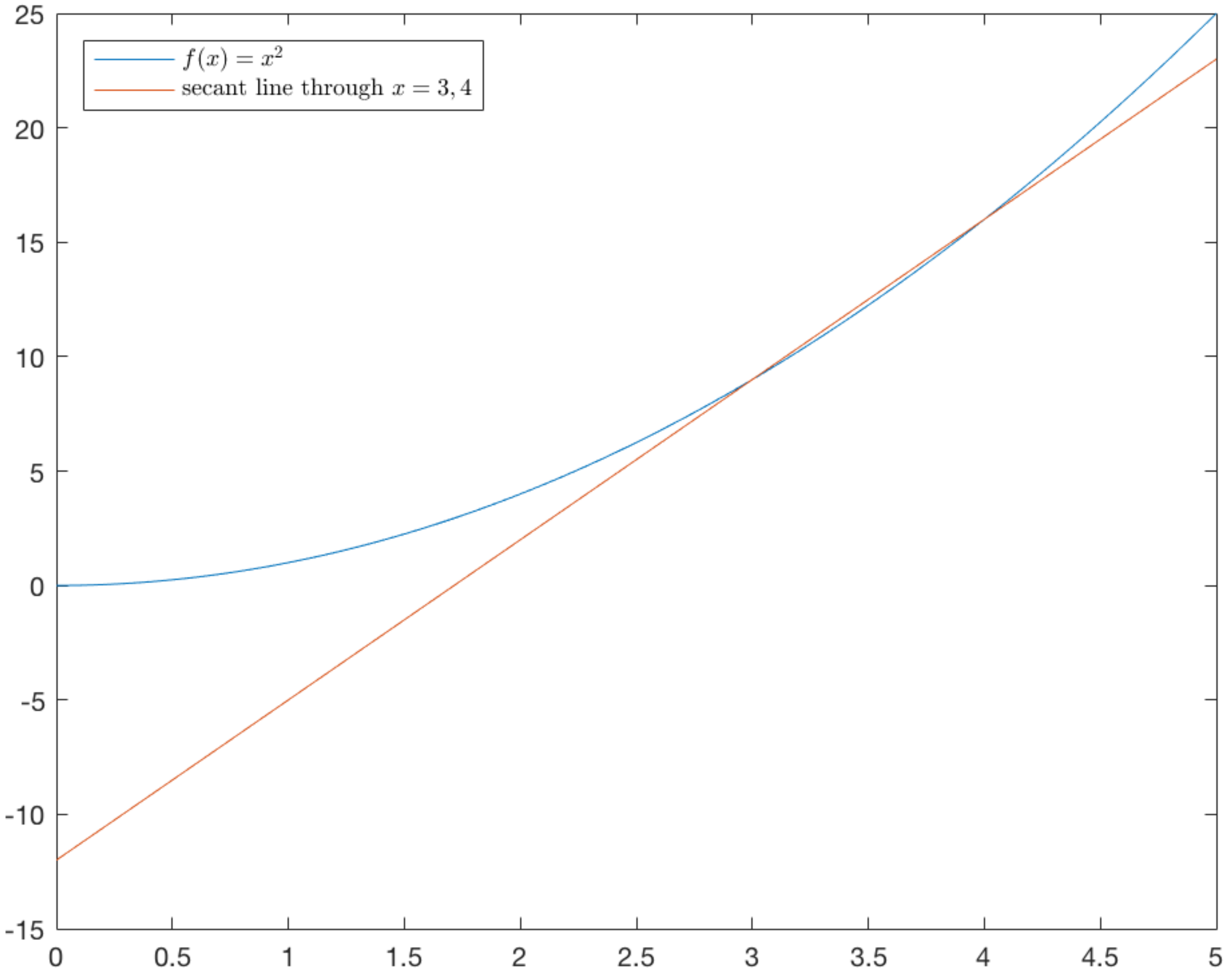
- Before we do any heavy lifting, let’s get a mental picture.
- One of the classical ideas behind calculus is the notion of *tangent line to* a function.
- This will motivate the limit definition of a derivative in the next submodule.

- **A line is *tangent* to a function if it intersects it only once.**
- **This is somewhat of a simplification, in that the line is allowed to intersect multiple times outside of some small interval, but that is more advanced and theoretical than we will get into.**

