Mechanics Corner

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Good News — Bad News
Rolling Disk in a Rolling Ring

1 Introduction

Well, it looks like *Mechanics Corner* is back, at least in terms of an occasional post. It will probably be less frequent than previously, but there are just too many interesting things to talk about to remain entirely silent! The title for this post may leave you wondering what is the Good News, and what is the Bad News? Why is there both? Well, let me tell you about it ...

2 The Good News

The Good News to which I am referring is information about another web site that I think will be of interest to some of you. I need to lay a bit of background before I actually get to it. In addition to the **ME Forum**, there is another similar web site where I participate called **Physics Forums** (https://www.physicsforums.com/), usually simply called **PF**, that is almost entirely a question and answer site. People post questions about anything having to do with physics, and anyone else is welcome to try to provide an answer. I post there using the same screen name I use here, DrD. I encourage all of you to have a look at the site; it is very broad, and some of the discussions are very interesting (some are also pretty dumb, also!!).

Recently, on PF, there was a dynamics question that caught my eye. It involved a ring on rollers, with a smaller disk rolling inside the ring. The problem had originated at yet another web site, and the person posting on PF was asking for help understanding the material posted at the other site.

Now, let me digress for a moment. *Mathematica*® is a software product from Wolfram. It has a wide variety of symbolic and numerical mathematical capabilities, plotting, animations, etc. It is a very powerful program, although not particularly easy to use (I presume this has improved since I last tried it some years ago). It is also extremely expensive! It finds application in many situations where users need computing power to crunch through large amounts of data, do very exotic mathematics, and so forth. It is highly recommended, if you can afford it.

In an effort to show the many capabilities of the *Mathematica* software, Wolfram has a portion of their website called *Demonstrations* where users are invited to post their work. There are thousands of solved problems posted there, and it is well worth looking at. You will see some amazing solutions to a host of interesting problems. The URL for the mechanics portion of the site is:

http://demonstrations.wolfram.com/topic.html?topic=Mechanical+Engineering&limit=20

The post on PF that led me to this site was

https://www.physicsforums.com/threads/equation-of-motion-for-a-disk-inside-a-rotating

-ring.909368/#post-5728207 *(all run together, no gaps)*

which contains a link to the Wolfram problem

http://demonstrations.wolfram.com/DiskRollingInsideARotatingRing/

I sincerely want to encourage all readers to look both at PF and at the Wolfram site. There is a lot there for you!

3 The Bad News

Now, after all that build up, here is the bad news. **The solution presented on the Wolfram site for this particular problem is wrong. It has an error.** There appears to be no way to notify the author of his error, so it stands without correction.

What is the significance of all of this, then? Three things:

(1) Having the fanciest, most expensive mathematical software in the world is no guarantee of a correct result. If you don't know what you are doing, or if you make a careless mistake, this elaborate software will simply enable you to make mistakes more rapidly and with color animation!

(2) Because the formulation is wrong, the beautiful animation presented on the Wolfram site is meaningless. It looks pretty, it looks like it might be correct, but in fact it is meaningless because it is based on the wrong equations of motion.

(3) The problem is not at all impossible, and it is interesting and instructive to look into the correct formulation. That forms the remainder of this post.

4 Disk Rolling in a Rolling Ring

The development given below is believed to be the correct formulation of the problem given on the Wolfram Mathematica web site, with a slight modification to make it more realistic. A full problem statement follows, and then the formulation is developed.

4.1 Problem Statement

The system considered here consists of a small disk inside a large ring. The large ring is supported on a pair of identical idler rollers. The small disk rolls without slipping in the large ring, while the large ring rolls without slipping on the idlers. The known data include

r = radius of the small disk

R_i = inside radius of the ring

R_o = outside radius of the ring

r_{roller} = radius of the idler rollers

m = mass of the disk

M = mass of the ring

J_{disk} = mass moment of inertia of the disk with respect to its center of mass

J_{ring} = mass moment of inertia of the ring with respect to its center of mass

J_{roller} = mass moment of inertia of a single idler roller with respect to its axis of rotation

Gravity acts vertically downward.

The problem is to find the equations of motion for this system.

Note that this is slightly more involved than the original problem posted at the Wolfram Mathematica web site in that the supporting rollers are assumed to be massive, rather than massless. It is well known that the less the mass of the roller, the more expensive it becomes until, in the limit, massless rollers are infinitely expensive (wink!!)[1]

4.2 Degrees of Freedom and Constraints

The first question that must be addressed is "How many degrees of freedom are involved?"

At first glance, it appears that four coordinates are required to describe the system configuration:

1. The rotation of the ring, ϕ
2. The locating of the ring center, θ

[1] "*wink*" means that the writer is joking with the reader, and that what has just been said should not be taken seriously.

