Mathematics for the Liberal Arts

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MATH FOR LIBERAL ARTS

Course Overview

This course was originally developed for the Open Course Library project. The text used is *Math in Society,* edited by David Lippman, Pierce College Ft Steilacoom. Development of this book was supported, in part, by the Transition Math Project and the Open Course Library Project. Topics covered in the course include problem solving, voting theory, graph theory, growth models, finance, data collection and description, and probability. Student learning outcomes include:

1. Describe how mathematics can contribute to the solution of problems in the natural world or human society.
2. Employ critical thinking skills, drawing upon prior knowledge when possible, to analyze and explore new and unfamiliar problems
3. Form and communicate generalizations of patterns discovered through individual or group investigations.
4. Solve problems using algorithms or formulas
5. Model and solve problems using graphical methods
6. Communicate methods of solutions and solutions to problems for the clarity of the receiver.
7. Analyze and interpret data, including calculating numerical summaries and creating graphical representations, to propose possible implications
8. Identify multicultural perspectives of, or multicultural contributions to, at least one mathematical topic studied

GENERAL PROBLEM SOLVING

MODULE 1 OVERVIEW

What You'll Learn To Do: Solve problems involving percents, geometry, and estimation.

Learning Objectives

- Solve problems involving percents, proportions, and rates.
- Solve problems using basic geometry.
- Use problem solving and estimation techniques.

Learning Activities

The learning activities for this module include:

- **Discuss:** Introduce Yourself to the Class (20 points)

Reading Assignments and Videos

- **Read:** Problem Solving
- **Watch:** Supplemental Videos

Homework Assignments

- **Review:** Optional Discussion Boards
- **Submit:** Problem Solving Homework #1 (15 points)
- **Submit:** Problem Solving Homework #2 (20 points)
- **Discuss:** Problem Solving Application (20 points)
- **Discuss:** Math Editor Practice (10 points)

DISCUSS: INTRODUCE YOURSELF TO THE CLASS

Go to the **Introduction** forum. Click on the **Create Thread** button to post your introduction. Include your *first* and *last* name in the **Subject** line. Within your posting, please be sure to include the following information:

- Your interests including what you like to do outside of class.
- Why you are taking this class and what you hope to gain out of taking this class this semester.
- What your most successful mathematical experience in or out of class has been.

Respond to the introductions of **at least two** classmates. Your responses should be substantive. That is, they should contain more than just a "Hello."

This assignment is required and worth up to 20 points.

PROBLEM SOLVING

In previous math courses, you've no doubt run into the infamous "word problems." Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use, and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure, and determining what information you will need (and won't need) to answer the question.

In this chapter, we will review several basic but powerful algebraic ideas: percents, rates, and proportions. We will then focus on the problem solving process, and explore how to use these ideas to solve problems where we don't have perfect information.

Percents

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. ((Note: http://www.factcheck.org/cheney_edwards_mangle_facts.html)) Who is correct? How can we make sense of these numbers?

Percent literally means "per 100," or "parts per hundred." When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200}=\frac{10}{25}=\frac{40}{100}$.

Example 1

243 people out of 400 state that they like dogs. What percent is this?

Solution

$\frac{243}{400} = 0.6075 = \frac{60.75}{100}$. This is 60.75%.

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

Example 2

Write each as a percent:

1. $\frac{1}{4}$
2. 0.02
3. 2.35

Solutions

1. $\frac{1}{4} = 0.25$ = 25%
2. 0.02 = 2%
3. 2.35 = 235%

Percents

If we have a *part* that is some *percent* of a *whole*, then $\text{percent} = \frac{\text{part}}{\text{whole}}$, or equivalently,

$\text{part} \cdot \text{whole} = \text{percent}.$

To do the calculations, we write the percent as a decimal.

Example 3

The sales tax in a town is 9.4%. How much tax will you pay on a $140 purchase?

Solution

Here, $140 is the whole, and we want to find 9.4% *of* $140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute: tax = 0.094(140) = $13.16 in tax.

Example 4

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was $1200 per quarter, what will it be next year?

Solution

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition: $1200(1.07) = $1284.

Alternatively, we could have first calculated 7% of $1200: $1200(0.07) = $84.

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition: $1200 + $84 = $1284.

Try It Now

A TV originally priced at $799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

Example 5

The value of a car dropped from $7400 to $6800 over the last year. What percent decrease is this?

Solution

To compute the percent change, we first need to find the dollar value change: $6800 – $7400 = –$600. Often we will take the absolute value of this amount, which is called the **absolute change**: |–600| = 600.

Since we are computing the decrease relative to the starting value, we compute this percent out of $7400:

$\frac{600}{7400} = 0.081 =$ 8.1% decrease. This is called a **relative change**.

Absolute and Relative Change

Given two quantities,

Absolute change = $|\text{ending quantity} - \text{starting quantity}|$

Relative change: $\frac{\text{absolute change}}{\text{starting quantity}}$

Absolute change has the same units as the original quantity.
Relative change gives a percent change.
The starting quantity is called the **base** of the percent change.

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a "gateway" drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn't true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs. ((Note: http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics))

Example 6

There are about 75 QFC supermarkets in the United States. Albertsons has about 215 stores. Compare the size of the two companies.

Solution

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is 215 – 75 = 140. From this, we could say "Albertsons has 140 more stores than QFC." However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\frac{140}{75} = 1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\frac{140}{215} = 0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC: $\frac{215}{75} = 2.867$, which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: $\frac{75}{215} = 0.349$, which tells us that QFC is 34.9% of the size of Albertsons.

Example 7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Solution

To answer this question, suppose the value started at \$100. After one week, the value dropped by 60%: \$100 – \$100(0.60) = \$100 – \$60 = \$40.

In the next week, notice that base of the percent has changed to the new value, \$40. Computing the 75% increase: \$40 + \$40(0.75) = \$40 + \$30 = \$70.

In the end, the stock is still \$30 lower, or $\frac{\$30}{100}$ = 30% lower, valued than it started.

Try It Now

The US federal debt at the end of 2001 was \$5.77 trillion, and grew to \$6.20 trillion by the end of 2002. At the end of 2005 it was \$7.91 trillion, and grew to \$8.45 trillion by the end of 2006. ((Note: http://www.whitehouse.gov/sites/default/files/omb/budget/fy2013/assets/hist07z1.xls)) Calculate the absolute and relative increase for 2001–2002 and 2005–2006. Which year saw a larger increase in federal debt?

Example 8

A Seattle Times article on high school graduation rates reported "The number of schools graduating 60 percent or fewer students in four years—sometimes referred to as "dropout factories"—decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half."

1. Is the "decrease by 17" number a useful comparison?
2. Considering the last sentence, can we conclude that the number of "dropout factories" was originally 34?

Solution

1. This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of "dropout factories" dropped from 20 to 3, that'd be a very significant change, but if the number dropped from 217 to 200, that'd be less of an improvement.
2. The last sentence provides relative change, which helps put the first sentence in perspective. We can estimate that the number of "dropout factories" was probably previously around 34. However, it's possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example 9

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

Solution

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward's claim was a percent with coalition forces as the base of the percent, while Cheney's claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

Try It Now

In the 2012 presidential elections, one candidate argued that "the president's plan will cut $716 billion from Medicare, leading to fewer services for seniors," while the other candidate rebuts that "our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures." Are these claims in conflict, in agreement, or not comparable because they're talking about different things?

We'll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

Example 10

A politician's support increases from 40% of voters to 50% of voters. Describe the change.

Solution

We could describe this using an absolute change: $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Lastly, a caution against averaging percents.

Example 11

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then they made 200(0.40) = 80 2-point shots and 100(0.30) = 30 3-point shots. Overall, they made 110 shots out of 300, for a $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

Proportions and Rates

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A rate is the ratio (fraction) of two quantities.
A **unit rate** is a rate with a denominator of one.

Example 12

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Solution

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{ gallon}}$, which we could also write as $20 \frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

Example 13

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value *x*.

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$.

Example 14

A map scale indicates that ½ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, *x*, to represent the unknown quantity—the mile distance between the cities.

$\frac{\frac{1}{2}\text{map inch}}{3\text{ miles}} = \frac{2\frac{1}{4}\text{map inches}}{x\text{ miles}}$	Multiply both sides by *x* and rewriting the mixed number
$\frac{\frac{1}{2}}{3} \cdot x = \frac{9}{4}$	Multiply both sides by 3
$\frac{1}{2}x = \frac{27}{4}$	Multiply both sides by 2 (or divide by ½)
$x = \frac{27}{2} = 13\frac{1}{2}\text{ miles}$	

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example 15

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\dfrac{300 \text{ miles}}{15 \text{ gallons}} = \dfrac{x \text{ miles}}{40 \text{ gallons}}$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit "cancels" and we're left with a number of miles:

$$40 \text{ gallons} \cdot \frac{20 \text{ miles}}{\text{gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallons}} = 800 \text{ miles}$$

Notice if instead we were asked "how many gallons are needed to drive 50 miles?" we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we're left with gallons:

$$50 \text{ miles} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

Unit Conversions

Length

1 foot (ft) = 12 inches (in)	1 yard (yd) = 3 feet (ft)
1 mile = 5,280 feet	
1000 millimeters (mm) = 1 meter (m)	100 centimeters (cm) = 1 meter
1000 meters (m) = 1 kilometer (km)	2.54 centimeters (cm) = 1 inch

Weight and Mass

1 pound (lb) = 16 ounces (oz)	1 ton = 2000 pounds
1000 milligrams (mg) = 1 gram (g)	1000 grams = 1kilogram (kg)
1 kilogram = 2.2 pounds (on earth)	

Capacity

1 cup = 8 fluid ounces (fl oz) ((Note: Fluid ounces are a capacity measurement for liquids. 1 fluid ounce ≈ 1 ounce (weight) for water only.))	1 pint = 2 cups

1 quart = 2 pints = 4 cups	1 gallon = 4 quarts = 16 cups
1000 milliliters (ml) = 1 liter (L)	

Example 16

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour}$$

Now we can multiply by the 15 miles/hr

$$\frac{1}{180} \text{ hour} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} = \frac{1}{12} \text{ mile}$$

Now we can convert to feet

$$\frac{1}{12} \text{ mile} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{15 \text{ miles}}{1 \text{ miles}} = \frac{5280 \text{ feet}}{1 \text{ mile}} = \frac{1}{180} \text{ hour}$$

Try It Now

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 17

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100\text{ft}^2} = \frac{n \text{ tiles}}{400\text{ft}^2}$$

Other quantities just don't scale proportionally at all.

Example 18

Suppose a small company spends \$1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend \$10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 19

Compare the 2010 U.S. military budget of \$683.7 billion to other quantities.

Solution

Here we have a very large number, about \$683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over \$488,000.

There are about 300 million people in the U.S. The military budget is about \$2,200 per person.

If you were to put \$683.7 billion in \$100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 20

Compare the electricity consumption per capita in China to the rate in Japan.

Solution

To address this question, we will first need data. From the CIA ((Note: https://www.cia.gov/library/publications/the-world-factbook/rankorder/2042rank.html)) website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank, ((Note: http://data.worldbank.org/indicator/SP.POP.TOTL)) we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Computing the consumption per capita for each country:

China: $\frac{4,693,000,000,000\text{KWH}}{1,344,130,000 \text{ people}}$ ≈ 3491.5 KWH per person

Japan: $\frac{859,700,000,000\text{KWH}}{127,817,277 \text{ people}}$ ≈ 6726 KWH per person

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

Geometry

Geometric shapes, as well as area and volumes, can often be important in problem solving.

Example 21

You are curious how tall a tree is, but don't have any way to climb it. Describe a method for determining the height.

Solution

There are several approaches we could take. We'll use one based on triangles, which requires that it's a sunny day. Suppose the tree is casting a shadow, say 15 ft long. I can then have a friend help me measure my own shadow. Suppose I am 6 ft tall, and cast a 1.5 ft shadow. Since the triangle formed by the tree and its shadow has the same angles as the triangle formed by me and my shadow, these triangles are called **similar triangles** and their sides will scale proportionally. In other words, the ratio of height to width will be the same in both triangles. Using this, we can find the height of the tree, which we'll denote by *h*:

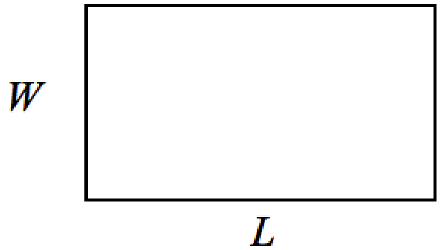
$$\frac{6\text{ft tall}}{1.5\text{ft shadow}} = \frac{h\text{ft tall}}{15\text{ft shadow}}$$

Multiplying both sides by 15, we get $h = 60$. The tree is about 60 ft tall.

It may be helpful to recall some formulas for areas and volumes of a few basic shapes.

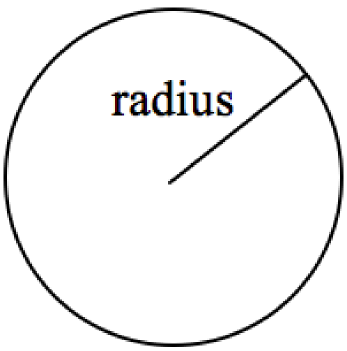
Areas

Rectangle



Area: $L \cdot W$
Perimeter: $2L + 2W$

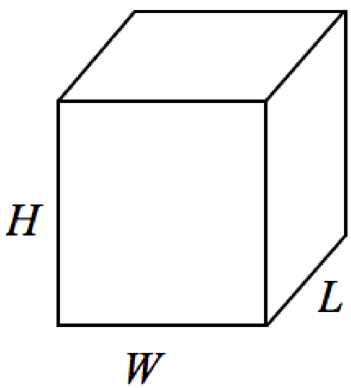
Circle



Radius: r
Area: πr^2
Circumference: $2\pi r$

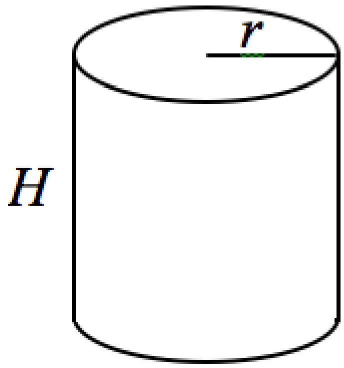
Volumes

Rectangular Box



Volume: $L{\cdot}W{\cdot}H$

Cylinder



Volume: $\pi r^2 H$

Example 22

If a 12 inch diameter pizza requires 10 ounces of dough, how much dough is needed for a 16 inch pizza?

Solution

To answer this question, we need to consider how the weight of the dough will scale. The weight will be based on the volume of the dough. However, since both pizzas will be about the same thickness, the weight will scale with the area of the top of the pizza. We can find the area of each pizza using the formula for area of a circle, $A = \pi r^2$:

> A 12″ pizza has radius 6 inches, so the area will be $\pi 6^2$ = about 113 square inches.

> A 16″ pizza has radius 8 inches, so the area will be $\pi 8^2$ = about 201 square inches.

Notice that if both pizzas were 1 inch thick, the volumes would be 113 in^3 and 201 in^3 respectively, which are at the same ratio as the areas. As mentioned earlier, since the thickness is the same for both pizzas, we can safely ignore it.

We can now set up a proportion to find the weight of the dough for a 16″ pizza:

$$\frac{10 \text{ ounces}}{113 \text{in}^2} = \frac{x \text{ ounces}}{201 \text{in}^2}$$

Multiply both sides by 201

$x = 201 \cdot \dfrac{10}{113}$ = about 17.8 ounces of dough for a 16″ pizza.

It is interesting to note that while the diameter is $\dfrac{16}{12}$ = 1.33 times larger, the dough required, which scales with area, is 1.33^2 = 1.78 times larger.

Example 23

A company makes regular and jumbo marshmallows. The regular marshmallow has 25 calories. How many calories will the jumbo marshmallow have?

Solution

We would expect the calories to scale with volume. Since the marshmallows have cylindrical shapes, we can use that formula to find the volume. From the grid in the image, we can estimate the radius and height of each marshmallow.

The regular marshmallow appears to have a diameter of about 3.5 units, giving a radius of 1.75 units, and a height of about 3.5 units. The volume is about $\pi(1.75)^2(3.5) = 33.7$ units^3.

The jumbo marshmallow appears to have a diameter of about 5.5 units, giving a radius of 2.75 units, and a height of about 5 units. The volume is about $\pi(2.75)^2(5) = 118.8$ units^3.

We could now set up a proportion, or use rates. The regular marshmallow has 25 calories for 33.7 cubic units of volume. The jumbo marshmallow will have:

$$118.8 \text{ units}^3 \cdot \frac{25 \text{ calories}}{33.7 \text{ units}^3} = 88.1 \text{ calories}$$

It is interesting to note that while the diameter and height are about 1.5 times larger for the jumbo marshmallow, the volume and calories are about $1.5^3 = 3.375$ times larger.

Try It Now

A website says that you'll need 48 fifty-pound bags of sand to fill a sandbox that measure 8ft by 8ft by 1ft. How many bags would you need for a sandbox 6ft by 4ft by 1ft?

Problem Solving and Estimating

Finally, we will bring together the mathematical tools we've reviewed, and use them to approach more complex problems. In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very few interesting questions can be answered in one mathematical step; often times you will need to chain together a solution pathway, a series of steps that will allow you to answer the question.

Problem Solving Process

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.

5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

Example 24

How many times does your heart beat in a year?

Solution

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:

$$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} = 42,048,000 \text{ beats per year}$$

Example 25

How thick is a single sheet of paper? How much does it weigh?

Solution

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.004 \text{ inches per sheet}$$

$$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.01 \text{ pounds per sheet, or } = 0.16 \text{ ounces per sheet.}$$

Example 26

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Solution

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$$12 \text{ muffins} \cdot \frac{250 \text{ calories}}{\text{muffin}} = 3000 \text{ calories for the whole recipe}$$

$$\frac{3000 \text{ calories}}{20 \text{ mini-muffins}} = \text{ gives 150 calories per mini-muffin}$$

$$4 \text{ mini-muffins} \cdot \frac{150 \text{ calories}}{\text{mini-muffin}} = \text{totals 600 calories consumed.}$$

Example 27

You need to replace the boards on your deck. About how much will the materials cost?

Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of 384 ft^2.

From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about \$7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$8 \text{ feet} \cdot 4 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2.667\text{ft}^2$. The cost per square foot is then $\frac{\$7.50}{2.667\text{ft}^2} = \$2.8125 \text{ per ft}^2$.

This will allow us to estimate the material cost for the whole 384 ft^2 deck

$$\$384\text{ft}^2 \cdot \frac{\$2.8125}{\text{ft}^2} = \$1080 \text{ total cost.}$$

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

Example 28

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata: $9000 \text{ city miles} \cdot \dfrac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ highway miles} \cdot \dfrac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$

Hybrid: $9000 \text{ city miles} \cdot \dfrac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ highway miles} \cdot \dfrac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$

If gas in your area averages about $3.50 per gallon, we can use that to find the running cost:

Sonata: $460.7 \text{ gallons} \cdot \dfrac{\$3.50}{\text{gallon}} = \$1612.45$

Hybrid: $332.1 \text{ gallons} \cdot \dfrac{\$3.50}{\text{gallon}} = \$1162.35$

The hybrid will save $450.10 a year. The gas costs for the hybrid are about $\dfrac{\$450.10}{\$1612.45}$ = 0.279 = 27.9% lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since "is it worth it" implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about $25,850, compared to the base model for the regular Sonata, at $20,895.

To better answer the "is it worth it" question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs $4965 more. With gas savings of $451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can't make it for you.

Try It Now

If traveling from Seattle, WA to Spokane WA for a three-day conference, does it make more sense to drive or fly?

SUPPLEMENTAL VIDEOS

This YouTube playlist contains several videos that supplement the reading on Problem Solving.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

OPTIONAL DISCUSSION BOARDS

There are two optional Discussion Boards in this course:

- Go to **Discussion Board > Open Discussion** to ask questions, answer questions, post current event examples, or contribute insights related to the reading. You can also use this forum to form study groups. *You will use this board throughout the duration course. It is **not** specific to this module.*
- Go to **Discussion Board > Problem Solving Homework Discussion** to discuss issues you are having with the homework or to help your classmates with their questions. *There will be a different homework discussion board for each module. They **are** specific to each module.*

DISCUSS: PROBLEM SOLVING APPLICATION

Pick a unique real problem and try to solve it using the general problem solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the General Problem Solving Strategies Application Directions for detailed directions.

Create a new thread in the **Problem-Solving Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

<table>
<tr><td>Grading Criteria</td><td>Points Possible</td></tr>
<tr><td>The problem:
Is it a real-life problem?
Is it challenging, not trivial?
Is it a unique problem instead of a copy of a classmate’s posting?</td><td>5</td></tr>
<tr><td>The strategies:
Are one or more general problem solving strategies used?
Are the strategies correctly identified?</td><td>5</td></tr>
<tr><td>The presentation:
Is the problem explained well?
Are the problem solving strategies explained well?
Are the appropriate terms used?</td><td>4</td></tr>
<tr><td>Your responses:
Did you post at least two responses?
Did you explain how the examples helped you better understand the math in this module?
Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?</td><td>6</td></tr>
</table>

DISCUSS: MATH EDITOR PRACTICE (BLACKBOARD)

You will need to become familiar with the use of the Math Editor in Blackboard. You may choose to use the Math Editor for some of your exam questions.