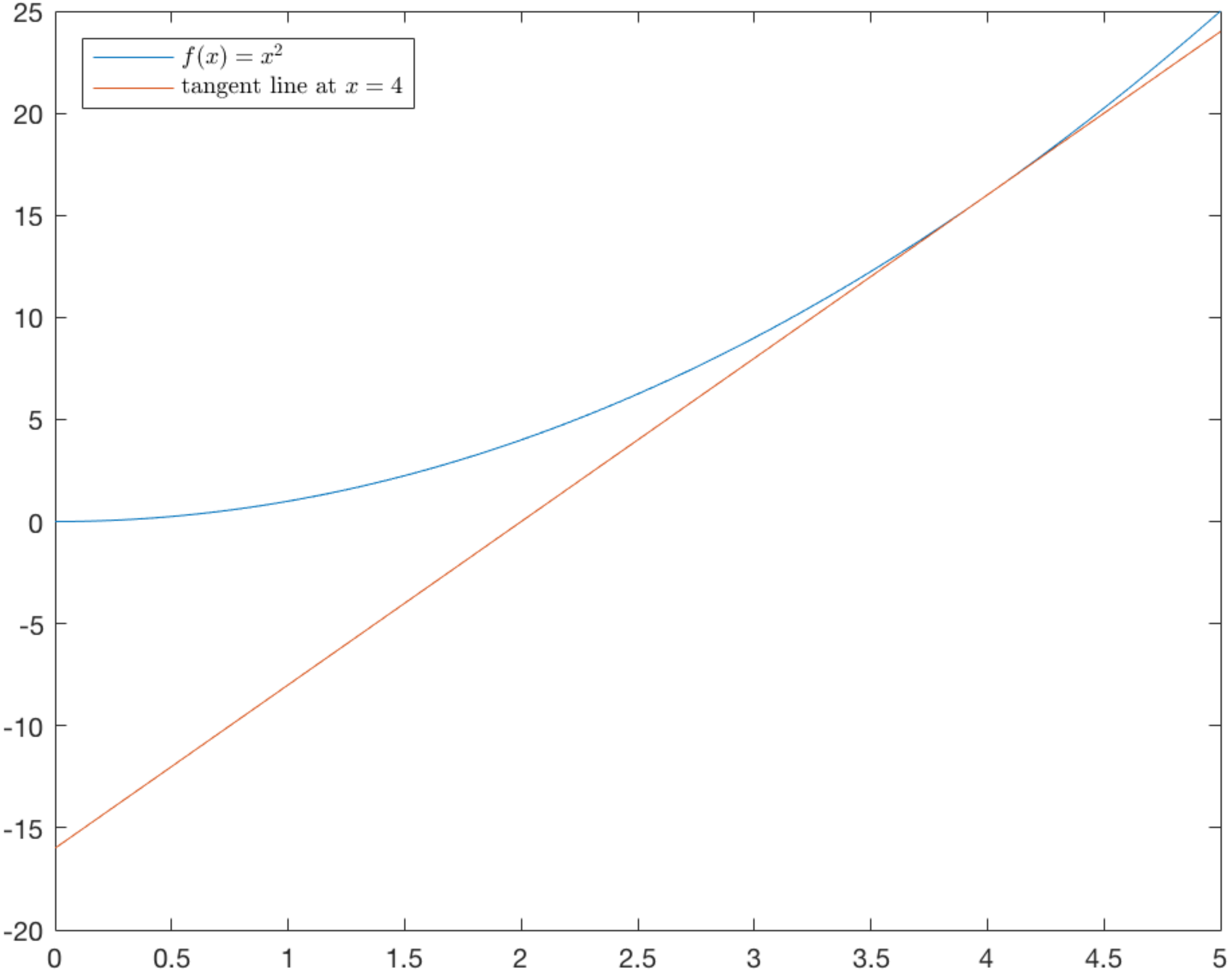
- **Tangent lines can be constructed as *limits of secant lines*, i.e. lines that intersect a function in exactly two points.**



$f(x) = x^2$
secant line through $x = 1, 4$
25
20
15
10
5
0
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



$f(x) = x^2$
secant line through $x = 3, 4$
25
20
15
10
5
0
-5
-10
-15
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



$f(x) = x^2$
tangent line at $x = 4$
25
20
15
10
5
0
-5
-10
-15
-20
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

- **The slopes of the secant lines are computed using the classical slope formula.**

- **If a line passes through:**

$$(a, f(a)), (b, f(b)),$$

then the slope of the line is

$$m = \frac{f(b) - f(a)}{b - a}$$

- **What is the slope of the tangent line? We need limits! This gives us the formal definition of the derivative!!!**

2.2 Definition of Derivative

- **The derivative is one of the two central objects in calculus.**

- **It measures *rate of change of a function.***

- In module 2, we will discuss methods for computing it, and discuss its geometric role.

- In module 3, we will use it as a tool to solve real-world problems.

- The slopes of the secant lines are computed using the classical slope formula.

- If a line passes through:

$$(a, f(a)), (b, f(b)),$$

then the slope of the line is

$$m = \frac{f(b) - f(a)}{b - a}$$

- **What is the slope of the tangent line? We need limits! This gives us the formal definition of the derivative!!!**

Let $f(x)$ be a function. The derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **So, the derivative is defined in terms of a *limit*.**
- **Notice that plugging in $h = 0$ yields 0/0, so we must be careful.**
- **In later submodules, we will develop some nice tricks and formulae.**

Let $f(x) = x$. Compute $f'(x)$.

Let $f(x) = x^2$. Compute $f'(x)$.

2.3 Rates of Change

- **Recall that for a general function $f(x)$, the slope of the secant line through a, b may be interpreted as the *average rate of change of* f *on* (a, b) .**

- **More precisely,**

$$\text{Average change of } f \text{ on } (a, b) = \frac{f(b) - f(a)}{b - a}$$

- **Let** $b = a + h$ **. Then we can say that**

$$\text{Average change of } f \text{ on } (a, a+h) = \frac{f(a+h) - f(a)}{h}$$

- **This looks an *awful lot like the definition of the derivative!***

- **Simply take the limit as** $h \to 0.$

- **This shrinks the interval in question to $\approx a$ alone.**

- **We conclude that**

$$\text{Instantaneous change of } f \text{ at } a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

- **So, derivatives are equal to *instantaneous rates of changes.***

Let $f(x) = x^2$. Find the average rate of change on $(-2, 2)$.

Find the instantaneous rate of change at $x = 2$.

Let $f(x) = cx,$ for some fixed constant c.

