Mechanics Corner

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Comments on A Calculus Challenge

For those who have read the Challenge problem, you will recall that I drew a circle with a flat side as the basis for five questions about this area. Let me begin by posting my results:

area:

$A = R^2 (\pi - \alpha + \sin\alpha\cos\alpha)$

centroid:

$x_c = 0$

$y_c = \frac{2}{3}\frac{R\sin\alpha\left(\cos^2\alpha - 1\right)}{(\pi-\alpha+\sin\alpha\cos\alpha)}$

area moments of inertia:

$I_{xxo} = \frac{1}{4}R^4 (\pi - \alpha - \sin\alpha\cos\alpha + 2\sin\alpha\cos^3\alpha)$

$I_{yyo} = \frac{1}{12}R^4 (3\pi - 3\alpha - 2\sin\alpha\cos^3\alpha + 5\sin\alpha\cos\alpha)$

$I_{xyo} = 0$

$J_{oo} = \frac{1}{6}R^4 (3\pi - 3\alpha + \sin\alpha\cos\alpha + 2\sin\alpha\cos^3\alpha)$

These are the results that I obtained (assuming I have not made any typos here!).

If I counted correctly, we had five people attempt some part of the challenge out of just under 2000 people who looked at it. Thus, the participation was 5/1972,just slightly over 1/4% of those who looked at the problem. I'll let you draw your own conclusions as to what that means.

One of the interesting things, to me, was the way that people attacked the problem. Everyone agreed that the area needed to be divided and considered in several parts, but there was considerable variety in the way people divided the area. Recall the basic definitions for the items of interest:

$A = \iint_A dA$

$x_c = \frac{1}{A}\iint_A x\, dA$

$y_c = \frac{1}{A}\iint_A y\, dA$

$$I_{xxo} = \iint_A y^2 dA$$

$$I_{yyo} = \iint_A x^2 dA$$

It is immediately evident that there are two common aspects to all of these integrations: (1) each requires that we express dA, and (2) each requires that we express the limits of integration in such a manner as to describe the enclosed area. Almost everyone set up the integrals in rectangular coordinates, but I'd like to suggest to you that this is not the best way to approach the problem.

When we look at the geometry, it is evident that the boundary consists of two parts: (1) a circular arc, and (2) a straight line. The existence of the circular arc suggests a description in polar coordinates, $r = R =$ constant. If we go that way, is it possible to describe the straight line with a reasonable amount of effort? The answer is "yes."

The particular straight line involved is

$$y = R\cos\alpha$$

Consider a radial line, starting at the origin and intersecting the straight part of the boundary. How long is that line? If θ is measured clockwise from the $y-$axis, then the line at angular position θ has length L,

$$L\cos\theta = y = R\cos\alpha$$

or

$$L = R\frac{\cos\alpha}{\cos\theta}$$

If the area is divided into two parts: (1) the triangle between $\theta = -\alpha$ and $\theta = +\alpha$ with the straight line for the upper edge, and (2) the remainder of the circle with boundary $r = R$, then the integration for the area is simply

$$A = \int_{-\alpha}^{+\alpha}\left(\int_0^{R\frac{\cos\alpha}{\cos\theta}} r\,dr\right)d\theta + \int_{+\alpha}^{2\pi}\left(\int_0^{R} r\,dr\right)d\theta$$

The first double integral covers the triangular area, while the second pair covers the remainder of the circle. Look particularly at the set-up of the interior integral in the first term. This extends the integration from the origin up to the straight line at any angle θ. For the calculation of the other items, where x and y factors are required in the integrand, they can simply be substituted with $x = r\sin\theta$ and $y = r\cos\dot{\theta}$. I would urge you to always consider polar coordinates when a circular boundary is involved; it is often the simplest way to work with things.

As a final observation, only one person noted that $x_c = 0$ by symmetry. This was a free give-away, an easy question. Everyone should have gotten that had you been seriously thinking about the problem.