Use the **Math Editor Practice** forum in the Discussion Board to complete this assignment. Click on the **Create Thread** button to begin a new thread.

With this assignment, you will accomplish two things:

- Practice using the math editor in Blackboard. Click here for instructions: BlackboardMathEditor.doc
- Troubleshooting the Math Editor: Upgrading JAVA
- Practice presenting a problem

Let *n* = your age + number of the month of your birth. Determine the n^{th} hexagonal number using the formula

$$H_n = \frac{n(4n-2)}{2}$$

Post your entire solution, including the original formula, in the discussion board **using the math editor**. Be sure to use proper fraction notation. Do not use / to represent the fraction bar. For example, if you are 25 and you were born in April, then n = 25 + 4 = 29 and the 29^{th} hexagonal number is

$$H_{29} = \frac{29((4 \times 29) - 2)}{2} = \frac{29(116-2)}{2} = \frac{29(114)}{2} = 1653$$

This assignment is required and worth up to 10 points.

You are strongly encouraged to review & repeat this practice exercise before taking each of the exams, but no extra points will be awarded.

PROBLEM SOLVING EXERCISES

Exercises

1. Out of 230 racers who started the marathon, 212 completed the race, 14 gave up, and 4 were disqualified. What percentage did not complete the marathon?
2. Patrick left an $8 tip on a $50 restaurant bill. What percent tip is that?
3. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?
4. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax), while employers paid another 6.2%. How much will someone earning $45,000 a year pay towards social security out of their gross wages?
5. A project on Kickstarter.com was aiming to raise $15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?
6. Another project on Kickstarter for an iPad stylus raised 1,253% of their goal, raising a total of $313,490 from 7,511 supporters. What was their original goal?
7. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and relative (percent) increase.
8. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year[6]. Find the absolute and relative (percent) decrease.

9. A company wants to decrease their energy use by 15%.
    a. If their electric bill is currently $2,200 a month, what will their bill be if they're successful?
    b. If their next bill is $1,700 a month, were they successful? Why or why not?
10. A store is hoping an advertising campaign will increase their number of customers by 30%. They currently have about 80 customers a day.
    a. How many customers will they have if their campaign is successful?
    b. If they increase to 120 customers a day, were they successful? Why or why not?
11. An article reports "attendance dropped 6% this year, to 300." What was the attendance before the drop?
12. An article reports "sales have grown by 30% this year, to $200 million." What were sales before the growth?
13. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements:
    a. Kaplan's enrollment was ___% larger than Walden's.
    b. Walden's enrollment was ___% smaller than Kaplan's.
    c. Walden's enrollment was ___% of Kaplan's.
14. In the 2012 Olympics, Usain Bolt ran the 100m dash in 9.63 seconds. Jim Hines won the 1968 Olympic gold with a time of 9.95 seconds.
    a. Bolt's time was ___% faster than Hines'.
    b. Hine' time was ___% slower than Bolt's.
    c. Hine' time was ___% of Bolt's.
15. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?
16. Which is better: having a stock that goes up 30% on Monday than drops 30% on Tuesday, or a stock that drops 30% on Monday and goes up 30% on Tuesday? In each case, what is the net percent gain or loss?
17. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "16.3% of Americans are without health insurance"[7]
    b. "only 55.9% of adults receive employer provided health insurance"[8]
18. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "We mark up the wholesale price by 33% to come up with the retail price"
    b. "The store has a 25% profit margin"
19. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "Every year since 1950, the number of American children gunned down has doubled."
    b. "The number of child gunshot deaths has doubled from 1950 to 1994."
20. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?[9]
    a. "75 percent of the federal health care law's taxes would be paid by those earning less than $120,000 a year"
    b. "76 percent of those who would pay the penalty [health care law's taxes] for not having insurance in 2016 would earn under $120,000"
21. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "The school levy is only a 0.1% increase of the property tax rate."
    b. "This new levy is a 12% tax hike, raising our total rate to $9.33 per $1000 of value."
22. Are the values compared in this statement comparable or not comparable? "Guns have murdered more Americans here at home in recent years than have died on the battlefields of Iraq and Afghanistan. In support of the two wars, more than 6,500 American soldiers have lost their lives. During the same period, however, guns have been used to murder about 100,000 people on American soil"[10]
23. A high school currently has a 30% dropout rate. They've been tasked to decrease that rate by 20%. Find the equivalent percentage point drop.
24. A politician's support grew from 42% by 3 percentage points to 45%. What percent (relative) change is this?
25. Marcy has a 70% average in her class going into the final exam. She says "I need to get a 100% on this final so I can raise my score to 85%." Is she correct?
26. Suppose you have one quart of water/juice mix that is 50% juice, and you add 2 quarts of juice. What percent juice is the final mix?
27. Find a unit rate: You bought 10 pounds of potatoes for $4.
28. Find a unit rate: Joel ran 1500 meters in 4 minutes, 45 seconds.
29. A crepe recipe calls for 2 eggs, 1 cup of flour, and 1 cup of milk. How much flour would you need if you use 5 eggs?

30. An 8ft length of 4 inch wide crown molding costs $14. How much will it cost to buy 40ft of crown molding?
31. Four 3-megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?
32. A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?
33. Convert 8 feet to inches.
34. Convert 6 kilograms to grams.
35. A wire costs $2 per meter. How much will 3 kilometers of wire cost?
36. Sugar contains 15 calories per teaspoon. How many calories are in 1 cup of sugar?
37. A car is driving at 100 kilometers per hour. How far does it travel in 2 seconds?
38. A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?
39. The table below gives data on three movies. Gross earnings is the amount of money the movie brings in. Compare the net earnings (money made after expenses) for the three movies.[11]

Movie	**Release Date**	**Budget**	**Gross earnings**
Saw	10/29/2004	$1,200,000	$103,096,345
Titanic	12/19/1997	$200,000,000	$1,842,879,955
Jurassic Park	6/11/1993	$63,000,000	$923,863,984

40. For the movies in the previous problem, which provided the best return on investment?
41. The population of the U.S. is about 309,975,000, covering a land area of 3,717,000 square miles. The population of India is about 1,184,639,000, covering a land area of 1,269,000 square miles. Compare the population densities of the two countries.
42. The GDP (Gross Domestic Product) of China was $5,739 billion in 2010, and the GDP of Sweden was $435 billion. The population of China is about 1,347 million, while the population of Sweden is about 9.5 million. Compare the GDP per capita of the two countries.
43. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.
44. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.
45. Your chocolate milk mix says to use 4 scoops of mix for 2 cups of milk. After pouring in the milk, you start adding the mix, but get distracted and accidentally put in 5 scoops of mix. How can you adjust the mix if:
    a. There is still room in the cup?
    b. The cup is already full?
46. A recipe for sabayon calls for 2 egg yolks, 3 tablespoons of sugar, and ¼ cup of white wine. After cracking the eggs, you start measuring the sugar, but accidentally put in 4 tablespoons of sugar. How can you compensate?
47. The Deepwater Horizon oil spill resulted in 4.9 million barrels of oil spilling into the Gulf of Mexico. Each barrel of oil can be processed into about 19 gallons of gasoline. How many cars could this have fueled for a year? Assume an average car gets 20 miles to the gallon, and drives about 12,000 miles in a year.
48. The store is selling lemons at 2 for $1. Each yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9-inch lemon pie requiring ½ cup of lemon juice?
49. A piece of paper can be made into a cylinder in two ways: by joining the short sides together, or by joining the long sides together[12]. Which cylinder would hold more? How much more?
50. Which of these glasses contains more liquid? How much more?

In the next 4 questions, estimate the values by making reasonable approximations for unknown values, or by doing some research to find reasonable values.

51. Estimate how many gallons of water you drink in a year.
52. Estimate how many times you blink in a day.

53. How much does the water in a 6-person hot tub weigh?
54. How many gallons of paint would be needed to paint a two-story house 40 ft long and 30 ft wide?
55. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?
56. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?
57. The flash of lightning travels at the speed of light, which is about 186,000 miles per second. The sound of lightning (thunder) travels at the speed of sound, which is about 750 miles per hour.
    a. If you see a flash of lightning, then hear the thunder 4 seconds later, how far away is the lightning?
    b. Now let's generalize that result. Suppose it takes *n* seconds to hear the thunder after a flash of lightning. How far away is the lightning, in terms of *n*?
58. Sound travels about 750 miles per hour. If you stand in a parking lot near a building and sound a horn, you will hear an echo.
    a. Suppose it takes about ½ a second to hear the echo. How far away is the building[13]?
    b. Now let's generalize that result. Suppose it takes *n* seconds to hear the echo. How far away is the building, in terms of *n*?
59. It takes an air pump 5 minutes to fill a twin sized air mattress (39 by 8.75 by 75 inches). How long will it take to fill a queen sized mattress (60 by 8.75 by 80 inches)?
60. It takes your garden hose 20 seconds to fill your 2-gallon watering can. How long will it take to fill
    a. An inflatable pool measuring 3 feet wide, 8 feet long, and 1 foot deep.[14]
    b. A circular inflatable pool 13 feet in diameter and 3 feet deep.[15]
61. You want to put a 2" thick layer of topsoil for a new 20'x30' garden. The dirt store sells by the cubic yards. How many cubic yards will you need to order?
62. A box of Jell-O costs $0.50, and makes 2 cups. How much would it cost to fill a swimming pool 4 feet deep, 8 feet wide, and 12 feet long with Jell-O? (1 cubic foot is about 7.5 gallons)
63. You read online that a 15 ft by 20 ft brick patio would cost about $2,275 to have professionally installed. Estimate the cost of having a 18 by 22 ft brick patio installed.
64. I was at the store, and saw two sizes of avocados being sold. The regular size sold for $0.88 each, while the jumbo ones sold for $1.68 each. Which is the better deal?
65. The grocery store has bulk pecans on sale, which is great since you're planning on making 10 pecan pies for a wedding. Your recipe calls for 1¾ cups pecans per pie. However, in the bulk section there's only a scale available, not a measuring cup. You run over to the baking aisle and find a bag of pecans, and look at the nutrition label to gather some info. How many pounds of pecans should you buy?
66. Soda is often sold in 20 ounce bottles. The nutrition label for one of these bottles is shown to the right. A packet of sugar (the kind they have at restaurants for your coffee or tea) typically contain 4 grams of sugar in the U.S. Drinking a 20 oz soda is equivalent to eating how many packets of sugar?[16]

For the next set of questions, *first* identify the information you need to answer the question, and *then* turn to the end of the section to find that information. The details may be imprecise; answer the question the best you can with the provided information. Be sure to justify your decision.

67. You're planning on making 6 meatloafs for a party. You go to the store to buy breadcrumbs, and see they are sold by the canister. How many canisters do you need to buy?
68. Your friend wants to cover their car in bottle caps, like in this picture.[17] How many bottle caps are you going to need?
69. You need to buy some chicken for dinner tonight. You found an ad showing that the store across town has it on sale for $2.99 a pound, which is cheaper than your usual neighborhood store, which sells it for $3.79 a pound. Is it worth the extra drive?
70. I have an old gas furnace, and am considering replacing it with a new, high efficiency model. Is upgradi3g worth it?
71. Janine is considering buying a water filter and a reusable water bottle rather than buying bottled water. Will doing so save her money?
72. Marcus is considering going car-free to save money and be more environmentally friendly. Is this financially a good decision?

For the next set of problems, research or make educated estimates for any unknown quantities needed to answer the question.

73. You want to travel from Tacoma, WA to Chico, CA for a wedding. Compare the costs and time involved with driving, flying, and taking a train. Assume that if you fly or take the train you'll need to rent a car while you're there. Which option is best?
74. You want to paint the walls of a 6ft by 9ft storage room that has one door and one window. You want to put on two coats of paint. How many gallons and/or quarts of paint should you buy to paint the room as cheaply as possible?
75. A restaurant in New York tiled their floor with pennies[18]. Just for the materials, is this more expensive than using a more traditional material like ceramic tiles? If each penny has to be laid by hand, estimate how long it would take to lay the pennies for a 12ft by 10ft room. Considering material and labor costs, are pennies a cost-effective replacement for ceramic tiles?
76. You are considering taking up part of your back yard and turning it into a vegetable garden, to grow broccoli, tomatoes, and zucchini. Will doing so save you money, or cost you more than buying vegetables from the store?
77. Barry is trying to decide whether to keep his 1993 Honda Civic with 140,000 miles, or trade it in for a used 2008 Honda Civic. Consider gas, maintenance, and insurance costs in helping him make a decision.
78. Some people claim it costs more to eat vegetarian, while some claim it costs less. Examine your own grocery habits, and compare your current costs to the costs of switching your diet (from omnivore to vegetarian or vice versa as appropriate). Which diet is more cost effective based on your eating habits?

Info for the breadcrumbs question

Nutrition Facts	
Serving Size: 1/3 cup (30g)	
Servings per Container: about 14	
Amount Per Serving	
Calories 110	Calories from Fat 15
	% Daily Value*
Total Fat 1.5g	2%

How much breadcrumbs does the recipe call for?

It calls for 1½ cups of breadcrumbs.

How many meatloaves does the recipe make?

It makes 1 meatloaf.

How many servings does that recipe make?

It says it serves 8.

How big is the canister?

It is cylindrical, 3.5 inches across and

7 inches tall.

What is the net weight of the contents of 1 canister?

15 ounces.

How much does a cup of breadcrumbs weigh?

I'm not sure, but maybe something from the nutritional label will help.

How much does a canister cost? $2.39

Info for bottle cap car

What kind of car is that?

A 1993 Honda Accord.

How big is that car / what are the dimensions? Here is some details from MSN autos:

Weight: 2800lb Length: 185.2 in Width: 67.1 in Height: 55.2 in

How much of the car was covered with caps?

Everything but the windows and the underside.

How big is a bottle cap?

Caps are 1 inch in diameter.

Info for chicken problem

How much chicken will you be buying?

Four pounds

How far are the two stores?

My neighborhood store is 2.2 miles away, and takes about 7 minutes. The store across town is 8.9 miles away, and takes about 25 minutes.

What kind of mileage does your car get?

It averages about 24 miles per gallon in the city.

How many gallons does your car hold?

About 14 gallons

How much is gas?

About $3.69/gallon right now.

Info for furnace problem

How efficient is the current furnace?

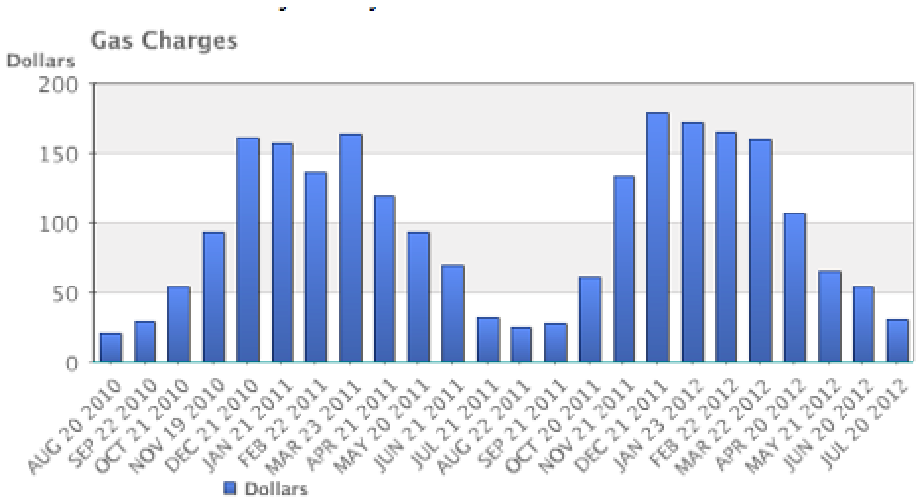
It is a 60% efficient furnace.

How efficient is the new furnace?

It is 94% efficient.

What is your gas bill?

Here is the history for 2 years:

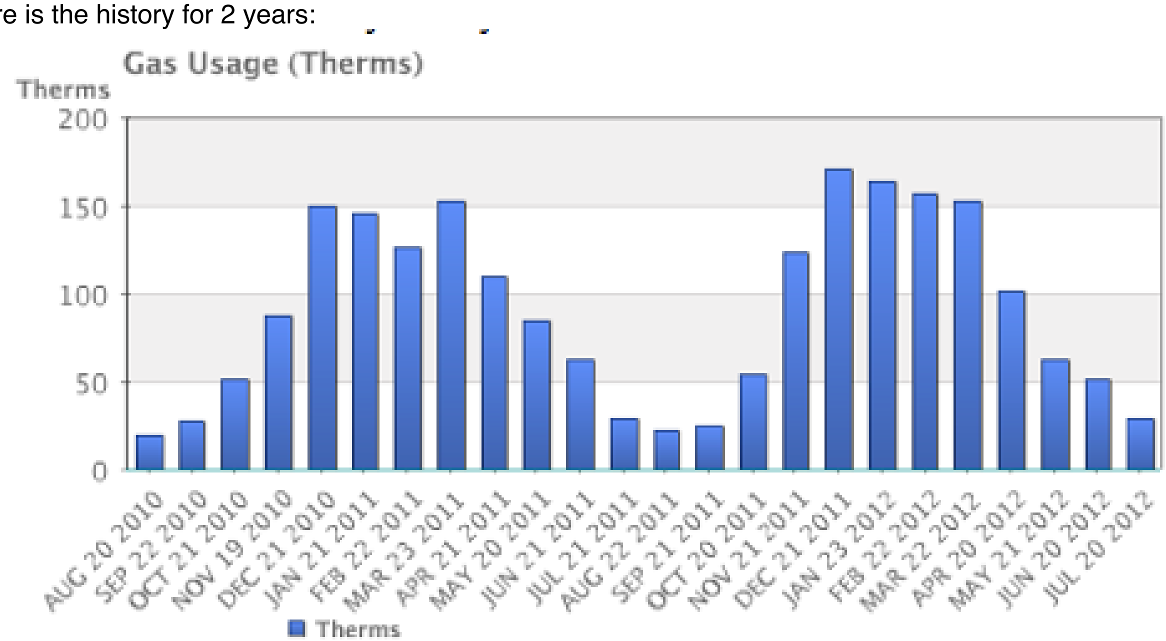


How much do you pay for gas?

There is $10.34 base charge, plus $0.39097 per Therm for a delivery charge, and $0.65195 per Therm for cost of gas.

How much gas do you use?

Here is the history for 2 years:



How much does the new furnace cost?

It will cost $7,450.

How long do you plan to live in the house?

Probably at least 15 years.

Info for water filter problem

How much water does Janine drink in a day?

She normally drinks 3 bottles a day, each 16.9 ounces.

How much does a bottle of water cost?

She buys 24-packs of 16.9 ounce bottles for $3.99.

How much does a reusable water bottle cost?

About $10.

How long does a reusable water bottle last?

Basically forever (or until you lose it).

How much does a water filter cost? How much water will they filter?

- A faucet-mounted filter costs about $28. Refill filters cost about $33 for a 3-pack. The box says each filter will filter up to 100 gallons (378 liters)
- A water filter pitcher costs about $22. Refill filters cost about $20 for a 4-pack. The box says each filter lasts for 40 gallons or 2 months
- An under-sink filter costs $130. Refill filters cost about $60 each. The filter lasts for 500 gallons.

Info for car-free problem

Where does Marcus currently drive? He:

- Drives to work 5 days a week, located 4 miles from his house.
- Drives to the store twice a week, located 7 miles from his house.
- Drives to other locations on average 5 days a week, with locations ranging from 1 mile to 20 miles.
- Drives to his parent's house 80 miles away once a month.

How will he get to these locations without a car?

- For work, he can walk when it's sunny and he gets up early enough. Otherwise he can take a bus, which takes about 20 minutes
- For the store, he can take a bus, which takes about 35 minutes.
- Some of the other locations he can bus to. Sometimes he'll be able to get a friend to pick him up. A few locations he is able to walk to. A couple locations are hard to get to by bus, but there is a ZipCar (short term car rental) location within a few blocks.
- He'll need to get a ZipCar to visit his parents.

How much does gas cost?

About $3.69/gallon.

How much does he pay for insurance and maintenance?

- He pays $95/month for insurance.
- He pays $30 every 3 months for an oil change, and has averaged about $300/year for other maintenance costs.

How much is he paying for the car?

- He's paying $220/month on his car loan right now, and has 3 years left on the loan.
- If he sold the car, he'd be able to make enough to pay off the loan.

- If he keeps the car, he's planning on trading the car in for a newer model in a couple years.

What mileage does his car get?

About 26 miles per gallon on average.

How much does a bus ride cost?

$2.50 per trip, or $90 for an unlimited monthly pass.