Find the average rate of change on $(-1, 3)$.

Find the instantaneous rate of change at $x = 1$.

2.4 Derivative Rules

2.4.1 Fundamental Derivative Rules

2.4.2 Chain Rule

2.4.3 Derivatives of Exponential and Logarithmic Functions

2.4.4 Trigonometric Derivatives

2.4.5 Derivatives of Inverse Trigonometric Functions

2.4.1 Fundamental Derivative Rules

- The limit definition of the derivative is not always very convenient.
- For practical purposes, it is nice to know exactly how this definition works for certain types of functions.
- The following results are not obvious, but we will not prove them in this course.

Derivative of a Constant

$$[a]' = 0$$

Derivative of a Polynomial

$$[x^a]' = ax^{a-1}, \text{ if } a \neq 0$$

Let $f(x) = x^4$. Compute $f'(x)$.

Derivative of a Sum

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

Let $f(x) = x^3 - 2x + 1$. Compute $f'(x)$.

Derivative of a Product

$$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Let $f(x) = (x+1)\sqrt{x}$. Compute $f'(x)$.

Derivative of a Quotient

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$$

Let $f(x) = \dfrac{2x-3}{x^4+1}$. Compute $f'(x)$.

2.4.2 Chain Rule

- **The *chain rule* is arguably to most foundational property of derivatives.**

- **It tells how to compute the derivation of a *composition of functions, i.e. a function of the form***

$$f(x) = g \circ h(x) = g(h(x))$$

$$[g \circ h(x)]' = [g' \circ h(x)] \cdot h'(x)$$

i.e. $[g(h(x))]' = [g'(h(x))] \cdot h'(x)$

Compute the derivative of $f(x) = (3x+2)^{-2}$

Compute the derivative of $f(x) = (x^2 + 2)^3\sqrt{4x+1}$

- **What if we are considering just plain old $f(x)$ that does not appear to have the form of a composition?**

- **Well, we may always write:**

$$f(x) = f(g(x)), g(x) = x$$

- **Taking derivatives and applying the chain rule yields:**

$$
\begin{aligned}
f'(x) &= f'(g(x)) \cdot g'(x) \\
&= f'(x) \cdot 1 \\
&= f'(x)
\end{aligned}
$$

- **This emphasizes that we are always *implicitly using the chain rule*, even when it might appear there is no composition.**

- **It may be necessary to apply the chain rule *iteratively:***

$$[f(g(h(x)))]' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Compute the derivative of $f(x) = (\sqrt{x^2 - 1} - 2)^{-1}$

2.4.3 Derivatives of Exponential and Logarithmic Functions

- **The exponential function with base e is rather simple from the calculus standpoint.**

$$[e^x]' = e^x$$

- **More general exponential functions have a slightly more delicate formula:**

$$[a^x]' = a^x \cdot \ln(a)$$

Compute $\dfrac{d}{dx}\left[e^{2x}\right]$

Compute $\dfrac{d}{dz}\left[e^{z^2} + 4z\right]$

Compute $\dfrac{d}{dx}\left[xe^{x^3}\right]$

- **By contrast, logarithms are somewhat trickier. Derivatives of logarithms do not stay as logarithms:**

$$[\ln(x)]' = \frac{1}{x}$$

$$[\log_a(x)]' = \frac{1}{\ln(a)x}$$

Compute $\dfrac{d}{dx}\left[\ln(x^2)\right]$

Compute $\dfrac{d}{dy}\left[\ln(y+y^4)\right]$

Compute $\dfrac{d}{dx}\left[\ln(e^{2x+1})\right]$

2.4.4 Trigonometric Derivatives

- **The trigonometric functions all have derivatives that related to other trigonometric functions.**

- **The foundational ones are:**

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

Compute $\dfrac{d}{dx}[\cos(x^2+1)]$

- **We can use decompose into $\sin(x), \cos(x)$ and then use the quotient rule to compute the derivatives of the remaining trigonometric functions.**

- **We will prove that**

$$\frac{d}{dx}[\tan(x)] = \sec(x)^2$$

- **Proving the rest of the trigonometric derivatives in a similar way is an *excellent* exercise.**

$$\frac{d}{dx}[\cot(x)] = -\csc(x)^2$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

Compute $[\tan(\theta + 1)]'$

Let $f(x) = \csc(x^2)$. Compute $f'(x)$.

2.4.5 Derivatives of Inverse Trigonometric Functions

- **The inverse trigonometric functions also have derivatives that ought to be committed to memory for the CLEP exam.**

- We will see in a later submodule how to *prove these formulae* starting from a general principle for derivatives of inverse functions.

- Until then, we will take the basic rules for granted.

$$\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}$$

Compute the derivative of $f(x) = \sin^{-1}(x^3 + 1)$.

Compute the derivative of $f(x) = \sec^{-1}(e^x)$.

Compute the derivative of $f(x) = \arctan(\sin(x))$.

2.5 Higher Order Derivatives

- **It is possible to *differentiate a function multiple times.***

- **The result of iterated differentiation is called a *higher order derivative.***

- **First derivative:** $f'(x)$
- **Second derivative:** $f''(x)$
- **Third derivative:** $f^{(3)}(x)$
- n^{th} **derivative:** $f^{(n)}(x)$

Let $f(x) = x^3 - 4x + 1$. Compute $f', f'', f^{(3)}$

Let $f(x) = e^{x^2}$. Compute $f', f'', f^{(3)}$

Let $f(x) = \sin(2x)$. Find all values x for which $f'' = 1$.

