Mechanics Corner

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Balls Rolling On An Incline

A Problem Where I Learned Something New

1 Introduction

In previous articles, I have mentioned another web site called Physics Forums (PF) where people post problems for which they need help. In this note, I want to present to you one such problem and it solution, along with a new insight that came from another commenter at PF, one of the advisory folk on that site. At first, I thought the adviser was wrong, but it turns out that he was correct and had something new that I had never seen before. Here is the problem.

1.1 Problem Statement

A thin wall spherical shell with a mass of 0.605 kg and a radius of 0.0402 m is released from rest at the top of an incline. The spherical shell rolls down the incline without slipping. The spherical shell takes 7.49 s to get to the bottom of the incline.

A solid sphere with mass of 0.127 kg and a radius of 0.1123 m is released from rest at the top of the same incline. The solid sphere rolls down the incline without slipping. How much time does it take for the solid sphere to reach the bottom of the incline.

Note that —

Thin spherical shell	$I = \frac{2}{3}MR^2$
Solid sphere	$I = \frac{2}{5}MR^2$

1.2 Discussion

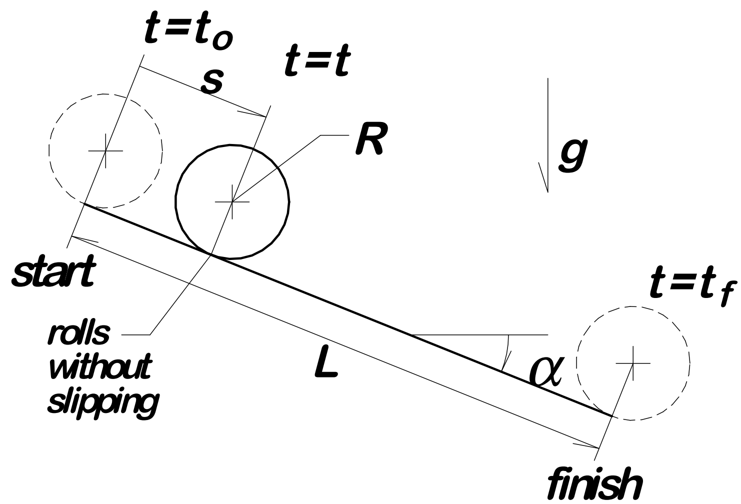
The original problem statement is above. Note what is given, and perhaps more importantly, what is not given. In particular, we are not given

1. The time for the solid sphere to reach the bottom – this is the item to be determined;
2. The angle of the incline;
3. The length of the incline;
4. The local value of g, the acceleration of gravity.

The last three items are things that we might expect to have given in such a problem, but here they are not. This is the major difficulty in this problem, and the solution must find a way to work around this missing information.

2 Analysis

The figure below shows the situation, including the missing information L and α, followed by a short table that summarizes the available information.



Ball on a Ramp

Given Data:

Data Item		Shell	Solid
Ball Mass	M	$= 0.605$ kg	$= 0.127$ kg
Ball Radius	R	$= 0.0402$ (m)	$= 0.1123$ (m)
MMOI	I	$= \frac{2}{3}MR^2$ (kg-m^2)	$= \frac{2}{5}MR^2$ (kg-m^2)
Final Time	t_f	$= 7.49$ s	=??

Equation of Motion

The kinetic energy of the rolling ball is T

$$T = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}I\omega^2 \tag{1}$$

but the rolling constraint is

$$s = R\theta \tag{2}$$

or

$$\dot{s} = R\omega \tag{3}$$

Keeping s as the generalized coordinate gives

$$T = \frac{1}{2}\dot{s}^2\left(M + \frac{I}{R^2}\right) \tag{4}$$

after the constraint is substituted.

If we take zero gravitational potential energy at the starting point, then the potential energy V is

$$V = -Mgs\sin\alpha \tag{5}$$

and the equation of motion for the ball rolling without slipping is

$$\left(M + I/R^2\right)\ddot{s} - Mg\sin\alpha = 0 \tag{6}$$

or

$$\ddot{s} = \frac{Mg\sin\alpha}{M + I/R^2} \tag{7}$$

Integrating twice with zero initial position and velocity gives

$$\begin{aligned} \dot{s} &= \frac{Mg\sin\alpha}{M + I/R^2}t & (8) \\ s &= \frac{1}{2}\frac{Mg\sin\alpha}{M + I/R^2}t^2 & (9) \end{aligned}$$