How much does a ZipCar rental cost?

- The "occasional driving plan": $25 application fee and $60 annual fee, with no monthly commitment. Monday-Thursday the cost is $8/hour, or $72 per day. Friday-Sunday the cost is $8/hour or $78/day. Gas, insurance, and 180 miles are included in the cost. Additional miles are $0.45/mile.
- The "extra value plan": Same as above, but with a $50 monthly commitment, getting you a 10% discount on the usage costs.

Extension: Taxes

Governments collect taxes to pay for the services they provide. In the United States, federal income taxes help fund the military, the environmental protection agency, and thousands of other programs. Property taxes help fund schools. Gasoline taxes help pay for road improvements. While very few people enjoy paying taxes, they are necessary to pay for the services we all depend upon.

Taxes can be computed in a variety of ways, but are typically computed as a percentage of a sale, of one's income, or of one's assets.

Example 1

The sales tax rate in a city is 9.3%. How much sales tax will you pay on a $140 purchase?

The sales tax will be 9.3% of $140. To compute this, we multiply $140 by the percent written as a decimal: $140(0.093) = $13.02.

When taxes are not given as a fixed percentage rate, sometimes it is necessary to calculate the **effective rate.**

Effective rate

The effective tax rate is the equivalent percent rate of the tax paid out of the dollar amount the tax is based on.

Example 2

Joan paid $3,200 in property taxes on her house valued at $215,000 last year. What is the effective tax rate?

We can compute the equivalent percentage: 3200/215000 = 0.01488, or about 1.49% effective rate.

Taxes are often referred to as progressive, regressive, or flat.

Tax categories

A **flat tax**, or proportional tax, charges a constant percentage rate.

A **progressive tax** increases the percent rate as the base amount increases.

A **regressive tax** decreases the percent rate as the base amount increases.

Example 3

The United States federal income tax on earned wages is an example of a progressive tax. People with a higher wage income pay a higher percent tax on their income.

For a single person in 2011, adjusted gross income (income after deductions) under $8,500 was taxed at 10%. Income over $8,500 but under $34,500 was taxed at 15%.

A person earning $10,000 would pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850	10% of $8500
1500(0.15) = 225	15% of the remaining $1500 of income
Total tax: = $1075	

The effective tax rate paid is 1075/10000 = 10.75%

A person earning $30,000 would also pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850   10% of $8500
21500(0.15) = 3225   15% of the remaining $21500 of income
Total tax: = $4075

The effective tax rate paid is 4075/30000 = 13.58%.

Notice that the effective rate has increased with income, showing this is a progressive tax.

Example 4

A gasoline tax is a flat tax when considered in terms of consumption, a tax of, say, $0.30 per gallon is proportional to the amount of gasoline purchased. Someone buying 10 gallons of gas at $4 a gallon would pay $3 in tax, which is $3/$40 = 7.5%. Someone buying 30 gallons of gas at $4 a gallon would pay $9 in tax, which is $9/$120 = 7.5%, the same effective rate.

However, in terms of income, a gasoline tax is often considered a regressive tax. It is likely that someone earning $30,000 a year and someone earning $60,000 a year will drive about the same amount. If both pay $60 in gasoline taxes over a year, the person earning $30,000 has paid 0.2% of their income, while the person earning $60,000 has paid 0.1% of their income in gas taxes.

Try it Now 1

A sales tax is a fixed percentage tax on a person's purchases. Is this a flat, progressive, or regressive tax?

Try it Now Answers

1. While sales tax is a flat percentage rate, it is often considered a regressive tax for the same reasons as the gasoline tax.

Income Taxation

Many people have proposed various revisions to the income tax collection in the United States. Some, for example, have claimed that a flat tax would be fairer. Others call for revisions to how different types of income are taxed, since currently investment income is taxed at a different rate than wage income.

The following two projects will allow you to explore some of these ideas and draw your own conclusions.

Project 1: Flat tax, Modified Flat Tax, and Progressive Tax.

Imagine the country is made up of 100 households. The federal government needs to collect $800,000 in income taxes to be able to function. The population consists of 6 groups:

Group A: 20 households that earn $12,000 each

Group B: 20 households that earn $29,000 each

Group C: 20 households that earn $50,000 each

Group D: 20 households that earn $79,000 each

Group E: 15 households that earn $129,000 each

Group F: 5 households that earn $295,000 each

This scenario is roughly proportional to the actual United States population and tax needs. We are going to determine new income tax rates.

The first proposal we'll consider is a flat tax – one where every income group is taxed at the same percentage tax rate.

1) Determine the total income for the population (all 100 people together)

2) Determine what flat tax rate would be necessary to collect enough money.

The second proposal we'll consider is a modified flat-tax plan, where everyone only pays taxes on any income over $20,000. So, everyone in group A will pay no taxes. Everyone in group B will pay taxes only on $9,000.

3) Determine the total *taxable* income for the whole population

4) Determine what flat tax rate would be necessary to collect enough money in this modified system

5) Complete this table for both the plans

		Flat Tax Plan		Modified Flat Tax Plan	
Group	**Income per household**	**Income tax per household**	**Income after taxes**	**Income tax per household**	**Income after taxes**
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

The third proposal we'll consider is a progressive tax, where lower income groups are taxed at a lower percent rate, and higher income groups are taxed at a higher percent rate. For simplicity, we're going to assume that a household is taxed at the same rate on *all* their income.

6) Set progressive tax rates for each income group to bring in enough money. There is no one right answer here – just make sure you bring in enough money!

Group	Income per household	Tax rate (%)	Income tax per household	Total tax collected for all households	Income after taxes per household
A	$12,000				
B	$29,000				

C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				
				This better total to $800,000	

7) Discretionary income is the income people have left over after paying for necessities like rent, food, transportation, etc. The cost of basic expenses does increase with income, since housing and car costs are higher, however usually not proportionally. For each income group, estimate their essential expenses, and calculate their discretionary income. Then compute the effective tax rate for each plan relative to discretionary income rather than income.

Group	**Income per household**	**Discretionary Income (estimated)**	**Effective rate, flat**	**Effective rate, modified**	**Effective rate, progressive**
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				

E	$129,000				
F	$295,000				

8) Which plan seems the most fair to you? Which plan seems the least fair to you? Why?

Project 2: Calculating Taxes.

Visit www.irs.gov, and download the most recent version of forms 1040, and schedules A, B, C, and D.

Scenario 1: Calculate the taxes for someone who earned $60,000 in standard wage income (W-2 income), has no dependents, and takes the standard deduction.

Scenario 2: Calculate the taxes for someone who earned $20,000 in standard wage income, $40,000 in qualified dividends, has no dependents, and takes the standard deduction. (Qualified dividends are earnings on certain investments such as stocks.)

Scenario 3: Calculate the taxes for someone who earned $60,000 in small business income, has no dependents, and takes the standard deduction.

Based on these three scenarios, what are your impressions of how the income tax system treats these different forms of income (wage, dividends, and business income)?

Scenario 4: To get a more realistic sense for calculating taxes, you'll need to consider itemized deductions. Calculate the income taxes for someone with the income and expenses listed below.

Married with 2 children, filing jointly

Wage income: $50,000 combined

Paid sales tax in Washington State

Property taxes paid: $3200

Home mortgage interest paid: $4800

Charitable gifts: $1200

[1] http://www.factcheck.org/cheney_edwards_mangle_facts.html

[2] http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics

[3] http://www.whitehouse.gov/sites/default/files/omb/budget/fy2013/assets/hist07z1.xls

[4] https://www.cia.gov/library/publications/the-world-factbook/rankorder/2042rank.html

[5] http://data.worldbank.org/indicator/SP.POP.TOTL

[6] http://www.cnn.com/2010/SHOWBIZ/Music/07/19/cd.digital.sales/index.html

[7] http://www.cnn.com/2012/06/27/politics/btn-health-care/index.html

[8] http://www.politico.com/news/stories/0712/78134.html

[9] http://factcheck.org/2012/07/twisting-health-care-taxes/

[10] http://www.northjersey.com/news/opinions/lautenberg_073112.html?c=y&page=2

[11] http://www.the-numbers.com/movies/records/budgets.php

[12] http://vimeo.com/42501010

[13] http://vimeo.com/40377128

[14] http://www.youtube.com/watch?v=DIkwefReHZc

[15] http://www.youtube.com/watch?v=p9SABH7Yg9M

[16] http://www.youtube.com/watch?v=62JMfv0tf3Q

[17] Photo credit: http://www.flickr.com/photos/swayze/, CC-BY

[18] http://www.notcot.com/archives/2009/06/floor-of-pennie.php

GEOMETRY

MODULE 2 OVERVIEW

What You'll Learn To Do: Understand and use geometry to solve problems.

Learning Objectives

- Apply formulas to solve perimeter, area, surface area, and volume problems.
- Solve problems using similar triangles and the Pythagorean Theorem.
- Distinguish Euclidean and non-Euclidean geometry.
- Describe the study of topology and its importance.
- Describe the study of networks and its importance.
- Recognize and create tilings.
- Recognize when geometry is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

The learning activities for this module include:

Reading Assignments

- **Read:** Perimeter and Area
- **Read:** Triangles
- **Read:** Volume
- **Read:** Surface Area
- **Read:** Graph Theory
- **Watch:** Graph Theory Videos
- **Read:** Topology, Tiling, Non-Euclidean Geometry

Homework Assignments

- **Submit:** Perimeter and Area (12 points)
- **Submit:** Triangles (4 points)
- **Submit:** Surface Area and Volume (14 points)
- **Submit:** Graph Theory (2 points)
- **Submit:** Topology, Tiling, and Non-Euclidean Geometry (7 points)
- **Discuss:** Application of Geometry (20 points)

- **Complete:** Exam 1

PERIMETER AND AREA

Perimeter

Perimeter is a one-dimensional measurement that is taken around the outside of a closed geometric shape. Let's start our discussion of the concept of perimeter with an example.

Guided Example

Joseph does not own a car so must ride the bus or walk everywhere he goes. On Mondays, he must get to school, to work, and back home again. His route is pictured in figure 1.

The obvious question to ask in this situation is, "how many miles does Joseph travel on Mondays"? To compute, we each distance: 3 + 6 + 6 = 15.

Joseph travels 15 miles on Mondays.

Another way to work with this situation is to draw a shape that represents Joseph's travel route and is labeled with the distance from one spot to another.

Notice that the shape made by Joseph's route is that of a closed geometric figure with three sides (a triangle) (see figure 2). What we can ask about this shape is, "what is the *perimeter* of the triangle"?

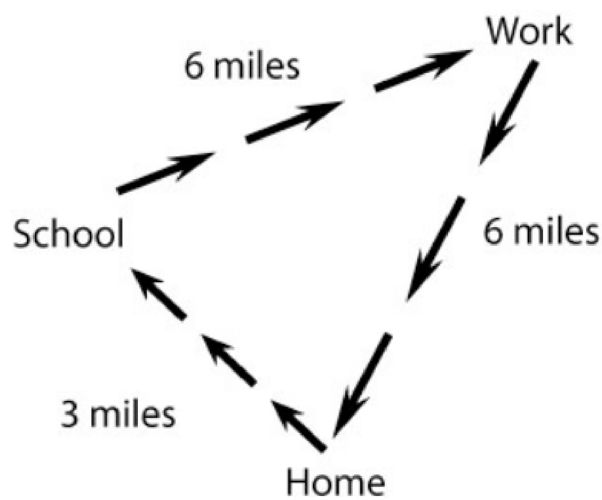


Figure 1.

Perimeter means "distance around a closed figure or shape" and to compute we add each length: 3 + 6 + 6 = 15

Our conclusion is the same as above: Joseph travels 15 miles on Mondays.

However, what we did was model the situation with a geometric shape and then apply a specific geometric concept (*perimeter*) to computer how far Joseph traveled.

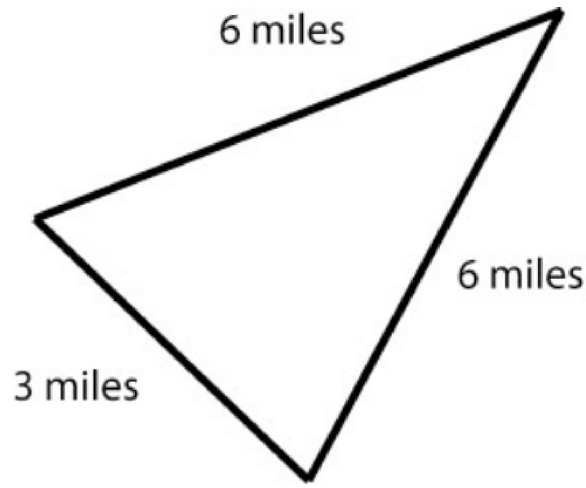
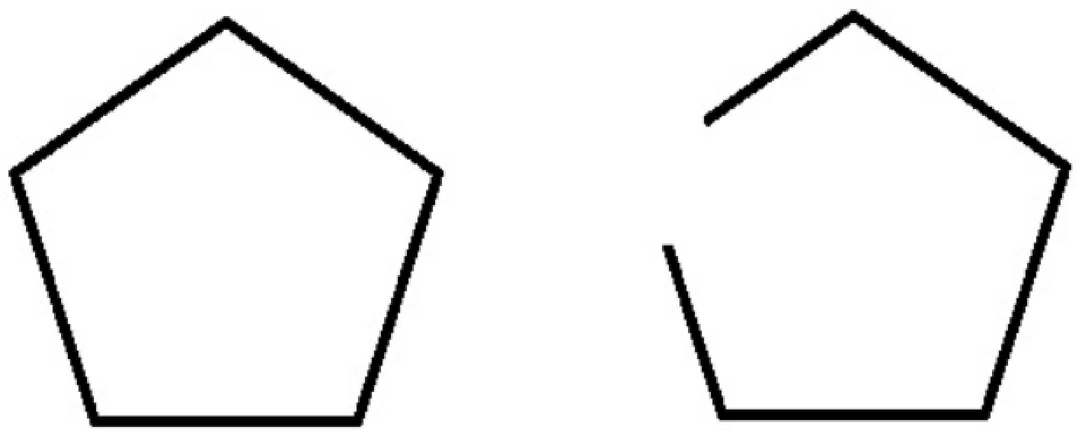


Figure 2.

Notes on Perimeter

- Perimeter is a one-dimensional measurement that represents the distance around a closed geometric figure or shape (no gaps).
- To find perimeter, add the lengths of each side of the shape.
- If there are units, include units in your final result. Units will always be of single dimension (i.e. feet, inches, yards, centimeters, etc…)

To compute perimeter, our shapes must be closed. Figure 3 shows the difference between a closed figure and an open figure.



Closed figure:
There is an inside and an outside to the shape. To get from inside to outside, you must cross the boundary of the shape

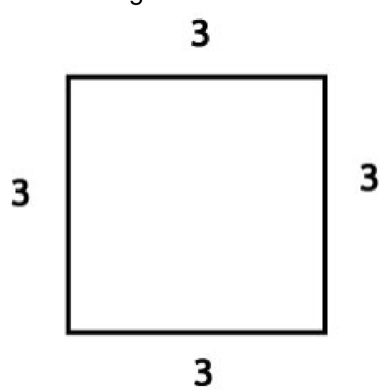
Open figure Not closed:
There isn't an inside or outside. Even a portion that seems enclosed can be reached without crossing the boundary of the shape.

Figure 3.

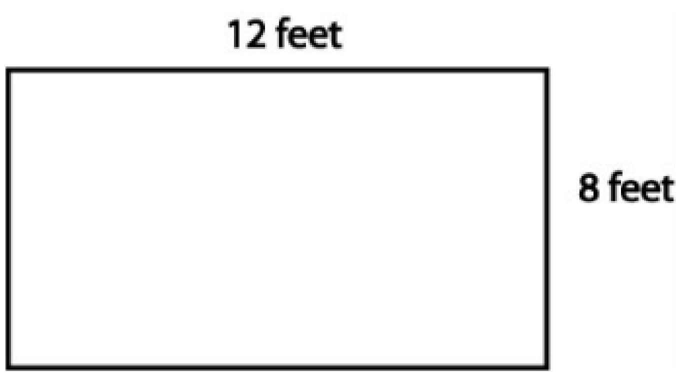
Example 1

Find the perimeter for each of the shapes below.

1. Add the lengths of each side.



2. Sometimes you have to make assumptions if lengths are not labeled.

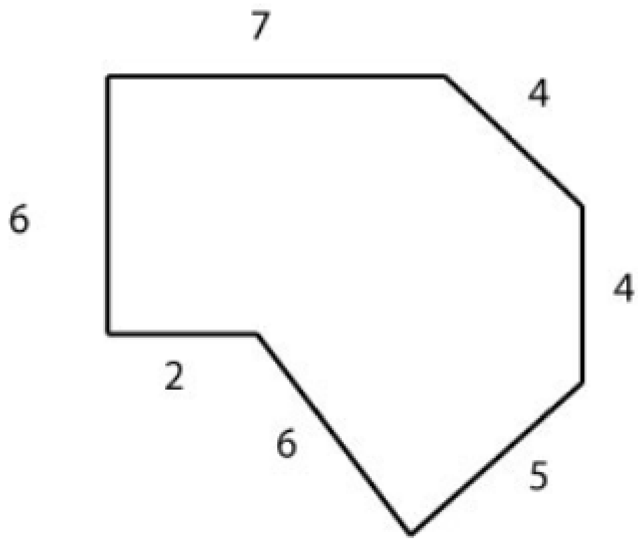


Solutions

1. 12 units
2. 40 feet

Example 2

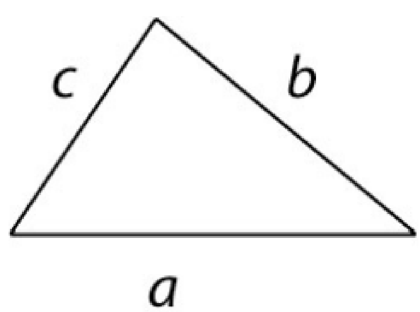
How do we find the perimeter of this more complicated shape?

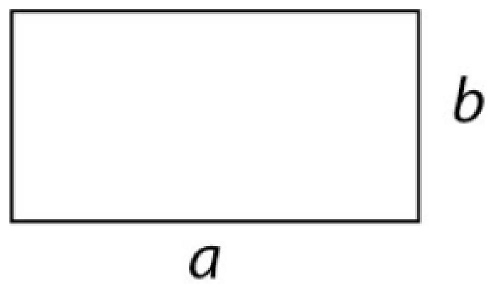


Solution

Just keep adding those side lengths. 6 + 7 + 4 + 4 + 5 + 6 + 2 = 34 units

If you look closely at the shapes in the previous examples, you might notice some ways to write each perimeter as a more explicit formula. See if the results from what we have done so far match the formulas below.

Shape		Perimeter
Triangle with side lengths, *a*, *b*, *c*:		$P = a + b + c$

Shape		Perimeter
Square with side length *a*:		$P = a + a + a + a$ $P = 4a$
Rectangle with side lengths *a*, *b*:		$P = a + b + a + b$ $P = a + a + b + b$ $P = 2a + 2b$

Circumference

You may realize that we have not yet discussed the distance around a very important geometric shape: a circle! The distance around a circle has a special name called the **circumference**. To find the circumference of a circle, we use this formula: $C = 2\pi r$

In this formula, π is pronounced "pi," and is defined as the circumference of a circle divided by its diameter: $\pi = \frac{C}{d}$. We usually replace π with the approximation 3.14. The letter *r* represents the **radius** of the circle.

Let's see where the formula for circumference comes from. Figure 4 shows a generic circle with radius *r.*

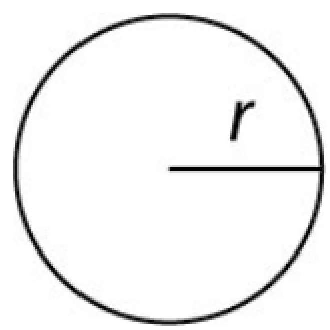


Figure 4.