Let $f(x) = \ln(g(x))$. Compute $f''(x)$ in terms of $g(x)$.

2.6 Implicit Differentiation

- **All of our work has so far focused on differentiating a function where there was only one variable:**

$$f(x) = \text{ something depending on } x$$

- **We may at times come across an expression involving both** x and y

- **In this case,** y **is** ***implicitly*** **a function of** x **.**

- **We differentiate in this case by noting that:**

$$\frac{d}{dx}[y] = y',$$
$$\frac{d}{dx}[x] = 1.$$

- **This allows us to differentiate both sides of an expression, and solve for the resulting y'.**

Solve for y' : $2xy + y^2 = 1$

Solve for y' : $\sqrt{y+1} + x^2 = y$

Solve for y' : $e^{xy-1} = x^2$

2.7 L’Hôpital’s Rule

- Recall that certain quantities are not well-defined: $$\frac{0}{0}, \frac{\infty}{\infty}$$
- These *indeterminate forms* sometimes arise when taking limits of rational functions, i.e. computing limits of the form $$\lim_{x \to y} \frac{f(x)}{g(x)}$$

- **In these special indeterminate cases, one can apply manipulations to $\dfrac{f(x)}{g(x)}$ in order to compute the limit.**

- **Another, slicker, trick is to use *L'Hôpital's rule,* which we state loosely as**

$$\text{If } \lim_{x\to y} f(x) = \lim_{x\to y} g(x) = 0 \text{ or } \pm\infty,$$

$$\text{then } \lim_{x\to y} \frac{f(x)}{g(x)} = \lim_{x\to y} \frac{f'(x)}{g'(x)}, \text{ provided the second limit exists.}$$

Compute $\displaystyle\lim_{x\to\infty} \frac{x+1}{3x-1}$

Compute $\lim_{x \to 0} \frac{e^x - 1}{x}$

Compute $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

Compute $\lim_{x \to 0} \frac{\sin(x)}{x}$

Compute $\lim_{x \to 0} \frac{\cos(x)}{x}$

2.8 Some Classic Theoretical Results

- This is not a course in theory, but certain results are important for the CLEP.

- Proving these would be an *excellent* learning experience, but is certainly not necessary. A basic understanding would suffice for the CLEP exam.

Differentiability Implies Continuity

Suppose a function f is differentiable at a point x.

Then f is continuous at x.

Rolle's Theorem

Suppose a function f is differentiable on an interval (a, b).

If $f(a) = f(b)$, then there is a point $c, a < c < b$ such that $f'(c) = 0$.

2.9 Derivatives of Inverse Functions

- **We have seen already some special examples of derivatives of inverse functions: inverse trigonometric functions.**

- **Recall that the inverse function of $f(x)$ is a function $f^{-1}(x)$ satisfying**

$$f^{-1} \circ f(x) = f \circ f^{-1}(x) = x.$$

Suppose $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$. Then

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}.$$

Suppose $f(x) = x^3 + x - 1$. Compute the derivative of f^{-1} at $x = 1$.

Suppose $f(x) = e^x + 2x + 3$. Compute the derivative of f^{-1} at $x = 4$.

3. Applications of the Derivative

3.1 Plotting with Derivatives

3.2 Rate of Change Problems

3.3 Some Physics Problems

3.1 Plotting with Derivatives

3.1.1 Increasing and Decreasing Functions

3.1.2 Extrema

3.1.3 Concavity

3.1.1 Increasing and Decreasing Functions

- Recall that the derivative of a function corresponds to *the rate of change of a function.*

- If the rate of change is positive, we say the function is increasing.

- **If it is negative, we say it is decreasing.**

- **We can quantify this by discussing the sign of the derivative.**

- Let $f(x)$ be a function.
- If $f'(x_0) > 0$, then $f(x)$ is increasing at x_0 .
- If $f'(x_0) < 0$, then $f(x)$ is decreasing at x_0 .
- If $f'(x_0) = 0$, no definitive conclusion can be made without further analysis.

- **Note that a function may not even be differentiable and still be increasing/ decreasing.**

Let $f(x) = \sin(x)$.

Is f increasing, decreasing at $x = 0, \dfrac{\pi}{4}, \pi$?

Let $f(x) = x - e^x$.

Is f increasing, decreasing at $x = -1, 0, 1$?

Let $f(x) = x^3 - 6x^2 + 3x - 2$.

Find where f is increasing and decreasing.

3.1.2 Extrema

- **We have seen that:**

$$f'(x) > 0 \Rightarrow f(x) \text{ increasing}$$
$$f'(x) < 0 \Rightarrow f(x) \text{ decreasing}$$

- **So, what about if** $f'(x) = 0$?

- **This is perhaps the most exciting aspect of differential calculus, and is a major reason it is studied by all kinds of people.**

- **Suppose** $f'(x) < 0, x < x_0$

  $f'(x_0) = 0$

  $f'(x) > 0, x > x_0$
- **Then** f ***transitions from decreasing to increasing at*** $x = x_0$.
- **This means** $f(x)$ **has a** ***local minimum at*** x_0.

Show $f(x) = x^2$ has a local minimum at $x = 0$.

- **Suppose** $$f'(x) > 0, x < x_0$$ $$f'(x_0) = 0$$ $$f'(x) < 0, x > x_0$$
- **Then** f ***transitions from increasing to decreasing at*** $x = x_0$.
- **This means** $f(x)$ **has a *local maximum at*** x_0.