If the length of the course is L, and the total time is t_f, then we may write

$$L = \frac{1}{2}\frac{Mg\sin\alpha}{M + I/R^2}t_f^2 \tag{10}$$

Now, regroup as

$$\frac{2L}{g\sin\alpha} = \frac{Mt_f^2}{M + I/R^2} \tag{11}$$

The quantity on the left is independent of the particular ball chosen, and therefore is the same for both balls. It follows then that

$$\frac{M_{solid}t_{f-solid}^2}{M_{solid} + I_{solid}/R_{solid}^2} = \frac{M_{shell}t_{f-shell}^2}{M_{shell} + I_{shell}/R_{shell}^2} \tag{12}$$

This can be solved for (the square of) $t_{f-solid}$ as follows:

$$t_{f-solid}^2 = t_{f-shell}^2 \cdot \frac{M_{shell}}{M_{solid}} \cdot \frac{M_{solid} + I_{solid}/R_{solid}^2}{M_{shell} + I_{shell}/R_{shell}^2} \tag{13}$$

3 Numerical Values

$M_{shell} = 0.605 \qquad M_{solid} = 0.127$

$R_{shell} = 0.0402 \qquad R_{solid} = 0.1123$

$I_{shell} = \frac{2}{3} M_{shell} R_{shell}^2 = \frac{2}{3}(0.605)(0.0402)^2 = 6.518 \cdot 10^{-4}$

$I_{solid} = \frac{2}{5} M_{solid} R_{solid}^2 = \frac{2}{5}(0.127)(0.1123)^2 = 6.406\,5 \cdot 10^{-4}$

$M_{shell} + I_{shell}/R_{shell}^2 = M_{shell} + \frac{2}{3} M_{shell} = \frac{5}{3}(0.605) = 1.008\,3$

$M_{solid} + I_{solid}/R_{solid}^2 = M_{solid} + \frac{2}{5} M_{solid} = \frac{7}{5}(0.127) = 0.177\,8$

$\frac{M_{shell}}{M_{solid}} \cdot \frac{M_{solid} + I_{solid}/R_{solid}^2}{M_{shell} + I_{shell}/R_{shell}^2} = \frac{0.605}{0.127} \cdot \frac{0.1778}{1.0083} = 0.840\,03$

$$t_{f-solid} = (0.84003)^{1/2}(7.49) = 6.864\,8 \text{ s} \tag{14}$$

This is the correct answer. Note how the missing data was resolved. We found an expression involving all of the missing data, $2L/(g \sin \alpha)$, that was the same for both balls and could thus be eliminated from the problem.

4 The Interesting Part

One of the advisers at PF said that this problem statement gave a lot of unnecessary data. Well, just what was unnecessary? My solution above used all of the given data! So I argued with him, and he showed me something new, something I had never noticed before. Let us return to eq(11) above:

$$\frac{2L}{g \sin \alpha} = \frac{M t_f^2}{M + I/R^2} \tag{11}$$

and note that the expressions for the mass moments of inertia can be written in the form

$$I = \beta M R^2 \tag{15}$$

in each case, that is for either the solid and hollow spheres ($\beta_{shell} = 2/3$, $\beta_{solid} = 2/5$). Let us make this substitution on the right in eq(11) to get the following result:

$$\frac{2L}{g \sin \alpha} = \frac{M t_f^2}{M + I/R^2} = \frac{M t_f^2}{M + \beta M R^2/R^2} = \frac{t_f^2}{1+\beta} \tag{16}$$

This is a considerable simplification of the expression, with all of the mass, radius, and acceleration of gravity information suddenly gone! All that remains on the right is the final time, and the factor β.

If we now argue as before that the quantity on the far left is common to both, then the following must be true:

$$\frac{t_{f-solid}^2}{1+\beta_{solid}} = \frac{t_{f-shell}^2}{1+\beta_{shell}} \tag{17}$$

or

$$t_{f-solid} = t_{f-shell}\sqrt{\frac{1+\beta_{solid}}{1+\beta_{shell}}} \tag{18}$$

When we substitute numerical values, we find

$$t_{f-solid} = (7.49)\sqrt{\frac{1+\frac{2}{5}}{1+\frac{2}{3}}} = 6.8647 \text{ s} \tag{19}$$

Note that this agrees with the previous result except by 1 in the last digit. If anything, this new calculation is slightly more accurate because there is less round-off error.

5 Conclusion

The PF adviser was in fact correct; it really was not necessary to provide all the mass and radius data given in the original problem statement. On the other hand, since the data was all given, there was no error in using it, only increased notational complexity.

The other important point is this: You can teach an old dog new tricks! I had never seen this before, and I certainly did not catch it on my own here. I am grateful to that PF adviser who showed me something new.

DrD is a retired Professor of Mechanical Engineering in the USA. He can be reached for comments or questions through the message facility. Be sure to check back soon at for more articles.