Notes about $C = 2\pi r$

Remember that in the formula, when computing the circumference $C = 2\pi r$, we multiply as follows **usually** substituting 3.14 in place of π:

$C = 2 \times 3.14 \times r$

Often, the use of ( ) will help make the different parts of the formula easier to see:

$C = (2)\times(3.14)\times(r)$

Origins of $C = 2\pi r$

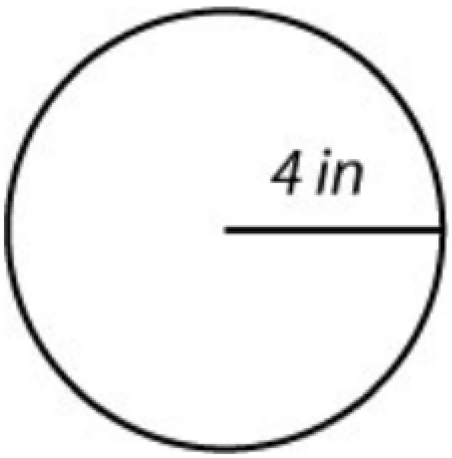
As mentioned earlier, the special number π is defined as the ratio of a circle's circumference to its diameter. We can write this in equation form as: $\frac{C}{d} = \pi$

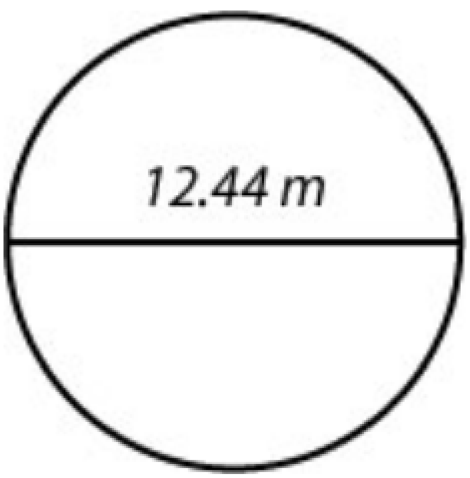
We know from our previous work that to identify the unknown, C, we can move d to the other side of the equation by writing $C = \pi d$. The diameter is all the way across the circle's middle so the diameter is twice the radius. We can update C in terms of the radius as $C = \pi(2r)$. With a little final rearranging of the order our parts are written in, we can say that $C = 2\pi r$.

Let's use the formula to find the circumference of a few circles.

Example 3

Find the circumference of each of the following circles. Leave your answers first in exact form and then in rounded form (to the hundredths place). (Note that when a radius is given, its value is centered above a radius segment. When a diameter is given, its value is centered above a diameter segment.)

1. 

2. 

Solutions

1. Exact 8π in; rounded from exact answer 25.13 in; rounded using 3.14 for π 25.12 in
2. Exact 12.44π m; rounded from exact answer 39.08 m; rounded using 3.14 for π 39.06 m

Exact Form vs. Rounded Form

- π is a number in exact form. It is not rounded.
- 3.14 is a rounded form approximation for π

Why does it matter which form we use? It matters because when we round, we introduce error into our final result. For this class, that error is usually acceptable. However, you will find in other subjects such as physics or chemistry, that level of accuracy is a concept of great importance. Let's see an example of the difference in forms.

Example 4

The radius of the moon is about 1079 miles. What is the circumference? Let's solve this using both the exact form and the rounded form:

Exact Solution

$C = 2\pi r = 2\pi(1079) = 2158\pi$

To round **from** the exact solution, use the π button on your calculator to get

$2158\pi \approx 6779.56$

Rounded Solution

$C = 2\pi r = 2(3.14)(1079) \approx 6776.12$

Notice that our final results are different. That difference is the error created by using 3.14 as an initial approximation for π. When doing homework and tests, read the directions carefully on each problem to see which form to use.

Example 5

Find the circumference or perimeter given in each described situation below. Include a drawing of the shape with the included information. Use the examples to help determine what shapes to draw. Show all work. As in the examples, if units are included then units should be present in your final result. Round to tenths unless indicated otherwise.

1. Find the perimeter of a square with side length 2.17 feet.
2. Find the perimeter of a rectangle with sides of length 4.2 and 3.8.
3. Find the perimeter of a triangle with sides of length 2, 5, 7.
4. Find the circumference of a circle with radius 6 inches. Present answer in exact form and also compute using 3.14 for π. Present rounded form to the nearest tenth.
5. Find the circumference of a circle with diameter 14.8 inches. Present answer in exact form and also compute using 3.14 for π. Present rounded form to the nearest tenth.

Solutions

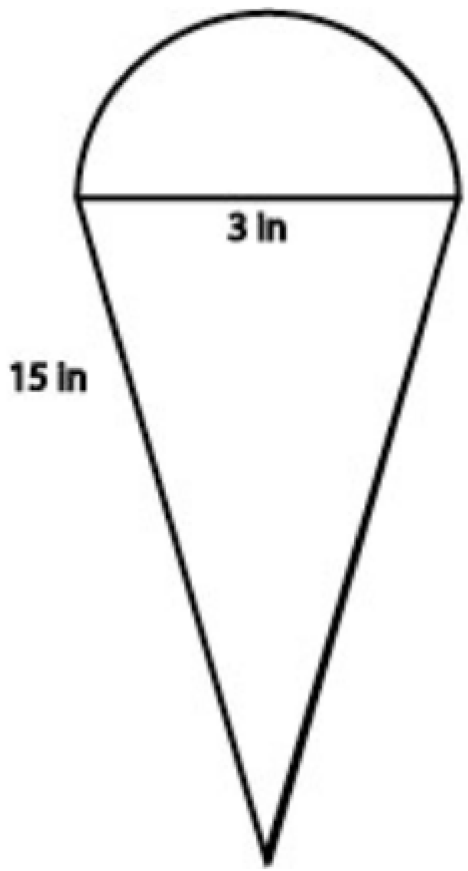
1. 8.68 feet
2. 16
3. 14
4. Exact 12π in, Rounded 37.7 in

5. Exact 14.8π in, Rounded 46.5 in

Example 6

Finding the Distance around Non-Standard Shapes

The basic formulas for perimeter of straight-line shapes and the circumference of a circle will help us find the distance around more complicated figures. Find the distance around the following shape. Round final answer to tenths and use 3.14 for π.



Solution

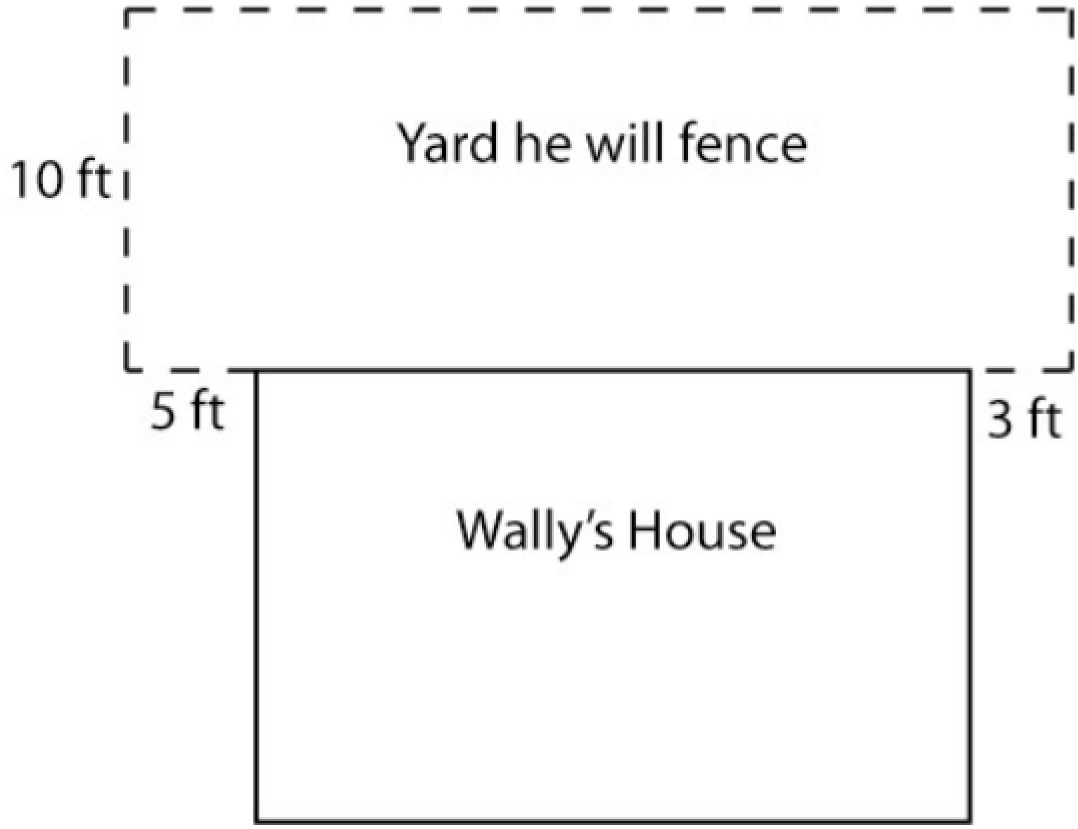
34.7 in

Example 7

Applications of Perimeter and Circumference

Our knowledge of basic geometric shapes can be applied to solve "real-life" problems.

Wally wants to add a fence to the back of his house to make some room for his children to play safely (see diagram below). He began measuring his yard but got distracted and forgot to finish measuring before he went to the store. If he remembers that the back wall of his house is 15 yards long, does he have enough information to buy the fencing he needs? If so, how many feet should he buy?



Solution

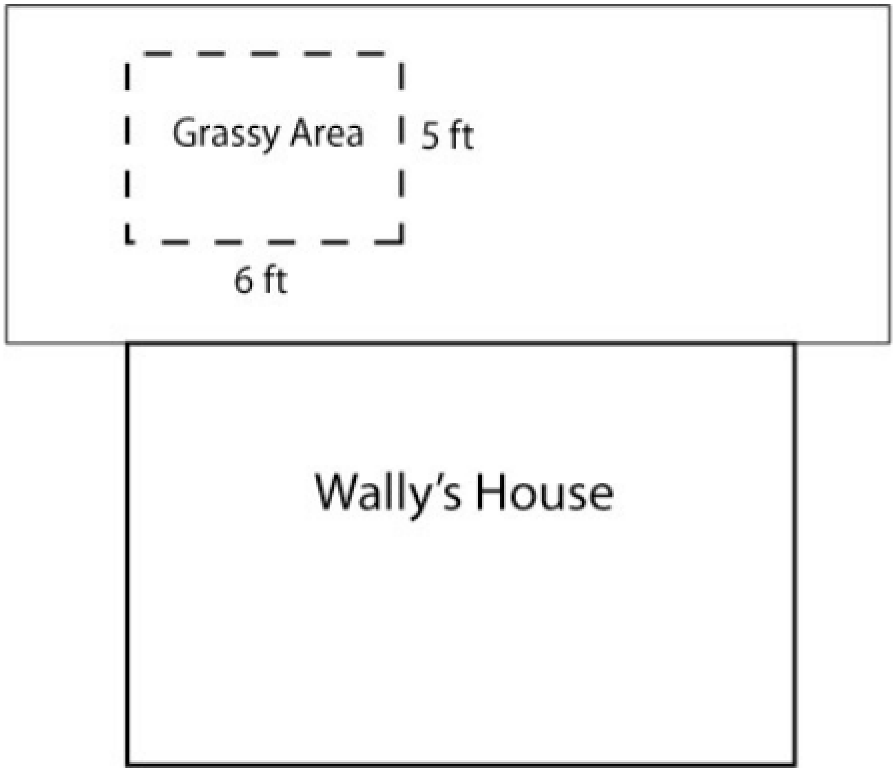
81 feet

Area

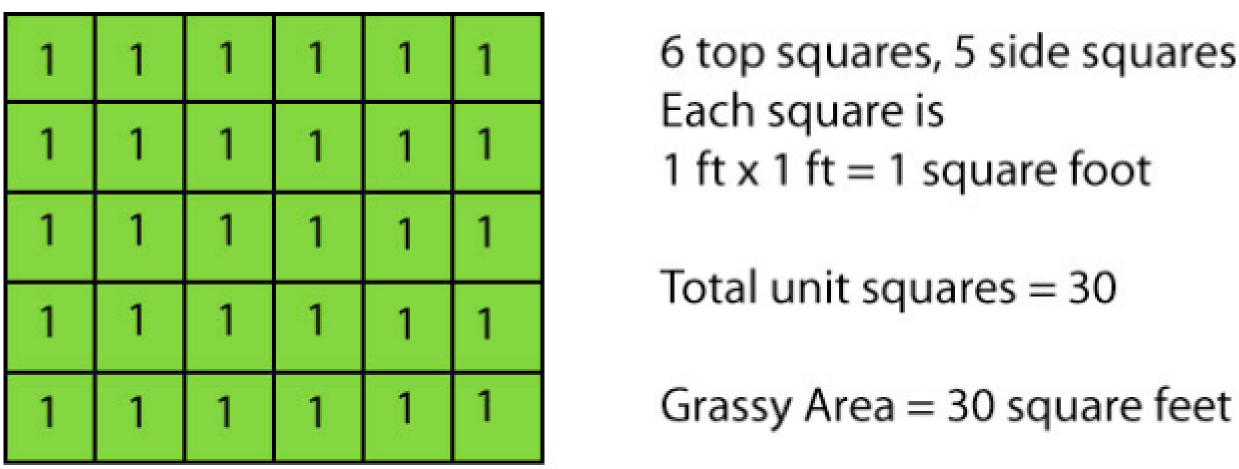
Let's take another look at Wally's backyard from Example 7 in order to introduce the next concept: area.

Guided Example

Wally successfully fenced his yard but now wants to add some landscaping and create a grassy area as shown below.



He heads down to the local lawn store and finds out that in order to determine how much sod he needs, he must figure out the square footage of the area he wants to add grass to. On his way home, he realizes that if he divides the grassy area into sections that are 1 foot by 1 foot and then counts them, he can determine the square footage. Here is the information Wally drew up when he got home.



Wally correctly determined the area of the rectangular grassy section to be 30 square feet.

Notes on Area

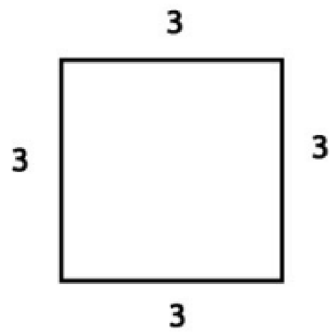
- **Area** is a two-dimensional measurement that represents the amount of space inside a two-dimensional shape.
- To find the area, count the number of unit squares inside the shape.

- If there are units, include units in your final result. Units will always be two-dimensional (i.e. square feet, square yards, square miles, etc…)

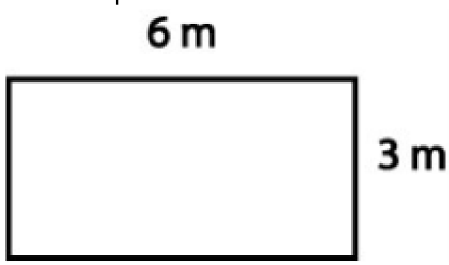
Example 8

Find the area for each of the shapes below.

1. Remember to count the unit squares inside the shape.

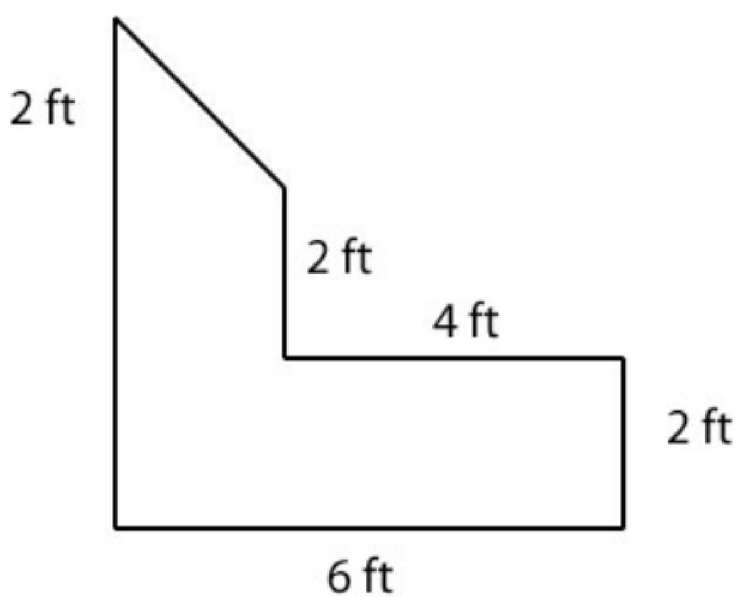


2. Is there a pattern here that would make our work easier?

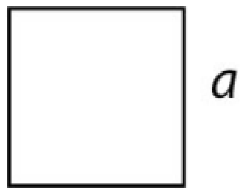
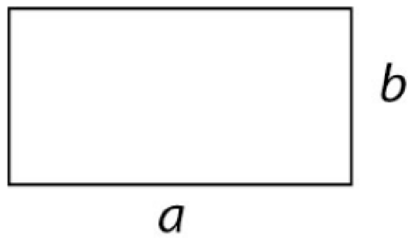


Example 9

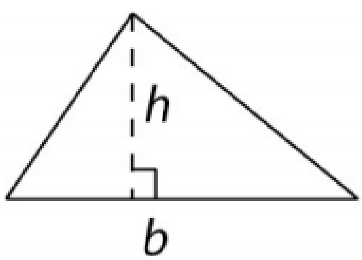
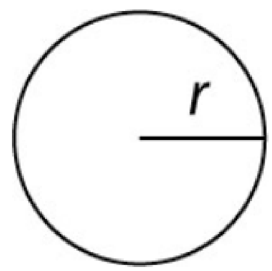
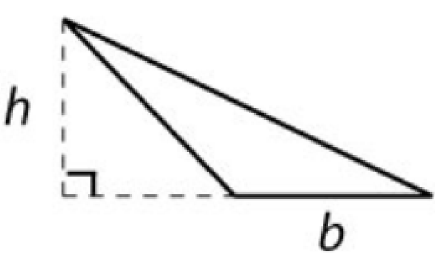
How do we find the area for shapes that are more complicated? Break up the areas into shapes that we recognize and add the area values together.



If you look closely at the shapes in the previous examples, you might notice some ways to write each area as a more explicit formula. See if the results from what we have done so far match the formulas below.

Shape		**Area**
Square with side length *a*		$A = a \cdot a$ $A = a^2$
Rectangle with side lengths *a*, *b*		$A = a \cdot b$ (You will also see this as $A = \text{length} \cdot \text{width}$)

The area formulas for the shapes below are more complicated to derive so the formulas are listed for you in the table.

Shape	**Shape**
Triangle with height h and base b  $A = \frac{1}{2}bh = \frac{bh}{2}$ Read as "one-half base times height" Note that h is the straight-line distance from top of the triangle directly to the other side. The small box next to *h* indicates this. In math terms the box indicates a 90° (right) angle.	Circle with radius *r*  $A = \pi r^2$ Read as "pi times radius squared"
	If your triangle is as pictured at left, then the height is drawn and measured outside the triangle. The area formula is the same.

Example 10

Find the area for each described situation. Create a drawing of the shape with the included information. Show all work. As in the examples, if units are included then units should be present in your final result. Use 3.14 for π and round answers to tenths as needed.

1. Find the area of a rectangle whose length is 12.9 meters and height is one-third that amount.
2. Find the area of a triangle with base $24\frac{1}{2}$ inches and height 7 inches.
3. Find the area of a circle with radius $2\frac{1}{3}$ inches. Present answer in exact form and also compute rounded form using 3.14 for π. Present rounded form to the nearest tenth.

Solutions

1. 55.5 m^2 or 55.5 square meters (rounded)
2. 85.8 in^2 or 85.8 square inches (rounded)
3. Exact 49/9 π in^2, Rounded 17.1 in^2

Example 11

Find the area given each described situation. Include a drawing of the shape with the included information. Show all work. As in the examples, if units are included then units should be present in your final result. Round answers to tenths unless otherwise indicated.

1. Find the area of a square with side length 4.2 feet.
2. Find the area of a rectangle with sides of length 4.2 and 3.8.
3. Find the area of a triangle with height 7 inches and base 12 inches.
4. Find the area of a circle with radius 6 inches. Present answer in exact form and also compute using 3.14 for π. Present rounded form to the nearest tenth.

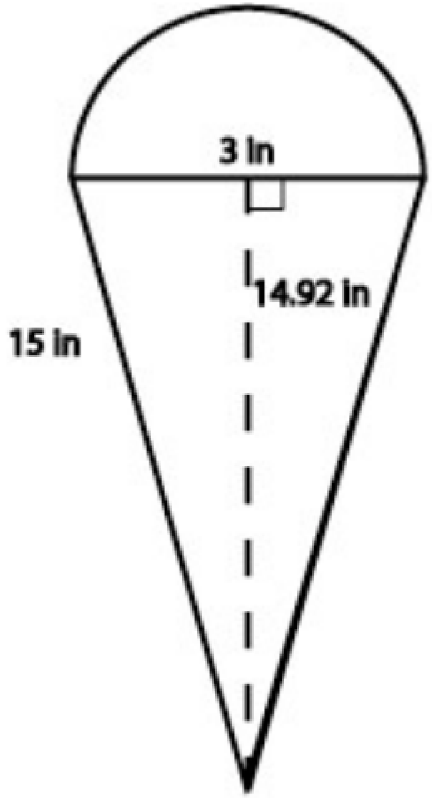
Solutions

1. 17.64 ft^2 or 17.64 square feet
2. 16.0
3. 42 in^2 or 42 square inches
4. Exact 36π in^2 or 36π square inches, Rounded using 3.14 for π 113.0 in^2 or 113.0 square inches

Example 12

Finding the Area of Non-Standard Shapes

The basic formulas for area will help us find the area of more complicated figures. This is the same problem we found the perimeter for earlier. Find the area of the given shape. Compute using 3.14 for π and round to the nearest tenth.