Show $f(x) = \cos(x)$ has a local maximum at $x = 0$.

- A classic calculus problem is to find the *local extrema (minima and maxima) of a function.*

- To do so, set the derivative equal to 0 and check how the derivative changes sign.

- Not every place the derivative equals zero is a local extrema, however.

Find the local extrema of $f(x) = \sin(x)$.

Find the local extrema of $f(x) = x^3$.

3.1.3 Concavity

- We saw in the previous submodule that the properties of a function being *increasing, decreasing,* and its *local extrema* are governed by its first derivative, $f'(x)$.

- A more subtle notion, *concavity*, is governed by the second derivative, $f''(x)$.

- **A loose metaphor is in order: when plotting a function, try pouring water on it.**

- **If the function holds the water, it is *concave up* there.**

- **If it doesn't hold water, it is *concave down* there.**

- A function $f(x)$ is *concave up* wherever $f''(x) > 0.$

- A function $f(x)$ is *concave down* wherever $f''(x) < 0.$

Determine the concavity and sketch $f(x) = x^3 - 12x + 1$

- **The second derivative can also be used to classify *critical points*, i.e. points where** $f'(x) = 0$.

- ***Second Derivative Test:***

Suppose $f'(x_0) = 0$.

If $f''(x_0) > 0, \ x_0$ is a local maximum.

If $f''(x_0) < 0, \ x_0$ is a local minimum.

Use the second derivative test to determine the nature of the critical points of $f(x) = 2\cos(4\pi x)$.

3.2 Rate of Change

- **A classic application of the derivate is to compute the *instantaneous rate of change* of a quantity.**

- **Recall that the instantaneous rate of change of $f(x)$ at $x = a$ is $f'(a)$.**

- **In contrast, the average rate of change of $f(x)$ on the interval $[a, b]$ is** $\dfrac{f(b) - f(a)}{b - a}$

Let $f(x) = x^4 - x^2 + 2$.

Find the average rate of change of f on $[0, 2]$.

Find the instantaneous rate of change of f at $x = 0, 2$.

Let the size of a population be given by $P(x) = 100 \cdot 2^{\frac{x}{100}}$.

Find the average rate of change of f on $[0, 200]$.

Find the instantaneous rate of change of f at $x = 0, 200$.

Let the value of an investment be $P(t) = 10 \cdot e^{\frac{x}{15}}$.

When will the instantaneous rate of growth of the investment first exceed 300?

3.3 Some Physics Problems

- **Another classic application of derivatives is related to the physical laws of motion.**

- **In this context, a one-dimensional particle's *position* is given by a function** $p(t)$

- **Related quantities, like its *velocity* $v(t)$ and its acceleration $a(t)$ may be understood as certain derivatives of the position.**

- **Let the position of a particle be given by $p(t)$.**

- **The velocity of the particle is given by $v(t) = p'(t)$.**

- **The acceleration of the particle is given by $a(t) = v'(t) = p''(t)$.**

- **So, velocity is the rate of change of position, and acceleration is the rate of change of velocity.**

Suppose a one-dimensional particle has position function $p(t) = 4 - 10t^2$.

When is the particle moving with velocity -10?

What is the acceleration of the particle?

Suppose a one-dimensional particle has position $p(t) = \ln(t^4 + t^2), t > 0$. Show that the particle never changes direction.

4. Theory of the Integral

4.1 Antidifferentiation

4.2 The Definite Integral

4.3 Riemann Sums

4.4 The Fundamental Theorem of Calculus

4.5 Fundamental Integration Rules

4.6 U-Substitutions

4.1 Antidifferentiation

- **We will begin our study of the *integral* by discussing antidifferentiation.**

- **As you might expect, this is the process of undoing a derivative.**

Let $f(x)$ be a function. A function $F(x)$ is an *antiderivative* of $f(x)$ if $F'(x) = f(x)$.

Let $f(x) = 1$. Find an antiderivative of $f(x)$.

Let $f(x) = \sin(x)$. Find an antiderivative of $f(x)$.

Let $f(x) = e^{2x}$. Find an antiderivative of $f(x)$.

- **Notice that I am asking to find *an* antiderivative, not *the* antiderivative.**

- **That is because antiderivatives are not unique!**

- **Indeed, if $F(x)$ is an antiderivative for $f(x)$, then $F(x) + C$ is also an antiderivative for any constant C .**

4.2 Definite Integral

- **We will relate the antiderivative to another important object: the *definite integral.***

- **This is a quantity that depends on two endpoint values,** a, b**, and a function,** $f(x)$.

- **It is written as** $\displaystyle\int_a^b f(x)dx.$

- **The definite integral has many important interpretations.**

- **The most significant for us is *area under the curve* $f(x)$ *from* a *to* b.**

- **It is not obvious how to compute the area under the curve of a general function—this is the power of calculus!**

- **Let's start with simple things.**

Compute $\int_0^2 3dx$.

Compute $\int_{-1}^{1} x dx$.

Compute $\int_0^5 2x dx$.