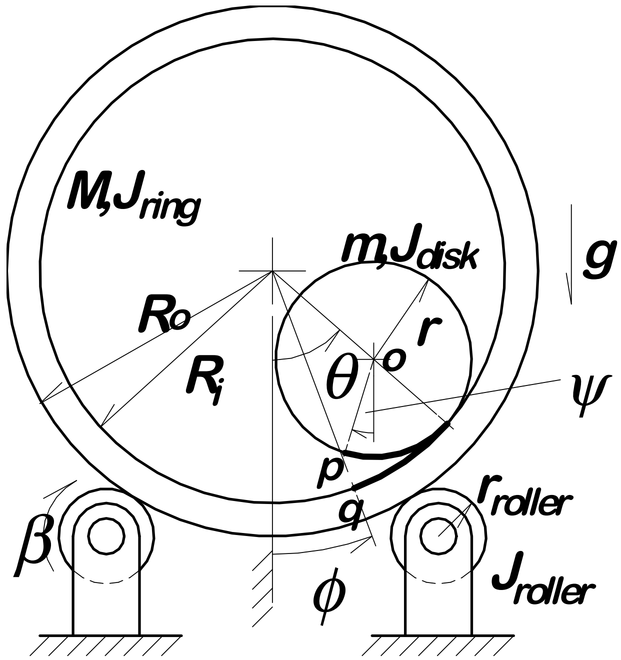

Figure 1: Disk Rolling in a Rolling Ring

3. The angular position of the ring
4. The angular position of the idlers (both assumed to be the same)

This number is reduced by the fact that there are two equations of constraint to describe the rolling without slipping condition (1) between the idlers and the ring, and (2) between the disk and the ring.

Let β describe the angular orientation of the idler roller. The equation describing rolling without slipping between the idler and the ring is

$$r_3\beta = R_o\phi$$

so that β can be expressed directly in terms of the ring rotation ϕ

$$\beta = \frac{R_o}{r_{roller}}\phi \tag{1}$$

Assume that for $\theta = \phi = 0$, the point p is in contact with the point q; this contact at the bottom of the arc. The condition of rolling without slipping between the disk and the ring requires that the two very heavy arc be the same length.

Let ψ describe the rotation of the disk, measured from the vertical to the point p. (It may help to consider $\phi = 0$ as a start.) Keeping in mind that the orientation of the line op is the quantity to be measured, as the disk rolls up the right side of the ring, the length of the arc of contact is $s = r\,(\theta + \psi)$ on the disk. On the ring, the contact length is $s = R\,(\theta - \phi)$.

Equating these two expressions gives the rolling constraint equation which can be solved for the disk rotation, ψ

$$\psi = \frac{1}{r}\left(R - r\right)\theta - \frac{R}{r}\phi \tag{2}$$

Thus it is evident that the two variables θ and ϕ are sufficient to describe all the entire system configuration. This is the hardest part of the entire problem, and it just gets easier from here on. (Many people find it difficult to write the second constraint equation in particular. Be sure th7at you understand exactly where this came from.)

4.3 Energy Expressions

The Lagrange formulation requires that the system kinetic and potential energies be written to begin the process.

4.3.1 Kinetic Energy

The system involves three bodies in pure rotation (the large ring and the two small idlers) and one with both translation and rotation.

$$T = \frac{1}{2}\left[J_{disk}\dot{\psi}^2 + m\left(R - r\right)^2\dot{\theta}^2 + J_{ring}\dot{\theta}^2 + 2J_{roller}\dot{\beta}^2\right] \tag{3}$$

$$\begin{aligned} &= \frac{1}{2}\{\left[J_{disk}\frac{\left(R_i - r\right)^2}{r^2} + m\left(R_i - r\right)^2\right]\dot{\theta}^2 \\ &- \left[2J_{disk}\frac{R_i\left(R_i - r\right)}{r^2}\right]\dot{\theta}\,\dot{\phi} \\ &+ \left[J_{disk}\left(\frac{R_i}{r}\right)^2 + J_{ring} + 2J_{roller}\left(\frac{R_o}{r_{roller}}\right)^2\right]\dot{\phi}^2\} \end{aligned} \tag{4}$$