Solution

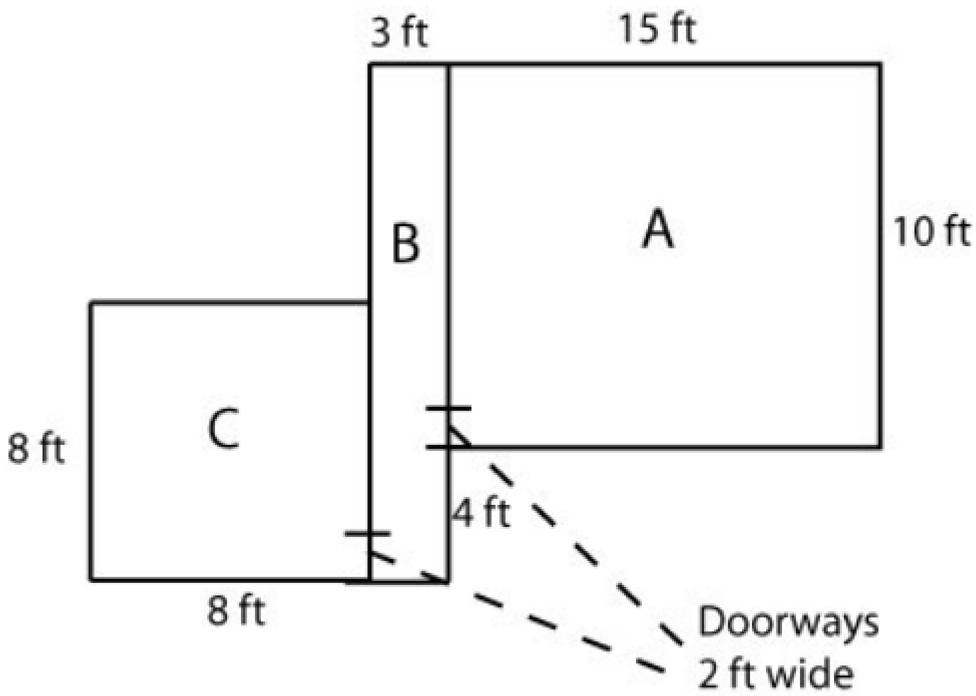
Rounded using 3.14 for π 25.9 in^2

Example 13

Applications of Area and Perimeter

We can combine our knowledge of area/perimeter to solve problems such as this one.

Wally is still fixing up his house and has a flooring project to complete. He wants to buy enough bamboo flooring to cover the floor space in rooms A, C and hallway B and enough bamboo edging for baseboards in all the spaces as well. How many square feet of flooring and how many feet of baseboards should he buy?



Solution

256 ft^2 flooring, 108 ft edging

TRIANGLES

Similar Triangles

Let's begin our discussion of similar triangles with an example.

Guided Example

Mary was out in the yard one day and had her two daughters with her. She was doing some renovations and wanted to know how tall the house was. She noticed a shadow 3 feet long when her daughter was standing 12 feet from the house and used it to set up figure 1.

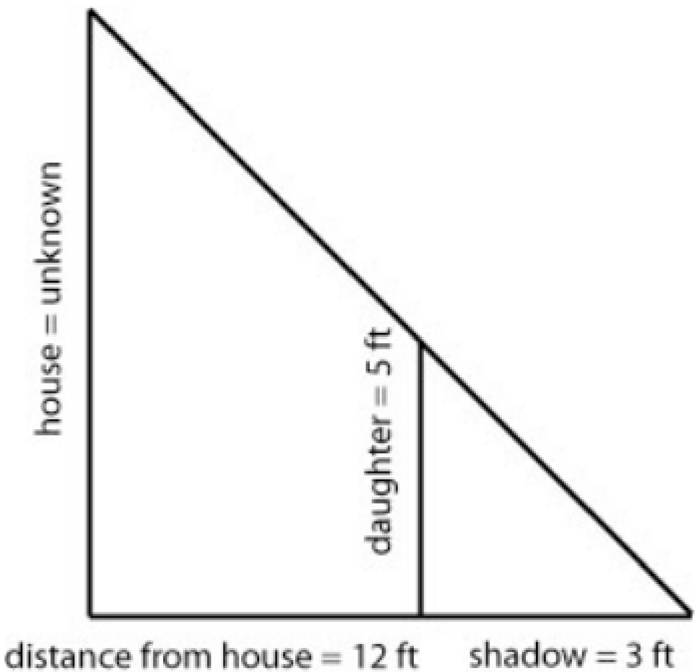


Figure 1.

We can take that drawing and separate the two triangles as follows allowing us to focus on the numbers and the shapes.

These triangles are what are called **similar triangles**. They have the same angles and sides in *proportion* to each other. We can use that information to determine the height of the house as seen in figure 2.

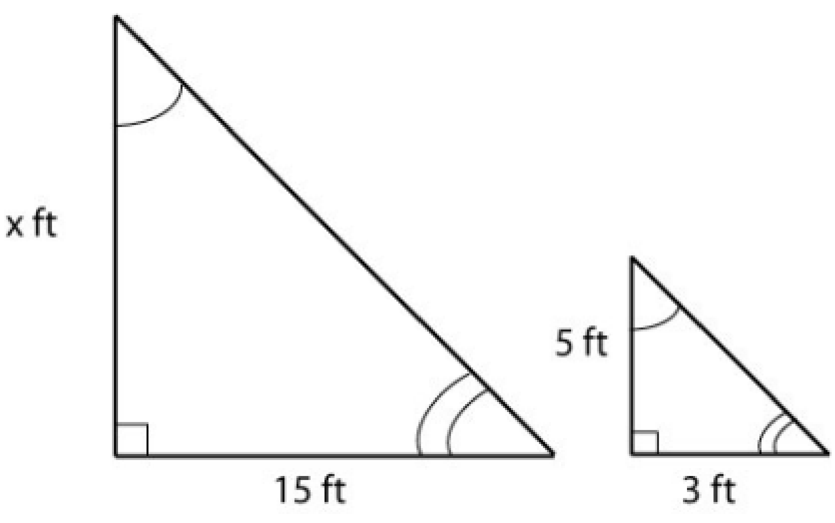


Figure 2.

To determine the height of the house, we set up the following proportion:

$$\frac{x}{15} = \frac{5}{3}$$

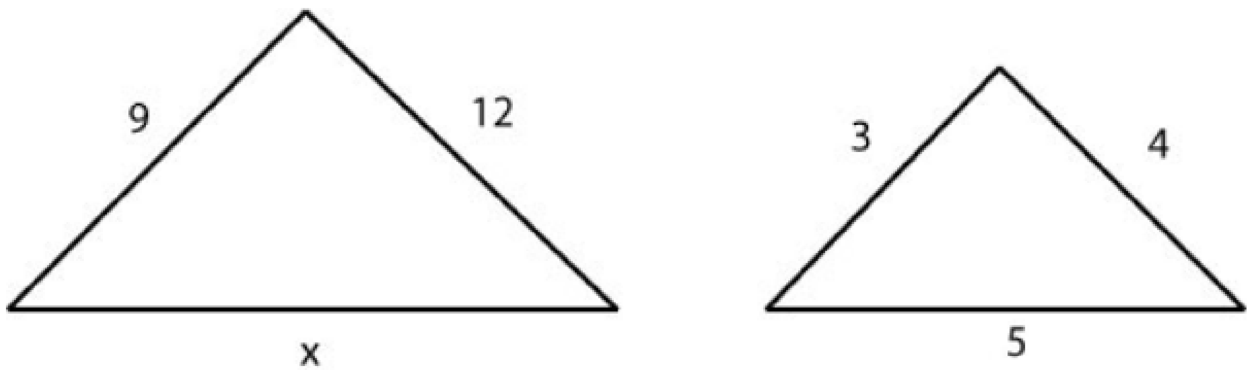
Then, we solve for the unknown *x* by using cross products as we have done before:

$$x = \frac{5 \times 15}{3} = \frac{75}{3} = 25$$

Therefore, we can conclude that the house is 25 feet high.

Example 1

Use the Similar Triangles process to determine the length of the missing side. Set up the proportions in as many ways as possible and show the results are all the same.

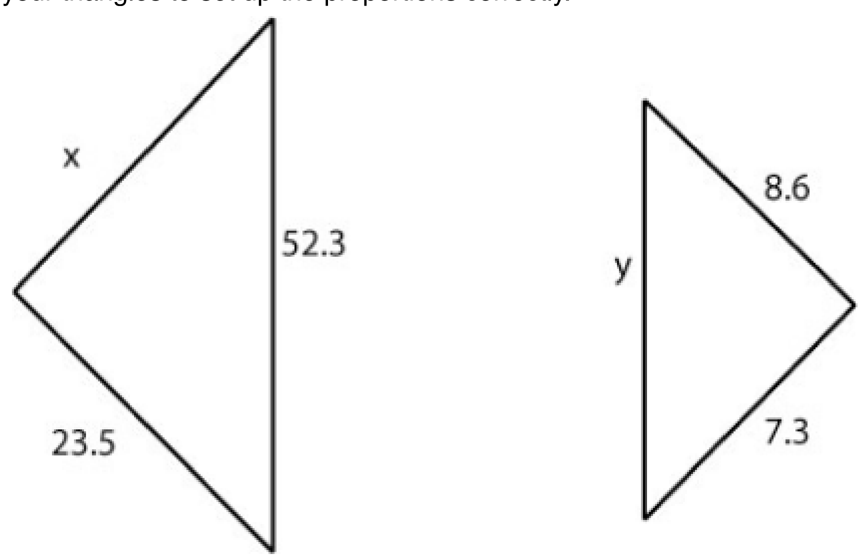


Solution

15 units

Example 2

Use the similar triangles process to determine the length of the missing sides. You may need to redraw your triangles to set up the proportions correctly.

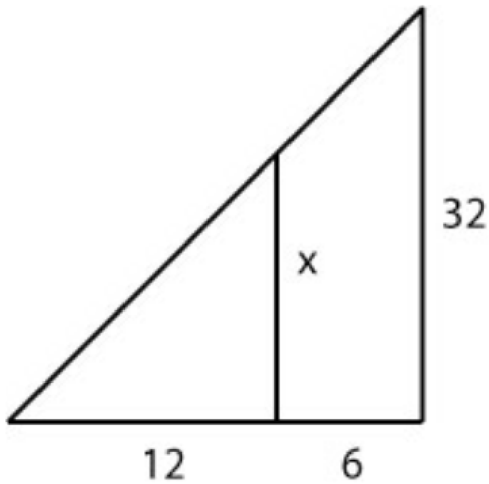


Solution

x = 27.7 (rounded), y = 16.2 (rounded)

Example 3

Given the similar triangles below, find the missing lengths. Round to tenths as needed. Feel free to redraw the triangles so you can see the proportional sides.



Solution

21.3 (rounded)

Example 4

Applications of Similar Triangles

Mary (from the application that started this topic), decides to use what she knows about the height of the roof to measure the height of her second daughter. If her second daughter casts a shadow that is 1.5 feet long when she is 13.5 feet from the house, what is the height of the second daughter? Draw an accurate diagram and use similar triangles to solve.

Solution

2.5 ft

Square Roots

Before we get to our last topic in this lesson—the Pythagorean Theorem,—we need to know a little bit about square roots.

- The **square root** of a number is that number which, when multiplied times itself, gives the original number. On your calculator, look for \sqrt\text{ }\\ to compute square roots. $\sqrt{16} = 4$ because 4 × 4 = 16
- A **perfect square** is a number whose square root is a whole number. The square root of a non-perfect square is a decimal value. $\sqrt{16}$ is a perfect square. $\sqrt{19}$ is **not** a perfect square.
- To obtain a decimal value for non-perfect square roots on your calculator, you may need to change the settings under your MODE button. Check your owner's manual for help if needed. $\sqrt{19} = 4.36$ (rounded

to hundredths)

Example 5

Find the square root of each of the following. Round to two decimal places if needed. Indicate those that are perfect squares and explain why.

1. $\sqrt{169}$
2. $\sqrt{31}$
3. $\sqrt{9}$

Solutions

1. 13 (perfect square)
2. 5.57 (not a perfect square)
3. 9 (perfect square)

Example 6

Find the square root of each of the following. Round to two decimal places if needed. Indicate those that are perfect squares and explain why.

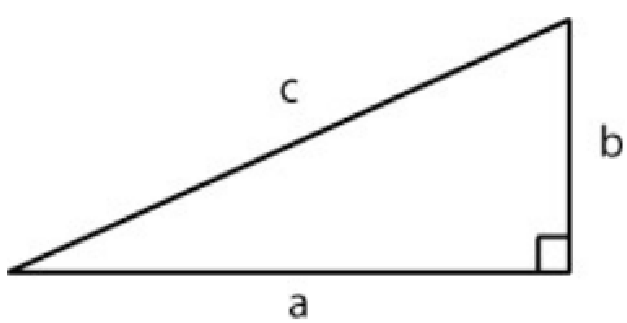
1. $\sqrt{225}$
2. $\sqrt{17}$
3. $\sqrt{324}$

Solutions

1. 15 (perfect square)
2. 4.12 (not a perfect square)
3. 18 (perfect square)

The Pythagorean Theorem

The mathematician Pythagoras proved the Pythagorean theorem. The theorem states that given any right triangle with sides *a*, *b*, and *c* as below, the following relationship is always true: $a^2 + b^2 = c^2$



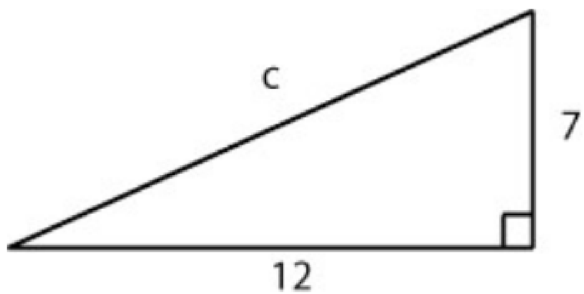
$a^2 + b^2 = c^2$

Notes about the Pythagorean theorem:

- The triangle must be a **right** triangle (contains a 90º angle).
- The side *c* is called the **hypotenuse** and *always* sits across from the right angle.
- The lengths *a* and *b* are interchangeable in the theorem, but *c* cannot be interchanged with *a* or *b*. In other words, the location of *c* is very important and cannot be changed.

Example 7

Use the Pythagorean theorem to find the missing sides length for the triangle given below. Round to the tenths place.

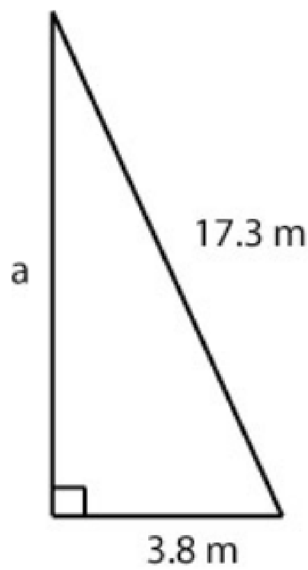


Solution

13.9 (rounded)

Example 8

Use the Pythagorean theorem to find the missing sides length for the triangle given below. Round to the tenths place.

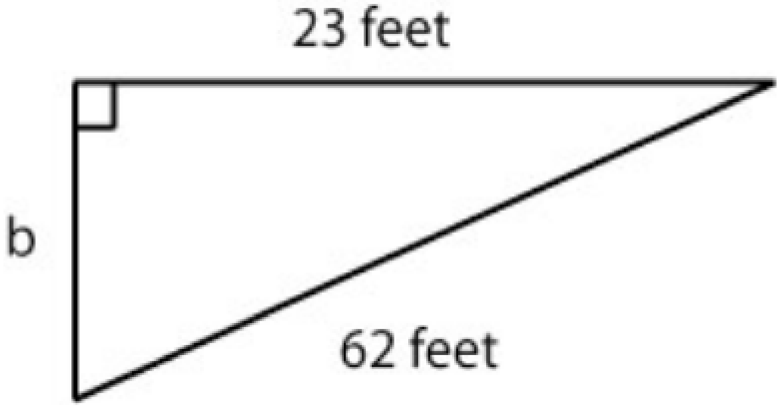


Solution

16.9 m (rounded)

Example 9

Use the Pythagorean theorem to find the missing sides length for the triangle given below. Round to hundredths.



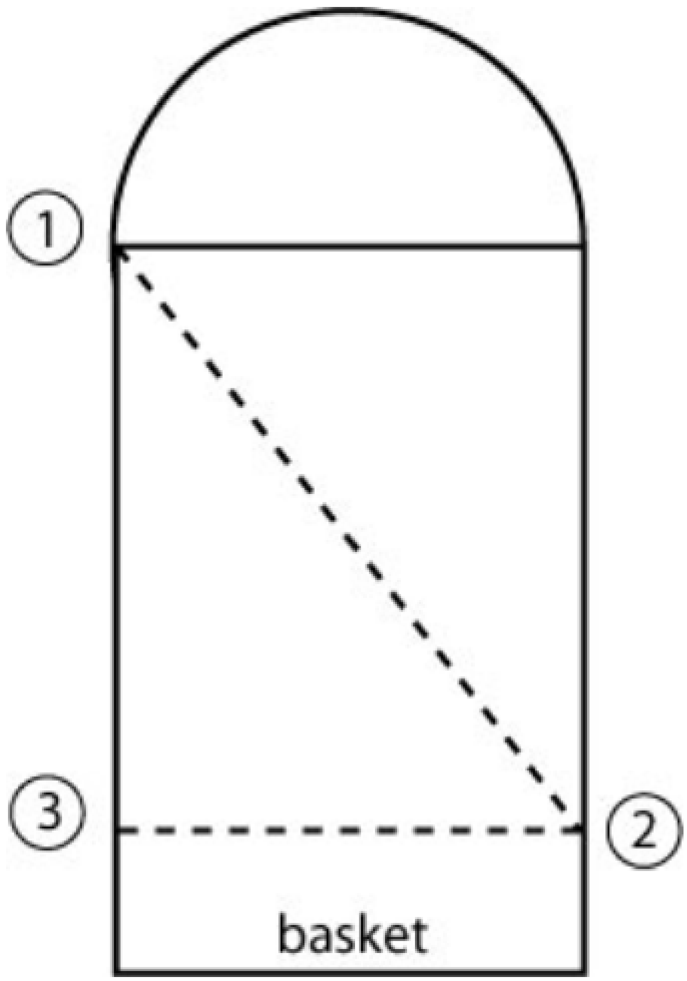
Solution

57.58 ft (rounded)

Example 10

Applications of the Pythagorean Theorem

In NBA Basketball, the width of the free-throw line is 12 feet. ((Note: http://www.sportsknowhow.com)) A player stands at one exact corner of the free throw line (Player 1) and wants to throw a pass to his open teammate across the lane and close to the basket (Player 2). If his other teammate (Player 3—heavily guarded) is directly down the lane from him 16 feet, how far is his pass to the open teammate? Fill in the diagram below and use it to help you solve the problem.



Solution

20 ft

VOLUME

Let's revisit our friend Wally from "Area and Perimeter" and use another aspect of his yard to introduce the concept of **volume**. Wally is a swimmer and wants to install a lap pool in his backyard. Because he has some extra space, he is going to build a pool that is 25 yards long, 2 yards wide, and 2 yards deep. How many cubic yards of water must be used to fill the pool (assuming right to the top).

Much as we did with area (counting unit squares), with **volume** we will be counting unit cubes. What is the volume of a unit cube? Let's look at figure 1:

Finding the Volume of a Cube

The shape at left is a cube (all sides are equal length). In particular, because all sides are of length 1, this cube is called a unit cube.
We know the area of the base from our previous work (1 yd × 1 yd or 1 square yard). We are going to take that area and extend it vertically through a height of 1 yard so our volume becomes
Volume = 1 yd × 1yd × yd = 1 cubic yard

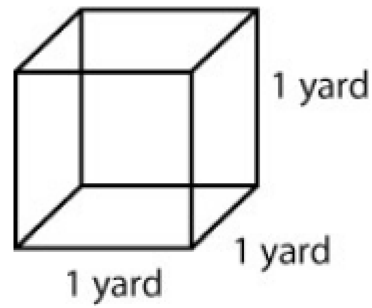


Figure 1. Volume = 1 yd × 1yd × yd = 1 cubic yard

How does this help Wally? Well, if he can count the number of unit cubes in his pool, he can determine the volume of water needed to fill the pool.