4.3 Riemann Sums

4.3.1 Riemman Sums Part I

4.3.2 Riemman Sums Part II

4.3.1 Riemann Sums Part I

- **We have seen how to compute definite integrals of functions with certain simple properties, by exploiting well-known area formulas from geometry.**

- **What can we do in general? Not much yet.**

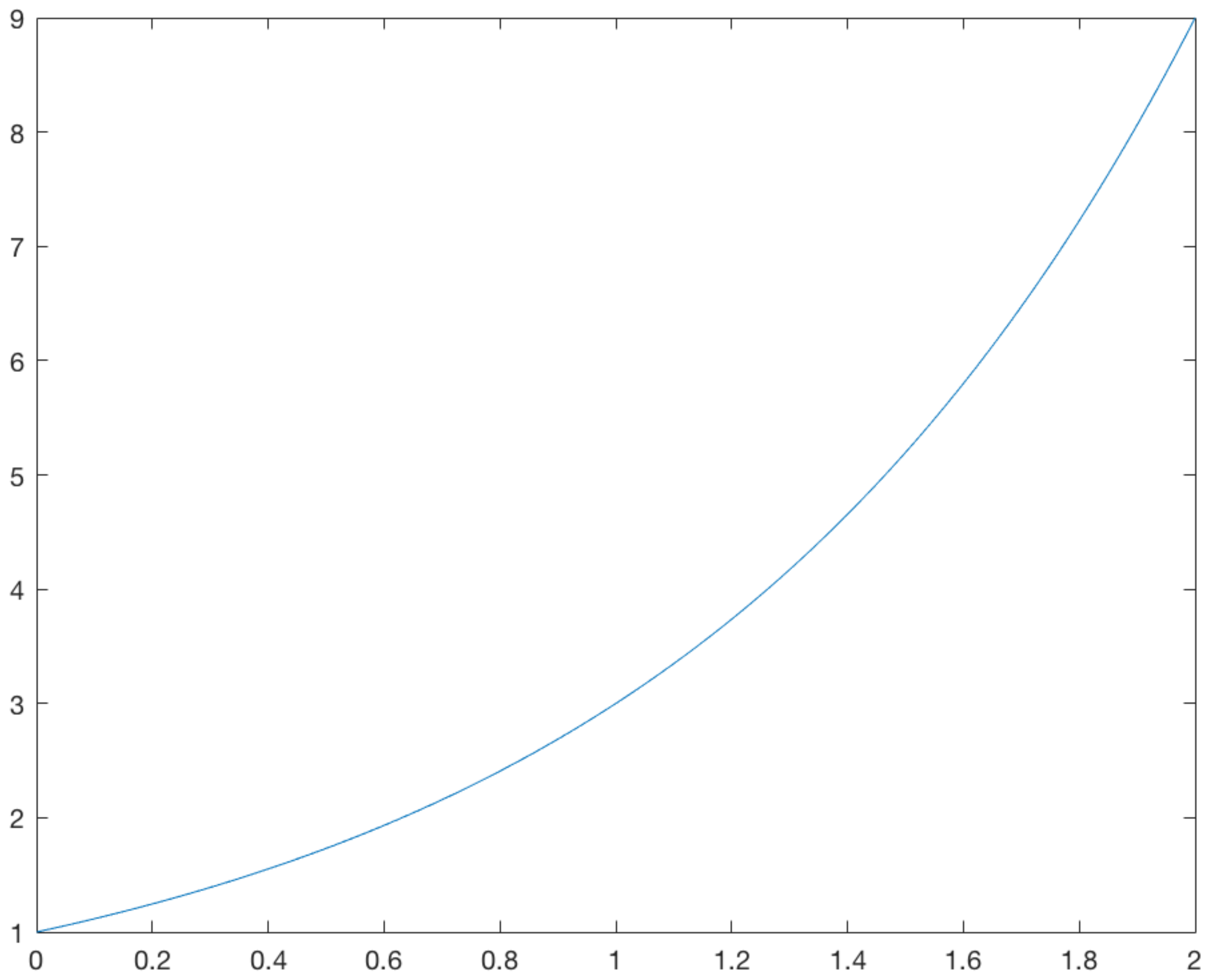
- **We can, however, approximate the area with *Riemann sums*.**

- A Riemann sum approximates an integral by covering the area beneath the curve with rectangles.

- The areas of the these rectangles are more easily computed.

- This is because the width of these rectangles is fixed, and the height is given by the value of the function at a given point.

- Programmers—try coding this! It's a classic.