Note that there are no time dependent quantities in the kinetic energy expression aside from the coordinate velocities. In anticipation of the Lagrange equations, it is useful to take partial derivatives of the kinetic energy with respect to the coordinate velocities at this point:

$$\frac{\partial T}{\partial\dot{\theta}} = \left[J_{disk}\frac{\left(R_i - r\right)^2}{r^2} + m\left(R_i - r\right)^2\right]\dot{\theta} - \left[J_{disk}\frac{R_i\left(R_i - r\right)}{r^2}\right]\dot{\phi} \tag{5}$$

$$\frac{\partial T}{\partial\dot{\phi}} = -\left[J_{disk}\frac{R_i\left(R_i - r\right)}{r^2}\right]\dot{\theta} + \left[J_{disk}\left(\frac{R_i}{r}\right)^2 + J_{ring} + 2J_{roller}\left(\frac{R_o}{r_{roller}}\right)^2\right]\dot{\phi} \tag{6}$$

4.3.2 Potential Energy

The disk is the only body for which there is a change in center of mass elevation as time passes, and consequently the only body that needs to be considered in the expression of gravitational potential energy.

$$V = -mg\left(R - r\right)\cos\theta \tag{7}$$

Taking partial derivatives gives

$$\frac{\partial V}{\partial \theta} = mg\left(R - r\right)\sin\theta \tag{8}$$

$$\frac{\partial V}{\partial \phi} = 0 \tag{9}$$

4.3.3 Lagrange Equations of Motion

The Lagrange form for the equation of motion may be written as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i^{nc} \tag{10}$$

where

$q_i = i^{th}$ generalized coordinate

Q_i^{nc} = nonconservative generalized force associated with q_i

For the case at hand, there are three important observations to be made:

(1) There is no direct coordinate dependence in T, so that $\partial T/\partial q_i = 0$ for all i;

(2) The only coordinate dependence in the potential energy is the θ dependence, reflected above in equations (7) and (8).

(3) For the current problem, there is no nonconservative generalized force acting in either coordinate, so the right side of both equations of motion will be zero.

Applying equation (10) for each coordinate gives the equations of motion:

$$\left[J_{disk}\frac{\left(R_i - r\right)^2}{r^2} + m\left(R_i - r\right)^2\right]\ddot{\theta} - \left[J_{disk}\frac{R_i\left(R_i - r\right)}{r^2}\right]\ddot{\phi} + mg\left(R - r\right)\sin\theta = 0 \tag{11}$$

$$-\left[J_{disk}\frac{R_i\left(R_i - r\right)}{r^2}\right]\ddot{\theta} + \left[J_{disk}\left(\frac{R_i}{r}\right)^2 + J_{ring} + 2J_{roller}\left(\frac{R_o}{r_{roller}}\right)^2\right]\ddot{\phi} = 0 \tag{12}$$

4.3.4 Solution Considerations

If a numerical solution is to be developed for this system of equations, it will be necessary to write expressions for $\ddot{\theta}$ and $\ddot{\phi}$, that is, to solve the equations of motion for these accelerations. Note that the coefficients of the accelerations are all constants, so that these equations can be written in an abbreviated form as

$$\begin{bmatrix} A & -B \\ -B & C \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} + \begin{Bmatrix} mg\,(R-r)\sin\theta \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{13}$$

where A, B, and C are constants,

$A = \left[J_{disk}\frac{(R_i-r)^2}{r^2} + m\,(R_i-r)^2\right]$

$B = \left[J_{disk}\frac{R_i(R_i-r)}{r^2}\right]$

$C = \left[J_{disk}\left(\frac{R_i}{r}\right)^2 + J_{ring} + 2J_{roller}\left(\frac{R_o}{r_{roller}}\right)^2\right]$

The coefficient matrix for the accelerations can be inverted symbolically to give the accelerations as

$$\begin{Bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} = \frac{-1}{AC-B^2}\begin{bmatrix} C & B \\ B & A \end{bmatrix} \begin{Bmatrix} mg\,(R-r)\sin\theta \\ 0 \end{Bmatrix} \tag{14}$$

$$= \frac{-1}{AC-B^2}\begin{Bmatrix} C\cdot mg\,(R-r)\sin\theta \\ B\cdot mg\,(R-r)\sin\theta \end{Bmatrix} \tag{15}$$

The last result above, equation (15), shows that both $\ddot{\theta}$ and $\ddot{\phi}$ vary sinusoidally with the angle θ. It should be possible to solve directly for $\theta\,(t)$ in terms of elliptic integrals (the equation for $\ddot{\theta}$ is in the form of the equation of motion of the large amplitude pendulum, a known solution involving elliptic functions), and then to determine $\phi\,(t)$ by integrating that solution twice with respect to time. The details have not been worked out, and there is always the possibility of an unexpected difficulty, although none are foreseen at this point.

Provided I have not made any algebraic errors, that is the full solution for the motion. If any reader detects errors, the writer would appreciate knowing about them as soon as possible.

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