Guided Example

Filling Wally's Pool

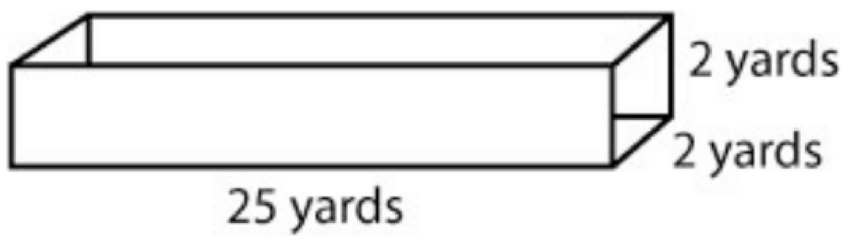
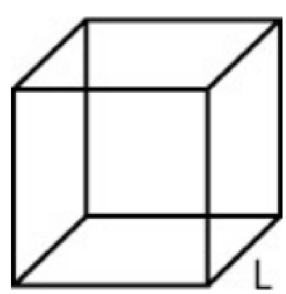


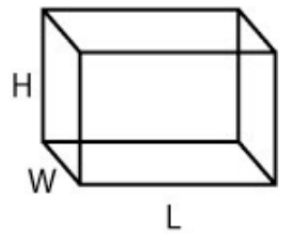
*Figure 2. Volume = Length × Width × Height **or** V = LWH*

If we fill the pool with unit cubes, we can fill 25 unit cubes along the length, 2 along the width and 2 along the height. That would give us 25 × 2 × 2 = 100 unit cubes or:

Volume = 25 yd × 2 yd × 2 yd = 100 cubic yd

Explicit formulas for the types of rectangular solids used in the previous section are as follows:

Shape		**Volume**
Cub of side length *L*		$V = L \times L \times L$ $V = L^3$

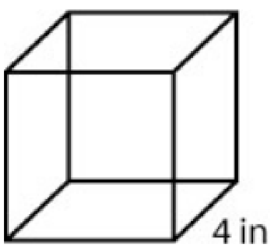
Shape		Volume
Box with sides of length L, W, H		$V = L \times W \times H$

Notes on Volume

- **Volume** is a three-dimensional measurement that represents the amount of space inside a closed three-dimensional shape.
- To find volume, count the number of unit cubes inside a given shape.
- If there are units, include units in your final result. Units will always be three-dimensional (i.e. cubic feet, cubic yards, cubic miles, etc…)

Example 1

Find the volume of each shape below.

1. 

2. A box with sides of length 2 ft, 3 ft, $2\frac{1}{2}$ ft.

Solutions

1. 64 in^3
2. 15 ft^3

Volume of A Circular Cylinder

Can we use what we know about the area of a circle to formulate the volume of a can (also called a **cylinder**)? Take a look at figure 3.

The base circle is shaded. If we take the area of that circle (A = $A = \pi r^2$) and extend it up through the height h, then our volume for the can would be:

$V = \pi r^2 h$.

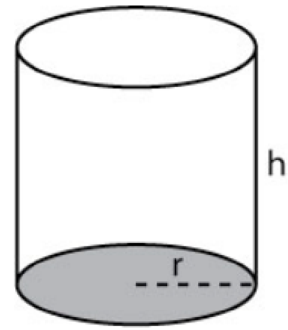
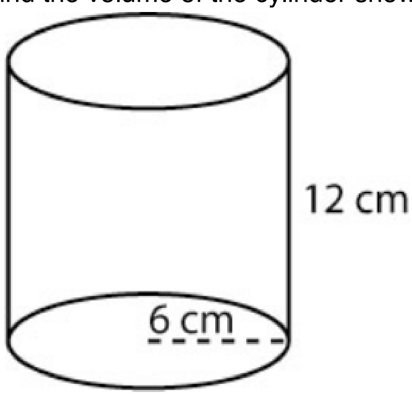


Figure 3.

Example 2

Find the volume of the cylinder shown below.



Solution

432π cm^3

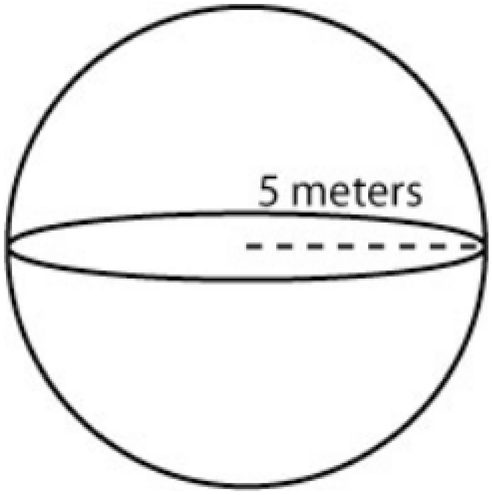
Volume of Other Shapes

The chart below shows the volumes of some other basic geometric shapes.

Shape	Volume
Sphere	$V = \frac{4}{3}\pi r^3$
Cone	$V = \frac{1}{3}\pi r^2 h$
Pyramid	$V = \frac{1}{3}LWH$

Example 3

Find the volume of a sphere with radius 5 meters.



Solution

$\frac{500}{3}\pi \text{ cm}^3$

Example 4

Determine the volume of each of the following. Include a drawing of the shape with the included information. Show all work. As in the examples, if units are included then units should be present in your final result. Use 3.14 for π and round answers to tenths as needed.

1. Find the volume of a cube with side 3.25 meters.
2. Find the volume of a box with sides of length 4 feet by $2\frac{1}{2}$ feet by 6 feet.
3. Find the volume of a can with radius 4.62 cm and height 10 cm.
4. Find the volume of a sphere with diameter 12 yards.

Solution

1. 34.3 m^3 or 34.3 cubic meters
2. 60 ft^3 or 60 cubic feet
3. 670.2 cm^3 or 670.2 cubic centimeters
4. 904.3 yd^3 or 904.3 cubic yards

Example 5

Applications of Volume

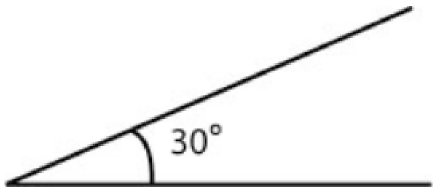
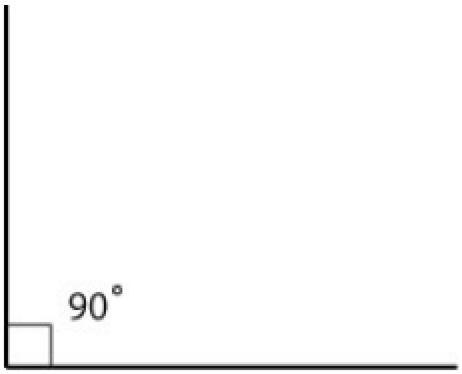
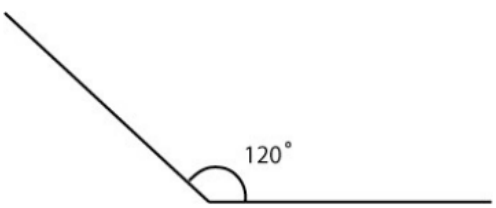
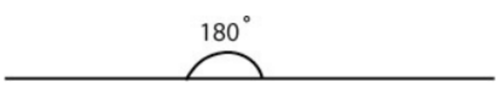
If you drank sodas from 5 cans each of diameter 4 inches and height 5 inches, how many cubic inches of soda did you drink? Use 3.14 for π and round to tenths.

Solution

314.0 in^3 (rounded)

Angles

Angles, often measured in *degrees*, measure the amount of rotation or “arc” between intersecting line segments. We need to have some sense of what an angle is before moving on to the next topic. See some examples and terminology below.

	• This angle measures 30° • This angle measure is less than 90° • Angles less than 90° are called **acute angles**
	• This angle measures 90° • Angles that measure exactly 90° are called **right angles** • The small box in the angle corner denotes a right angle
	• This angle measures 120° • This angle measure is more than 90° • Angles that measure more than 90° but less 180° than are called **obtuse angles**
	• This angle measures 180° • Angles that measure exactly 180° are called **straight angles**

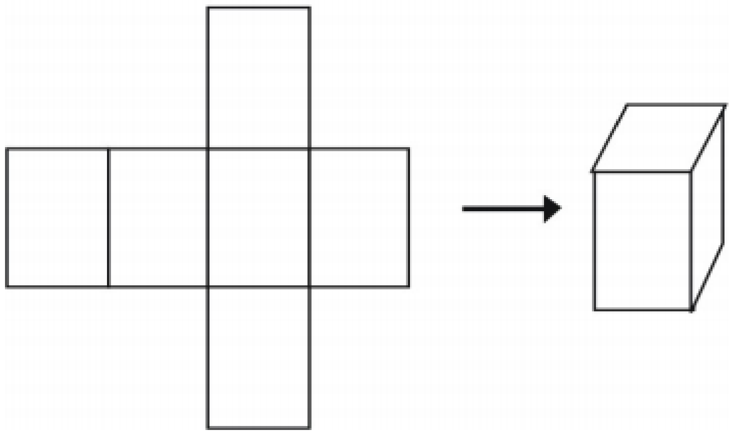
SURFACE AREA

Learning Objectives

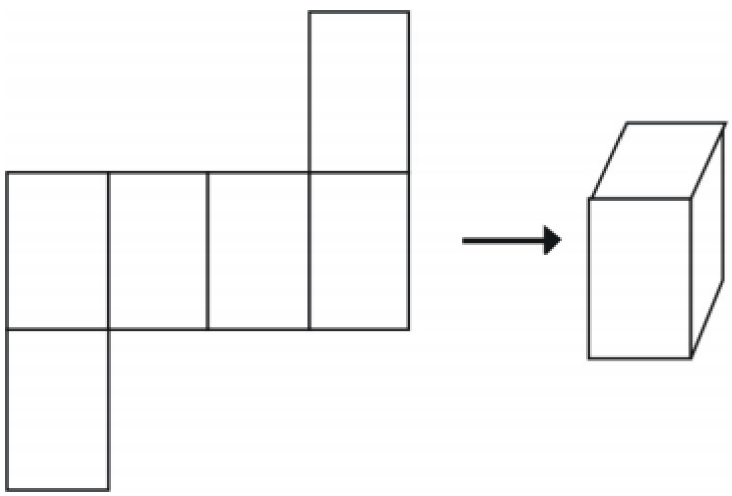
- Understand nets.
- Use nets to represent prisms.
- Find the surface area of a prism.
- Find the surface area of cylinders.
- Find the surface area of a sphere.

Nets

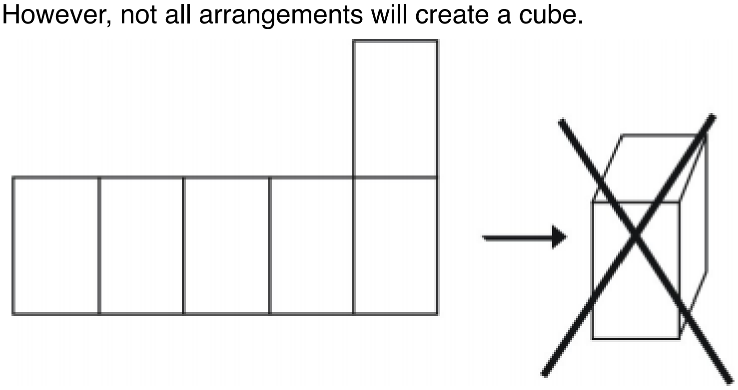
One final way to represent a solid is to use a net. If you cut out a net you can fold it into a model of a figure. Nets can also be used to analyze a single solid. Here is an example of a net for a cube.



There is more than one way to make a net for a single figure.

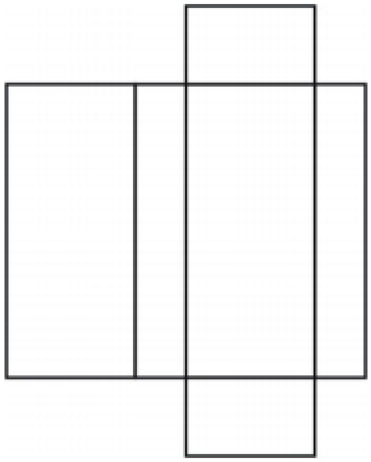


However, not all arrangements will create a cube.

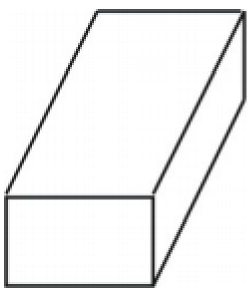


Example 1

What kind of figure does the net create? Draw the figure.

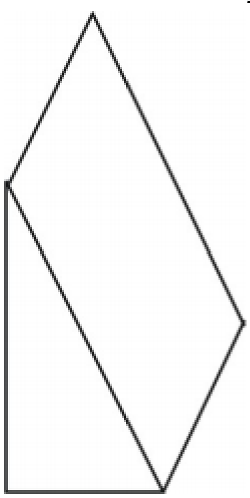


The net creates a box-shaped rectangular prism as shown below.

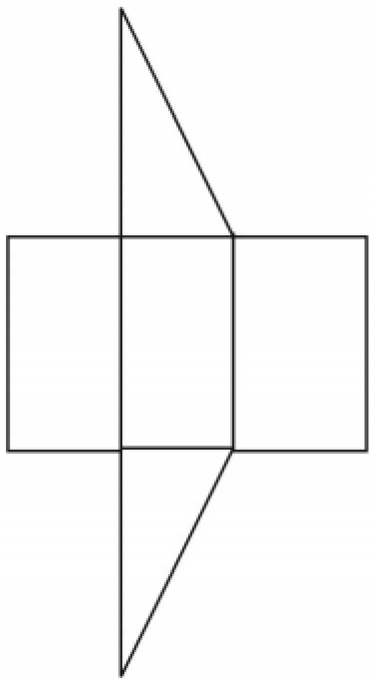


Example 2

What kind of net can you draw to represent the figure shown? Draw the net.

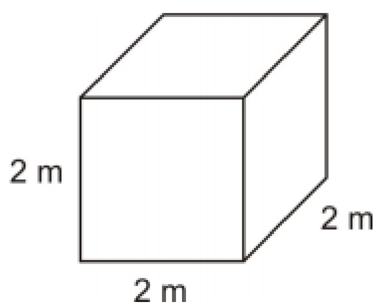


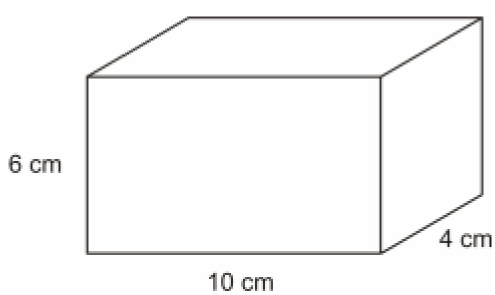
A net for the prism is shown. Other nets are possible.



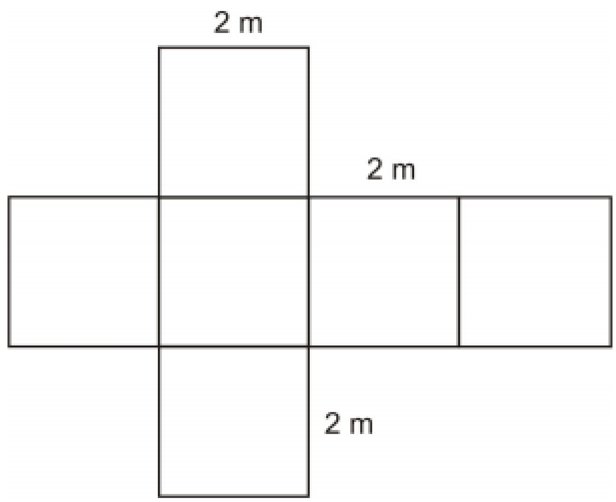
Review Exercises

Draw a net for each of the following:

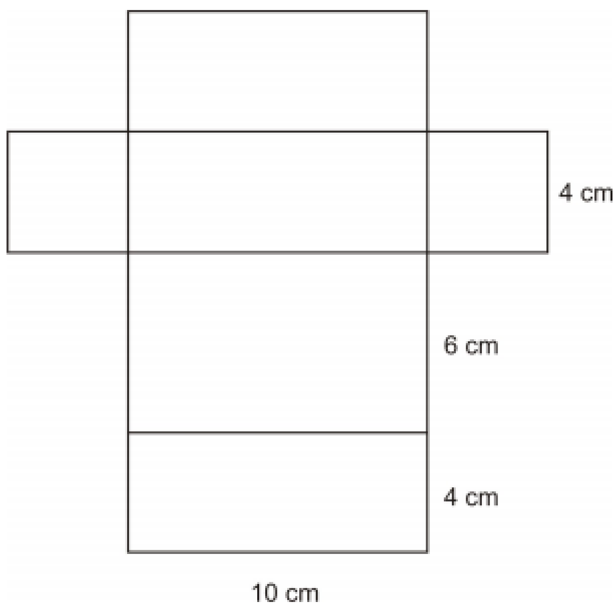
1. 

2. 

Answers



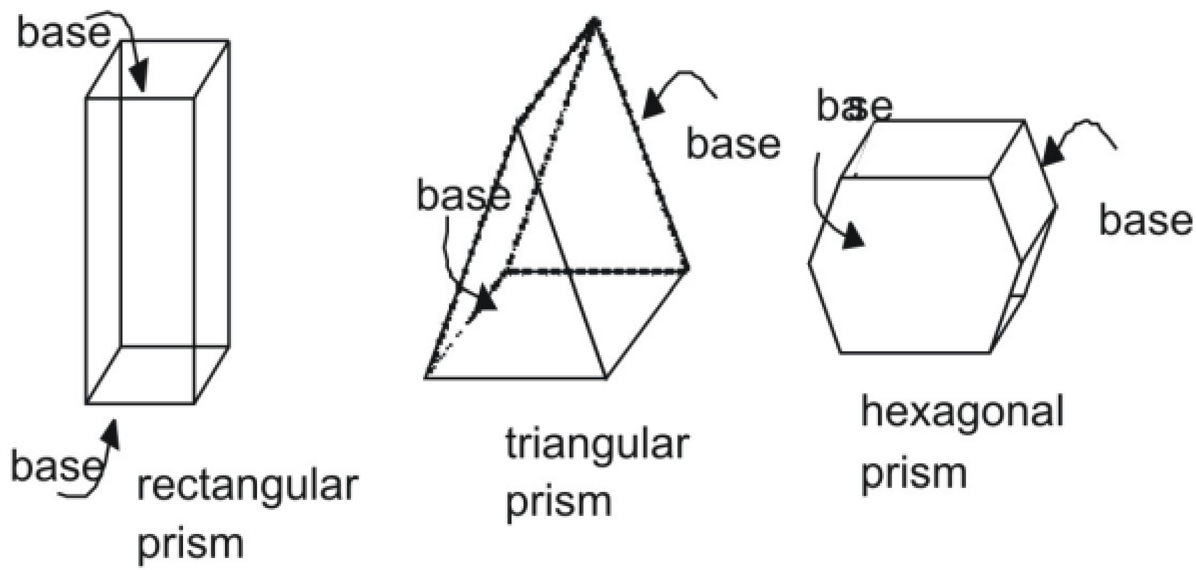
1.



2.

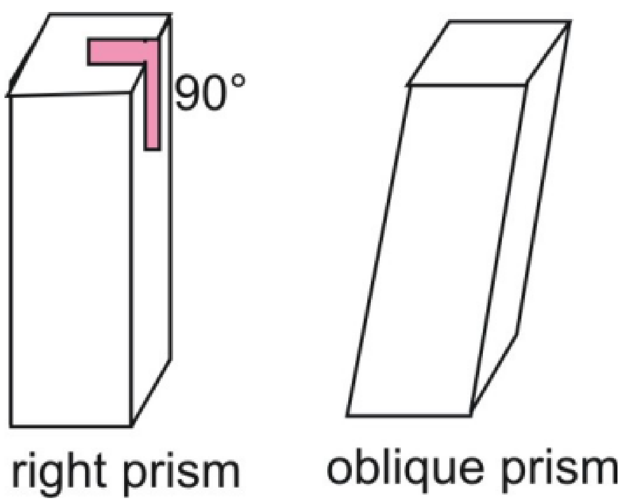
Prisms

A prism is a three-dimensional figure with a pair of parallel and congruent ends, or bases. The sides of a prism are parallelograms. Prisms are identified by their bases.



Surface Area of a Prism Using Nets

The prisms above are **right prisms**. In a right prism, the lateral sides are perpendicular to the bases of prism. Compare a right prism to an **oblique prism**, in which sides and bases are not perpendicular.