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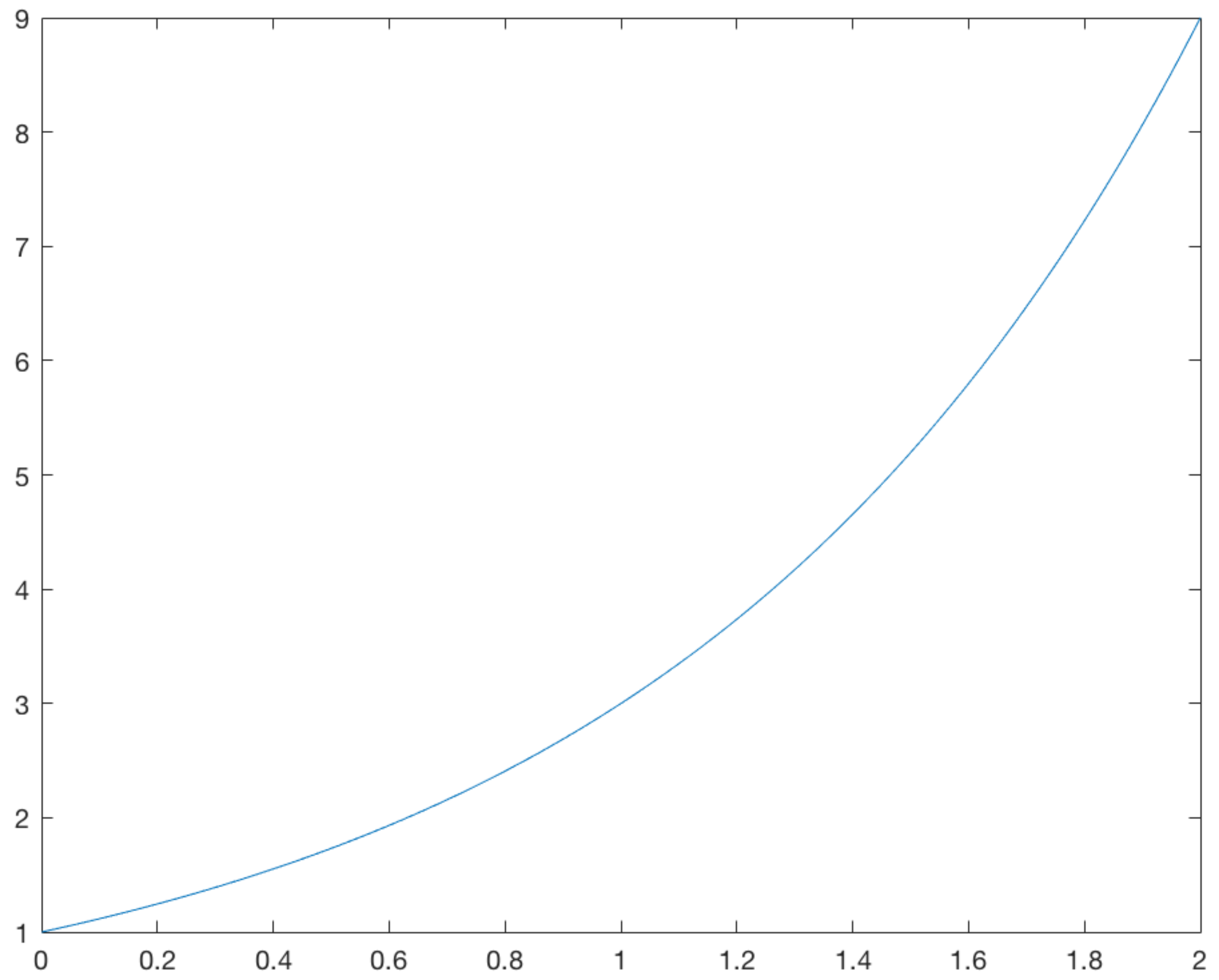

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Estimate $\displaystyle\int_0^4 x^2 dx$ with left and right Riemann sums of width 1.

4.3.2 Riemann Sums Part II

Estimate $\displaystyle\int_{-1}^{2}(1-x)dx$ with left and right Riemann sums of width 1.

4.4 The Fundamental Theorem of Calculus

- The *fundamental theorem of calculus* is a classic result.
- It links the derivative and the integral.

- **We will not prove it, though we will use it extensively to compute areas under curves.**

- **Intuitively, definite integrals can be computed by evaluating an antiderivative at the endpoints of integration.**

Suppose f has antiderivative $F(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Compute $\int_0^2 x^2 dx$.

Compute $\displaystyle\int_0^{2\pi} \cos(x)dx$.

- **When no particular endpoints are specified, the FTC suggests that we write** $$\int f(x) = F(x) + C$$

- **Here,** C **is an arbitrary constant.**

Compute $\int e^{3x}\,dx$.

Compute $\displaystyle\int \frac{2}{x}\,dx$.

- Another way to interpret the FTC is as stating that the derivative and integral *undo each other.*

- More precisely,

$$\frac{d}{dx}\int f(x)dx = f(x)$$

- This is valid for *all* $f(x)$ likely to appear on the CLEP exam.

4.5 Basic Integral Rules

4.5.1 Basic Integral Rules I

4.5.2 Basic Integral Rules II

4.5.1 Basic Integral Rules I

- Using the FTC, we see that all the basic *derivative rules* apply, in an inverted way, to *integrals.*

- This means that to know the basic rules for integrals, it suffices to know the basic rules for derivatives.

For constants a, b, $\int (af(x) + bg(x))dx = a \int f(x)dx + b \int g(x)dx$

$$\text{If } n \neq -1, \int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\text{If } n = -1, \int x^n dx = \ln(x) + C$$

Compute $\int (x^3 + 2x - 3)dx$

Compute $\displaystyle\int (x^{-1} + 1)dx$

$$\int e^x dx = e^x + C$$

Compute $\displaystyle\int \left(\frac{-4}{x} + 2e^{x} \right) dx$

4.5.2 Basic Integral Rules II

Compute $\int (\sin(x) + x^2)dx$

$$\int \sin(x)dx = -\cos(x) + C$$

$$\int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\tan(x) + \sec(x)| + C$$

Compute $\int (\tan(\theta) - \cos(\theta)) d\theta$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C$$

$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}(x) + C$$

Compute $\displaystyle \int \frac{-3dx}{\sqrt{4-4x^2}}$

Compute $\displaystyle\int \frac{dy}{2|y|\sqrt{y^2 - 1}}$

4.6 U-Substitutions

- There are many more sophisticated types of integration methods.
- These include those based on the product rule (integration by parts), special properties of trigonometric functions (trig. substitutions), and those based on tedious algebra (partial fraction decomposition).

- **We focus on a method based on the *chain rule*.**

- **Recall that to compute the derivative of a composition of functions, we use the chain rule:**

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

- **According to the FTC,**

$$\int \frac{d}{dx}f(g(x)) = f(g(x)) + C.$$

- **Hence,**

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Compute $\displaystyle\int xe^{x^2}\,dx$

Compute $\displaystyle\int \cos(4x+1)dx$

Compute $\int x^3 \sqrt{x^4+1}\,dx$

Compute $\int \tan(x)dx$

5. Applications of the Integral

5.1 Area Under Curves

5.2 Average Value

5.3 Growth and Decay Models

5.4 Return to Physics Problems

5.1 Area Under Curves

5.1.1 Area Under Curves Part I

5.1.2 Area Under Curves Part II

5.1.1 Area Under Curves Part I

- **One of the classic applications of the integral is to compute areas.**

- **We defined the integral to be the area under the curve:**

$$\int_a^b f(x)dx = \text{area under } f \text{ from } a \text{ to } b$$

Compute the area between x^2 and the x-axis from $x = 0$ to $x = 4$.

- **By convention, areas are positive. So if $f(x)$ is negative on $[a, b]$,**

$$-\int_a^b f(x)dx = \text{area under } f \text{ from } a \text{ to } b$$

- **Geometry also informs the following result:**

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \text{ if } a < c < b.$$

5.1.2 Area Under Curves Part II

- **One can also compute *the area between two curves* with the integral.**

- **Suppose** $f(x) \geq g(x)$ on $[a, b]$.

- **The area between** $f(x), g(x)$ **on** $[a, b]$ **is**

$$\int_a^b (f(x) - g(x))dx.$$

Compute the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on $\left[0, \frac{\pi}{4}\right]$.

Compute the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on $\left[0, \dfrac{\pi}{2}\right]$.

Compute the area between $f(x) = x$ and $g(x) = x^2$ on $[0, 1]$.

5.2 Average Value

- **The integral also has an interpretation as *the average of a function's value over an interval.***

- **This makes sense if you recall that an integral** $\int_a^b f(x)dx$ **is approximated by Riemann sums, which are just rectangles whose heights are the function's values.**

- **The following statement is also worth considering for constant functions, which clearly have constant average.**

- **The average value of $f(x)$ on the interval $[a, b]$ is**

$$\frac{1}{b-a}\int_a^b f(x)dx$$

- **So, we compute the integral, then divide by the length of the interval.**

- **Interpreting the integral as a sum, this bears resemblance to how the average of a finite set of numbers is computed.**

Compute the average value of $\ln(x)$ on $[1, 100]$.

Compute the average value of $\dfrac{1}{x^2+1}$ on $[-1, 1]$.

5.3 Growth and Decay Models

- **The integral allows us to solve certain basic *differential equations.***

- **Differential equations is a huge world of mathematics, and a subject with many problems without solutions.**

- It is a field of active research, including with computers.

- We will focus on an simple differential equation on the CLEP exam.

- **Consider the equation in terms of the unknown function $y(x)$:**

$$y' = ky, \quad \text{some constant } k.$$

- **To solve for $y(x)$, we do some algebra and recall the chain rule and formula for the derivative of $\ln(x)$.**

$$
\begin{aligned}
y' &= ky \\
\Leftrightarrow \frac{y'}{y} &= k \\
\Leftrightarrow \int \frac{y'}{y} dx &= \int k dx \\
\Leftrightarrow \ln(y) &= kx + C \\
\Leftrightarrow y(x) &= Ce^{kx}
\end{aligned}
$$

- **If $k > 0$, we have exponential growth.**

- **If $k < 0$, we have exponential decay.**

- **The constant $C > 0$ is determined based on details in the problem, noting that $y(0) = C$.**

Suppose $y' = 2y,\ y(0) = 100$. Find $y(5)$.

Suppose $y' = -5y,\ y(0) = 1000$. Find x such that $y(x) = 1$.

5.4 Return to Physics

- **Just as we used derivatives to understand *position, velocity, and acceleration* of a one-dimensional particle, so too can we use integrals.**

- **We simply follow the fundamental theory of calculus:**

$$\int_a^b f'(x)dx = f(b) - f(a).$$

- **Let $v(t)$ be the instantaneous velocity of a particle at time t.**

- **The position of the particle at time t is $p(t)$ and satisfies**

$$
\begin{aligned}
p'(t) &= v(t) \\
\Rightarrow \int_a^b p'(t)dt &= \int_a^b v(t)dt \\
\Rightarrow p(b) &= p(a) + \int_a^b v(t)dt.
\end{aligned}
$$

Suppose a particle has instantaneous velocity $v(t) = -t^2$ and initial position $p(0) = 10$. Find $p(5)$.

- **A similar game can be played with acceleration:**

$$
\begin{aligned}
v'(t) &= a(t) \\
\Rightarrow \int_{t_0}^{t_1} v'(t)dt &= \int_{t_0}^{t_1} a(t)dt \\
\Rightarrow v(t_1) &= v(t_0) + \int_{t_0}^{t_1} a(t)dt.
\end{aligned}
$$

- **With this formula for velocity, we can keep going and get a formula for position.**

Suppose a particle has instantaneous acceleration $a(t) = -10$, initial position $p(0) = 0,$ and initial velocity $v(0) = 0$. Find $p(5)$.