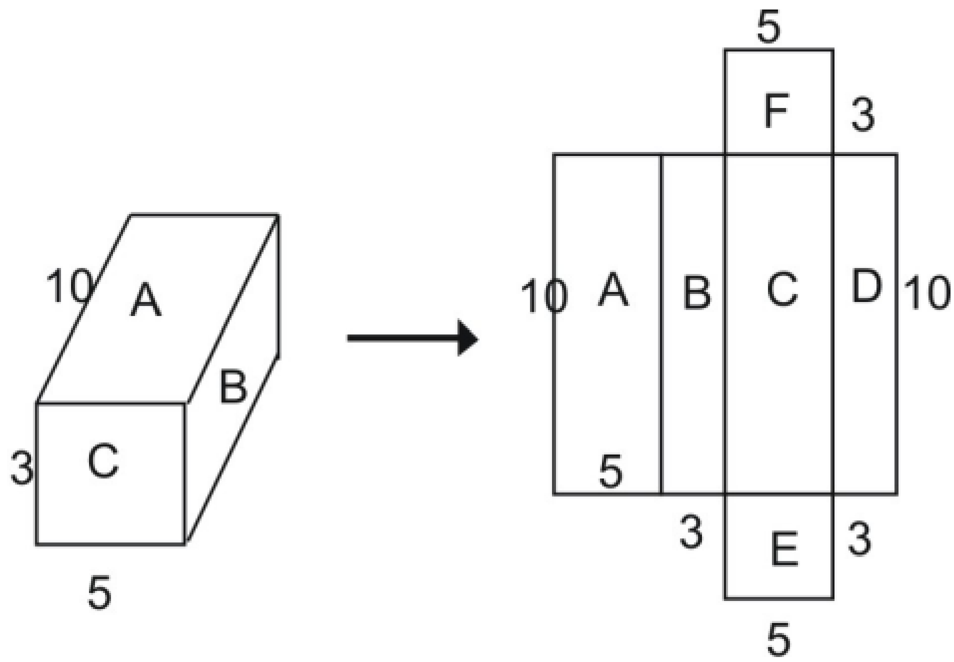


Two postulates that apply to area are the Area Congruence Postulate and the Area Addition Postulate.

Area Congruence Postulate: If two polygons (or plane figures) are congruent, then their areas are congruent.

Area Addition Postulate: The surface area of a three-dimensional figure is the sum of the areas of all of its non-overlapping parts.

You can use a net and the Area Addition Postulate to find the surface area of a right prism.



From the net, you can see that that the surface area of the entire prism equals the sum of the figures that make up the net:

Total surface area = area A + area B + area C + area D + area E + area F

Using the formula for the area of a rectangle, you can see that the area of rectangle A is:

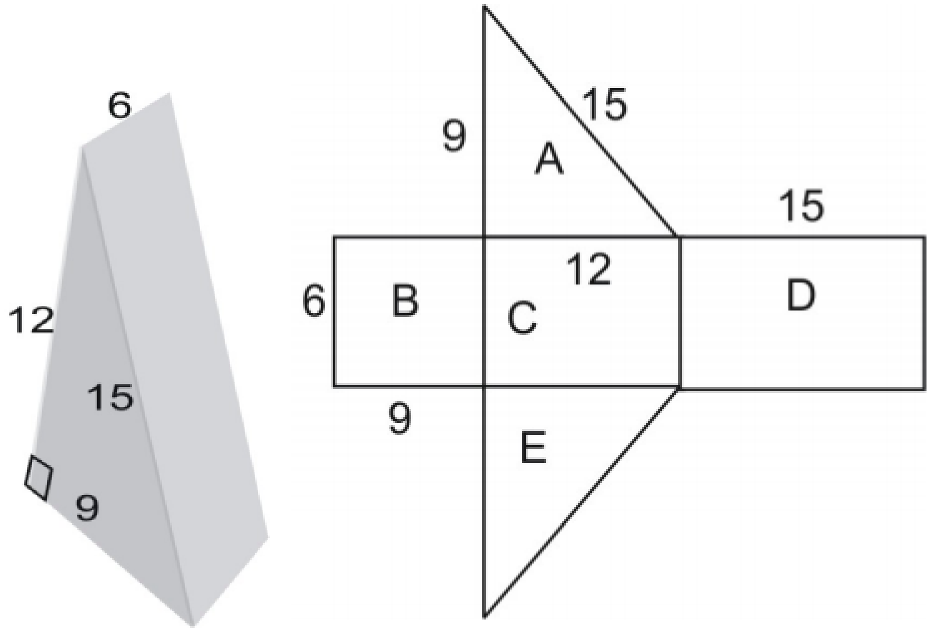
$A = l \cdot w$
$A = 10 \cdot 5 = 50$ square units

Similarly, the areas of the other rectangles are inserted back into the equation above.

Total surface area = area A + area B + area C + area D + area E + area F
Total surface area = $(10 \cdot 5) + (10 \cdot 3) + (10 \cdot 5) + (10 \cdot 3) + (5 \cdot 3) + (5 \cdot 3)$
Total surface area = $50 + 30 + 50 + 30 + 15 + 15$
Total surface area = 190 square units

Example 3

Use a net to find the surface area of the prism.



The area of the net is equal to the surface area of the figure. To find the area of the triangle, we use the formula:

$A = \frac{1}{2}hb$ where *h* is the height of the triangle and *b* is its base.

Note that triangles *A* and *E* are congruent so we can multiply the area of triangle A by 2.

$$\text{area} = \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E$$

$$= 2(\text{area } A) + \text{area } B + \text{area } C + \text{area } D$$

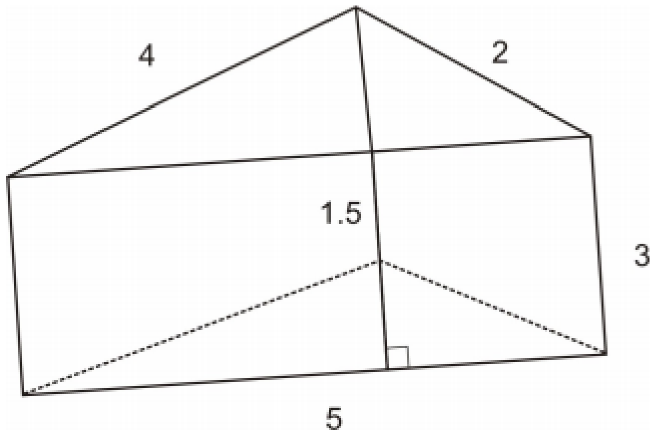
$$= 2[\frac{1}{2}(9 \cdot 12)] + (6 \cdot 9) + (6 \cdot 12) + (6 \cdot 12)$$

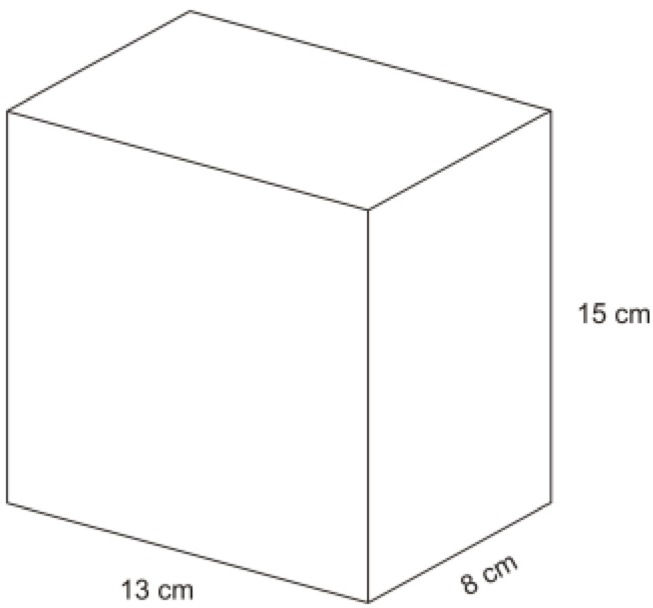
$$= 108 + 54 + 72 + 90 = 324$$

Thus, the surface area is 324 square units.

Review Exercises

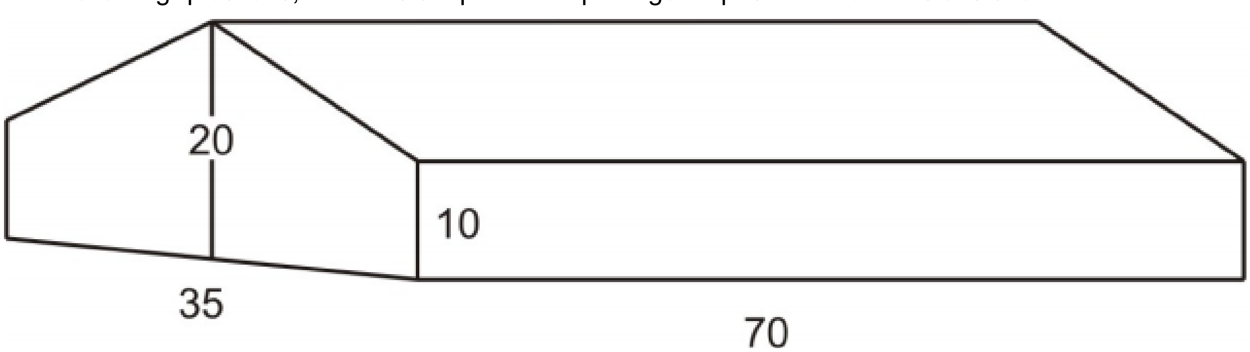
For each of the following find the surface are a using the method of nets and the perimeter

1. 

2. 

3. The base of a prism is a right triangle whose legs are 3 and 4 and show height is 20. What is the total area of the prism?
4. A right hexagonal prism is 24 inches tall and has bases that are regular hexagons measuring 8 inches on a side. What is the total surface area?
5. What is the volume of the prism in problem #4?

In the following questions, a barn is shaped like a pentagonal prism with dimensions shown in feet:



1. How many square feet (excluding the roof) are there on the surface of the barn to be painted?
2. If a gallon of paint covers 250 square feet, how many gallons of paint are needed to paint the barn?
3. A cardboard box is a perfect cube with an edge measuring 17 inches. How many cubic feet can it hold?
4. A swimming pool is 16 feet wide, 32 feet long and is uniformly 4 feet deep. How many cubic feet of water can it hold?

5. A cereal box has length 25 cm, width 9 cm and height 30 cm. How much cereal can it hold?

Answers

Nets and perimeter:

1. 40.5 in^2
2. 838 cm^2
3. 252 square units
4. 1484.6 square units
5. 3990.7 cubic inches

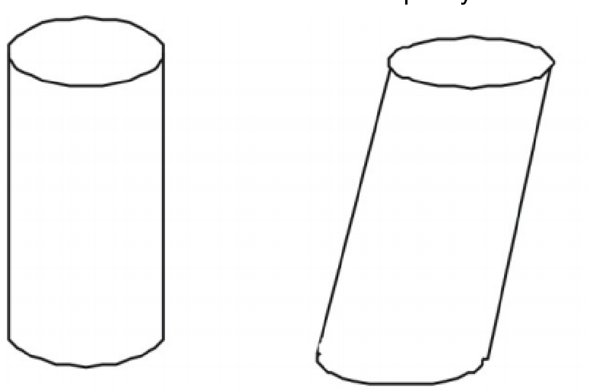
The barn:

1. 2450 square feet
2. 10 gallons of paint
3. 2.85 cubic feet (be careful here. The units in the problem are given in inches but the question asks for feet.)
4. 2048 cubic feet
5. 6750 cm^3

Cylinders

A **cylinder** is a three-dimensional figure with a pair of parallel and congruent circular ends, or **bases**. A cylinder has a single curved side that forms a rectangle when laid out flat.

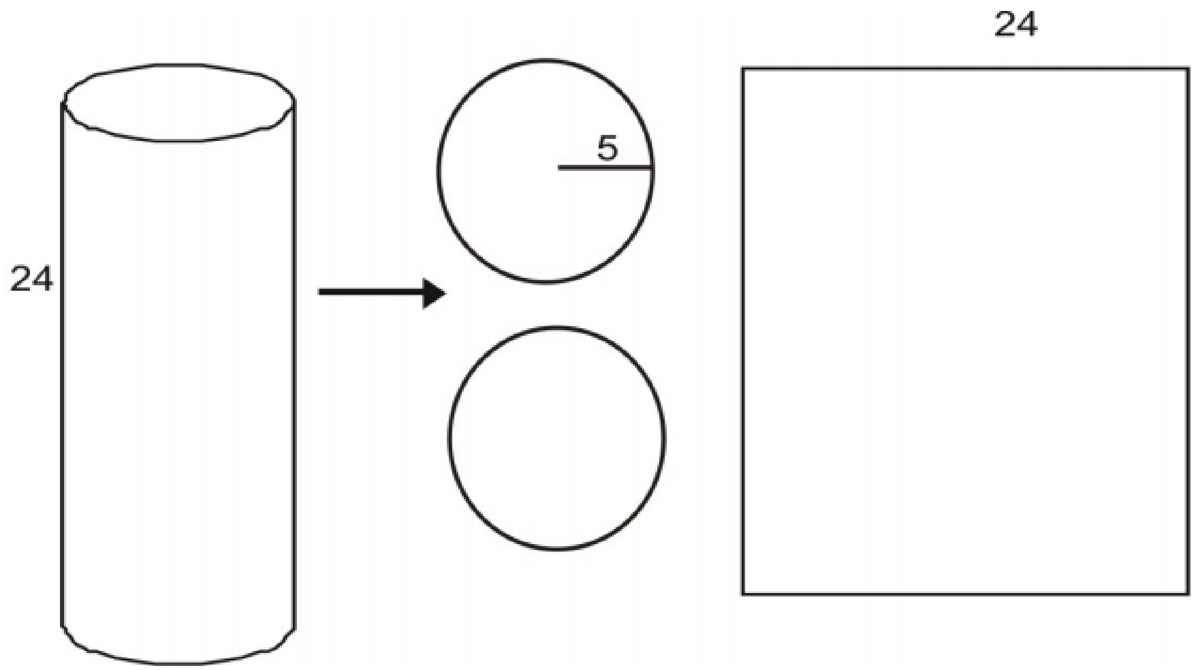
As with prisms, cylinders can be *right* or *oblique*. The side of a right cylinder is perpendicular to its circular bases. The side of an oblique cylinder is not perpendicular to its bases.



right cylinder oblique cylinder

Surface Area of a Cylinder Using Nets

You can deconstruct a cylinder into a net.



The area of each base is given by the area of a circle:

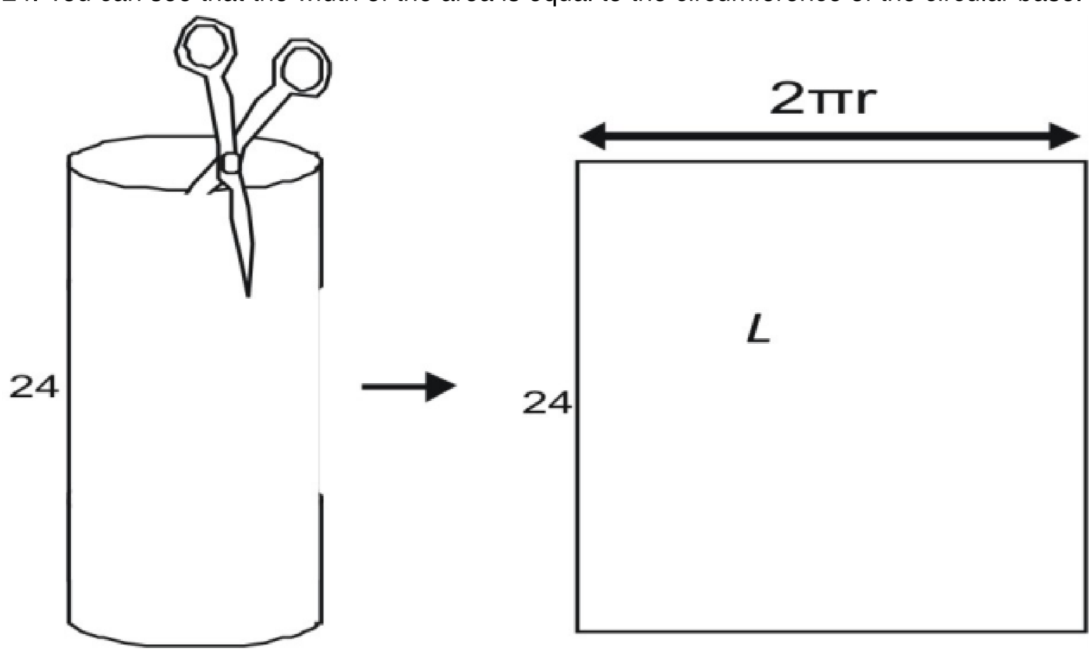
$$A = \pi r^2$$

$$A = \pi(5)^2$$

$$A = 25\pi$$

$$A \approx (25)(3.14) = 78.5$$

The area of the rectangular lateral area *L* is given by the product of a width and height. The height is given as 24. You can see that the width of the area is equal to the circumference of the circular base.



To find the width, imagine taking a can-like cylinder apart with a scissors. When you cut the lateral area, you see that it is equal to the circumference of the can's top. The circumference of a circle is given by $C = 2\pi r$, the lateral area, L, is

$$L = 2\pi rh$$

$$L = 2\pi(5)(24)$$

$$L = 240\pi$$

$$L \approx (240)(3.14) = 753.6$$

Now we can find the area of the entire cylinder using A = (area of two bases) + (area of lateral side).

$$A = 2(75.36) + 753.6$$

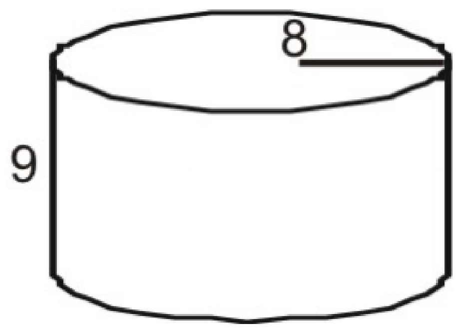
$$A = 904.32$$

You can see that the formula we used to find the total surface area can be used for any right cylinder.

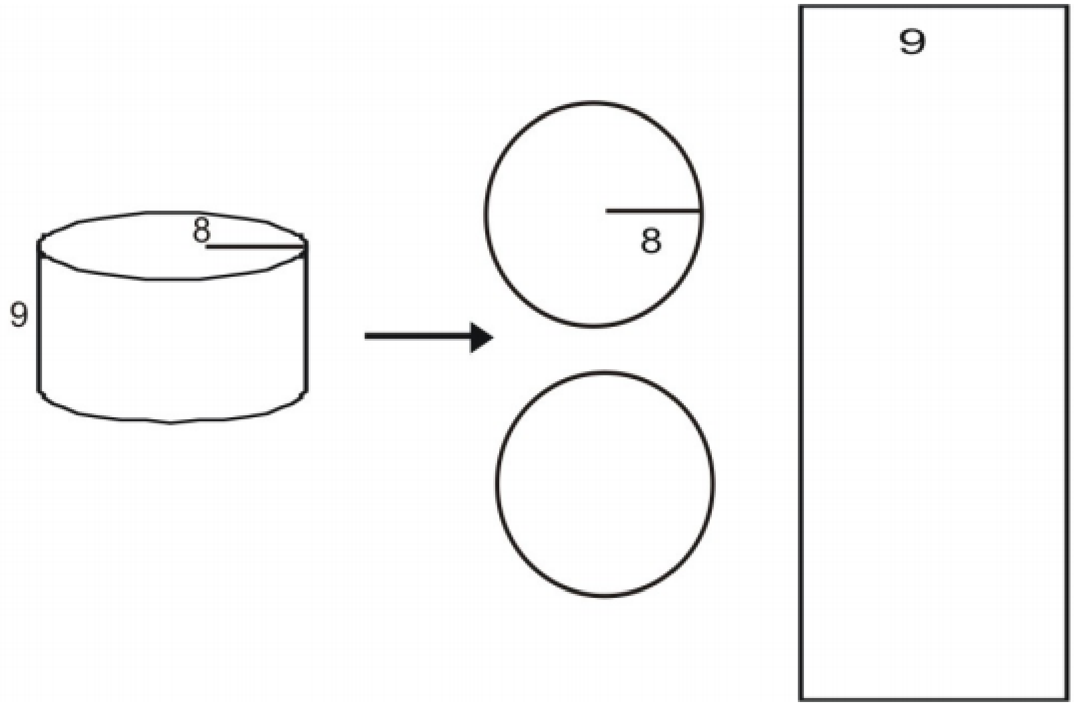
Area of a Right Cylinder: The surface area of a right cylinder, with radius and height h is given by $A = 2B + L$, where B is the area of each base of the cylinder and L is the lateral area of the cylinder.

Example 4

Use a net to find the surface area of the cylinder.



First draw and label a net for the figure.



Calculate the area of each base.

$$A = \pi r^2$$

$$A = \pi(8)^2$$

$$A = 64\pi$$

$$A \approx (64)(3.14) = 200.96$$

Calculate *L*.

$$L = 2\pi rh$$

$$L = 2\pi(8)(9)$$

$$L = 144\pi$$

$$L \approx (144)(3.14) = 452.16$$

Find the area of the entire cylinder.

$$A = 2(200.96) + 452.16$$

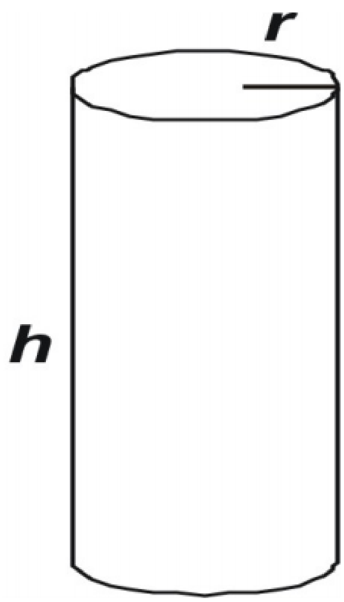
$$A = 854.08$$

Thus, the total surface area is approximately 854.08 square units

Surface Area of a Cylinder Using a Formula

You have seen how to use nets to find the total surface area of a cylinder. The postulate can be broken down to create a general formula for all right cylinders.

$A = 2B + L$



Notice that the base, B, of any cylinder is: $B = \pi r^2$

The lateral area, L, for any cylinder is:

$$L = \text{width of lateral area} \cdot \text{height of cylinder}$$

$$L = \text{circumference of base} \cdot \text{height of cylinder}$$

$$L = 2\pi r \cdot h$$

Putting the two equations together we get:

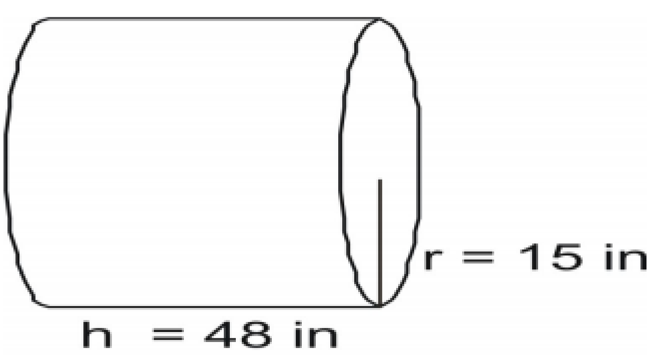
Factoring out a from the equation gives:

The Surface Area of a Right Cylinder: A right cylinder with radius r and height h can be expressed as:
$A = 2\pi r^2 + 2\pi rh$
or:
$A = 2\pi r(r + h)$

You can use the formulas to find the area of any right cylinder.

Example 5

Use the formula to find the surface area of the cylinder.



Write the formula and substitute in the values and solve.

$$A = 2(\pi r^2) + 2\pi rh$$

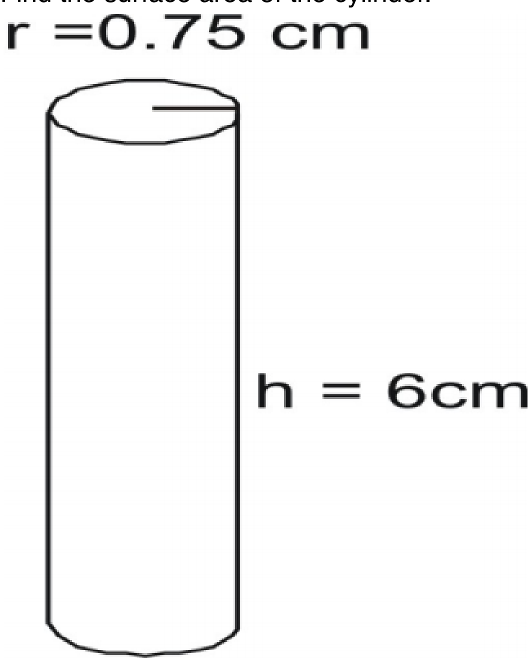
$$A = 2(3.14)(15)(15) + 2(3.14)(15)(48)$$

$$A = 1413 + 4521.6$$

$$A = 5934.6 \text{ square inches}$$

Example 6

Find the surface area of the cylinder.



Write the formula and substitute in the values and solve.

$$A = 2\pi r(r + h)$$

$$A = 2(3.14)(0.75)[0.75 + 6]$$

$$A = 31.7925 \text{ square inches}$$

Example 7

Find the height of a cylinder that has radius 4cm and surface area of 226.08 sq cm.

Write the formula with the given information and solve for *h*.

$$A = 2\pi r(r + h)$$

$$226.08 = 2(3.14)(4)[4 + h]$$

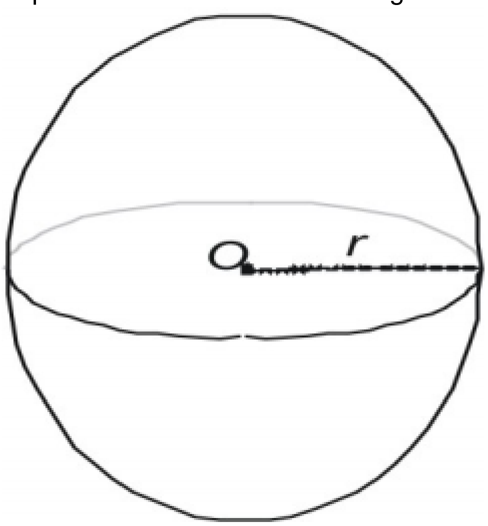
$$226.08 = 25.12[4 + h]$$

$$226.08 = 100.48 + 25.12h$$

$$5 = h$$

Spheres

A sphere is a three-dimensional figure that has the shape of a ball.

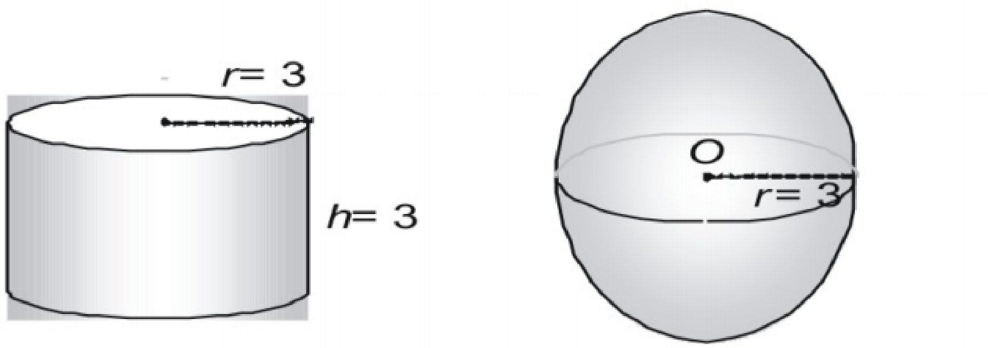


Spheres can be characterized in three ways.

- A sphere is the set of all points that lie a fixed distance *r* from a single center point *O*.
- A sphere is the surface that results when a circle is rotated about any of its diameters.

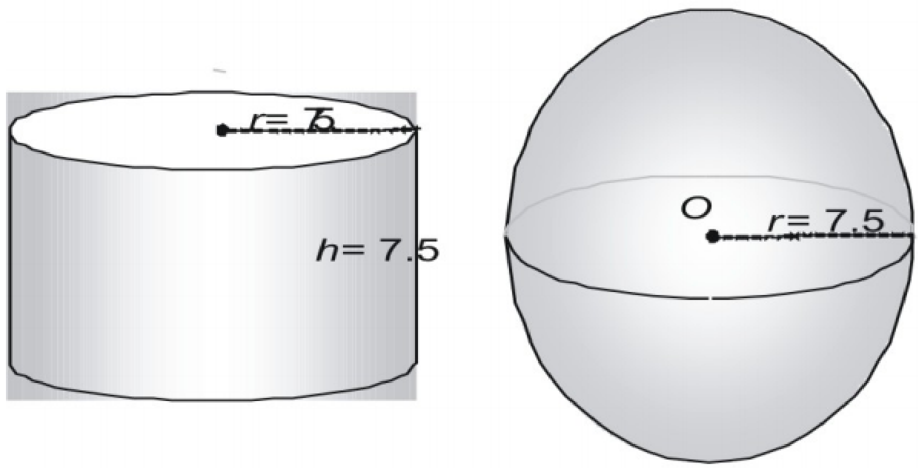
Surface Area of a Sphere

You can infer the formula for the surface area of a sphere by taking measurements of spheres and cylinders. Here we show a sphere with a radius of 3 and a right cylinder with both a radius and a height of 3 and express the area in terms of π.



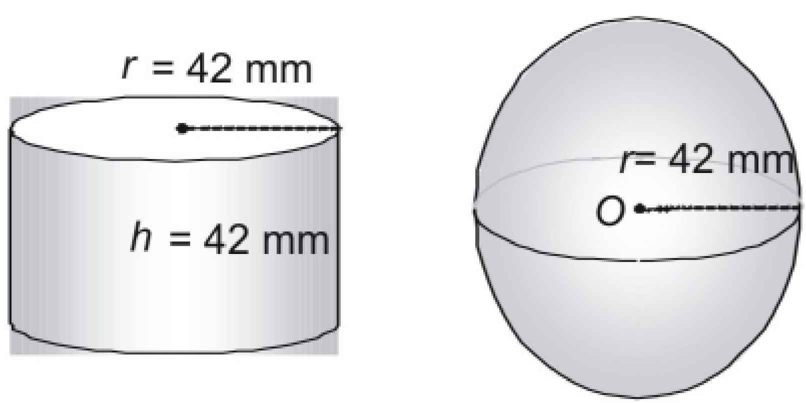
surface area = 36π   surface area = 36π

Now try a larger pair, expressing the surface area in decimal form.



surface area = 706.5   surface area = 706.5

Look at a third pair.



surface area = 22,155.84 sq mm   surface area = 22,155.84 sq mm

Is it a coincidence that a sphere and a cylinder whose radius and height are equal to the radius of the sphere have the exact same surface area? Not at all! In fact, the ancient Greeks used a method that showed that the following formula can be used to find the surface area of any sphere (or any cylinder in which).

The Surface Area of a Sphere is given by: $A = 4\pi r^2$

Example 8

Find the surface area of a sphere with a radius of 14 feet.

Use the formula.

$$A = 4\pi r^2$$

$$A = 4\pi (14)^2$$

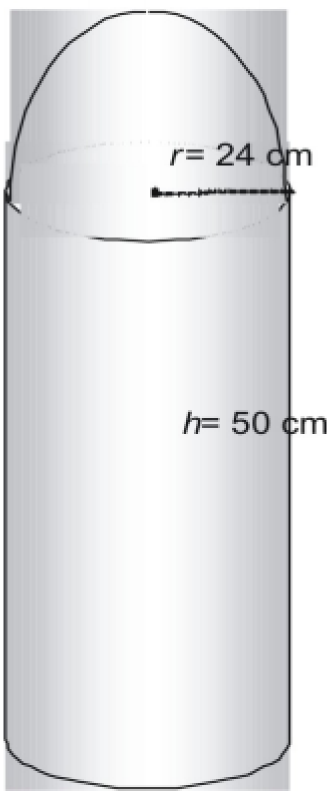
$$A = 4\pi (196)$$

$$A = 784\pi$$

$$2461.76 \text{ square feet using } 3.14 \text{ for } \pi$$

Example 9

Find the surface area of the following figure in terms of π.



The figure is made of one half sphere or hemisphere, and one cylinder without its top.

$$A(\text{half sphere}) = \frac{1}{2}A(\text{sphere})$$

$$A(\text{half sphere}) = \frac{1}{2} \cdot 4\pi r^2$$

$$A(\text{half sphere}) = 2\pi(576)$$

$$A(\text{half sphere}) = 1152\pi \text{ square cm}$$

Now find the area of the cylinder without its top.

$$A(\text{topless cylinder}) = A(\text{cylinder}) - A(\text{top})$$

$$A(\text{topless cylinder}) = 2(\pi r^2) + 2\pi rh - \pi r^2$$

$$A(\text{topless cylinder}) = \pi r^2 + 2\pi rh$$

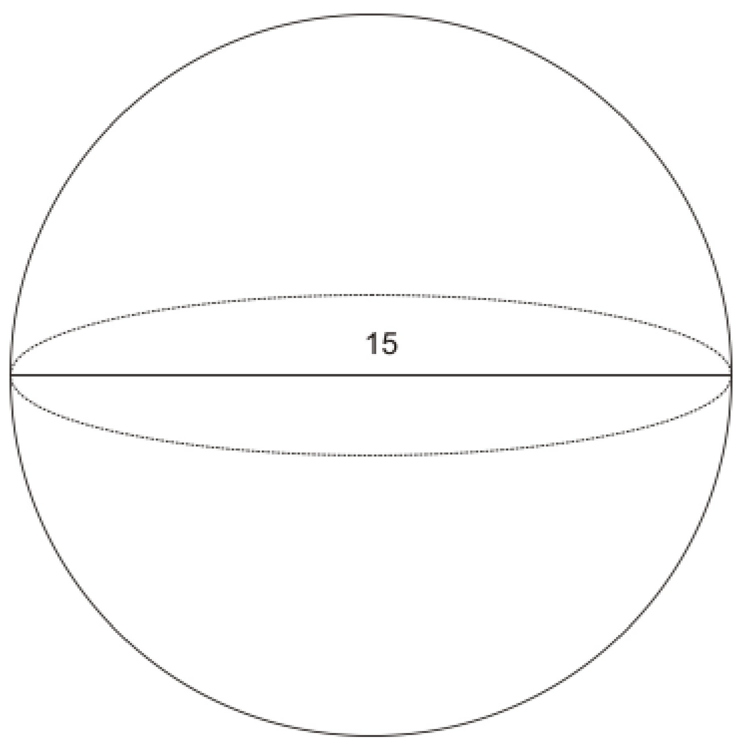
$$A(\text{topless cylinder}) = \pi(576) + 2\pi(24)(50)$$

$$A(\text{topless cylinder}) = 2976\pi \text{ square cm}$$

Thus, the total surface area is $1152\pi + 2976\pi = 4128\pi$

Review Exercises

1. Find the radius of the sphere that has a volume of 335 cm^3.
2. Determine the surface area and volume of this shape:



3. The radius of a sphere is 4. Find its volume and total surface area.

4. A sphere has a radius of 5. A right cylinder, having the same radius has the same volume. Find the height and total surface area of the cylinder.
5. Sphere: volume = 296 cm^3. Find the Diameter.
6. Sphere: surface area is 179 in^2. Find the Radius.
7. Tennis balls with a diameter of 3.5 inches are sold in cans of three. The can is a cylinder. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space not occupied by the tennis balls?
8. A sphere has surface area of 36π in2. Find its volume.
9. A giant scoop, operated by a crane, is in the shape of a hemisphere of radius = 21 inches. The scoop is filled with melted hot steel. When the steel is poured into a cylindrical storage tank that has a radius of 28 inches, the melted steel will rise to a height of how many inches?

Answers

Note that these problems use π not 3.14.

1. 1. Radius = 4.31 cm
2. Surface area = 706.86 cm2
   Volume = 1767.15 cm3
3. Volume = 268.08 units3
   Surface area = 201.06 units2
4. Height = 20/3 units total surface area = 366.52 units2
5. Diameter = 8.27cm
6. Radius = 3.77 inches
7. Volume of cylinder = 32.16π in3 volume of tennis balls = 21.44π in3
   Volume of space not occupied by tennis balls = 33.68 in3
8. Volume = 113.10 in3
9. Height of molten steel in cylinder will be 7.88 inches

GRAPH THEORY

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

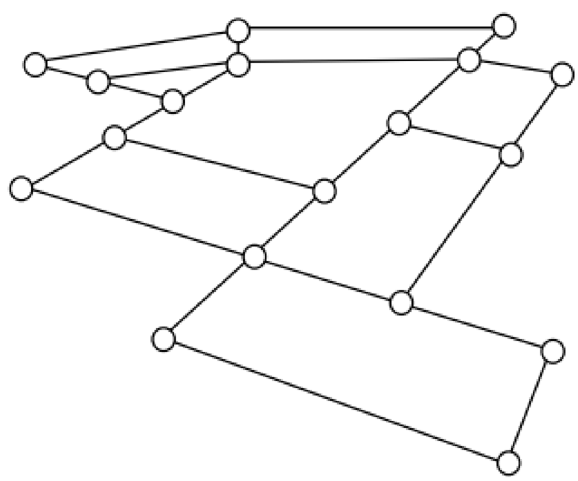
Drawing Graphs

Example 1

Here is a portion of a housing development from Missoula, Montana. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.

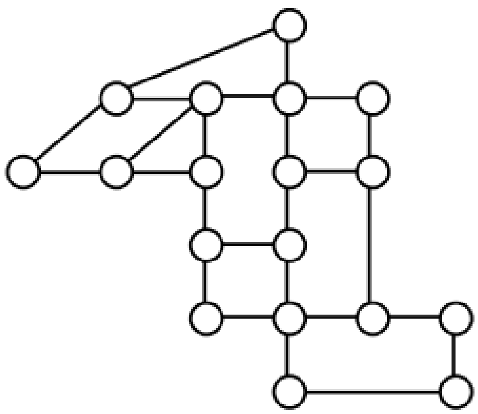


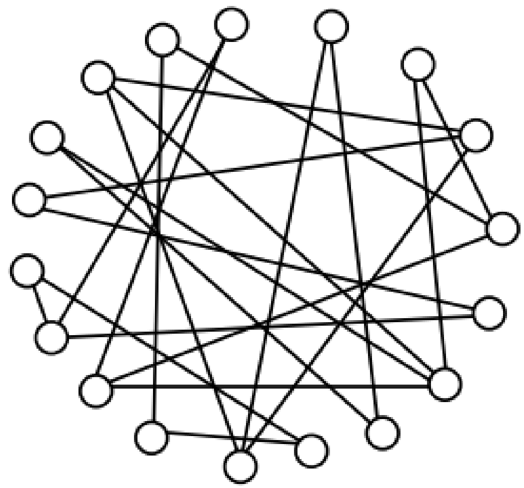
This type of simplified picture is called a **graph**.

Graphs, Vertices, and Edges

A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of the graphs below are equivalent to the one drawn above.

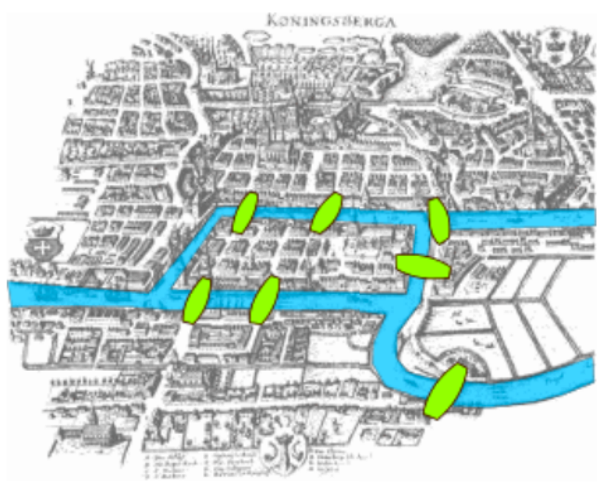




You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

Example 2

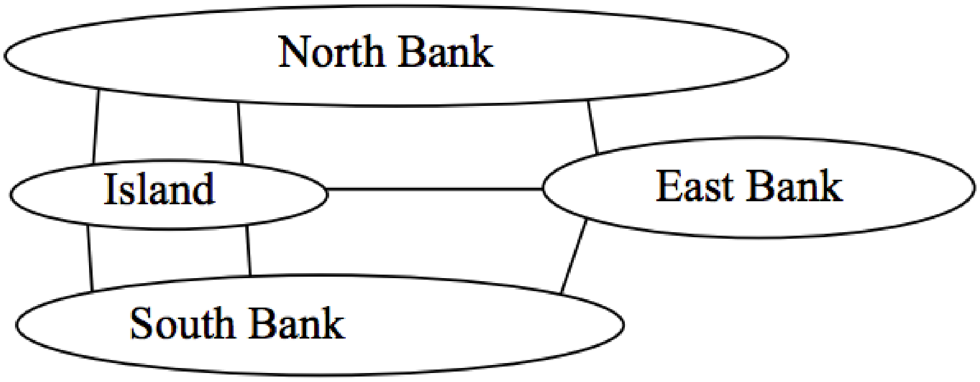
Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right. ((Note: Bogdan Giuşcă.



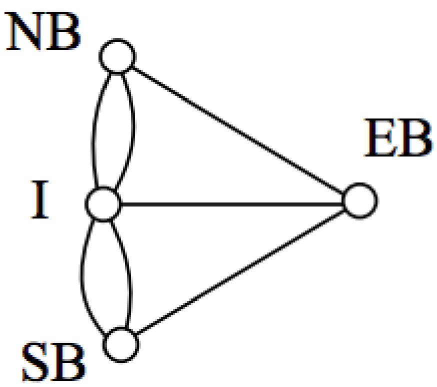
http://en.wikipedia.org/wiki/File:Konigsberg_bridges.png))

As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

While we haven't answered the actual question yet of whether or not there is a route which crosses every bridge once and returns to the starting location, the graph provides the foundation for exploring this question.

Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

Vertex

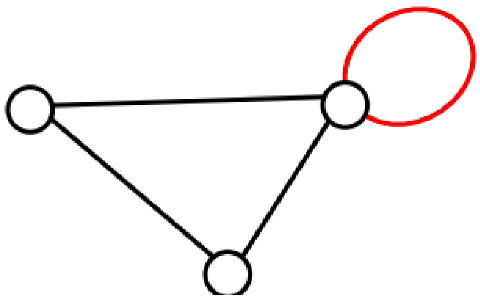
A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like "work" or "school". Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass—the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

Edges

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

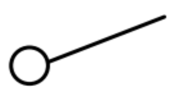
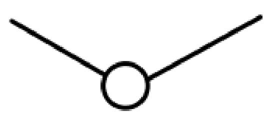
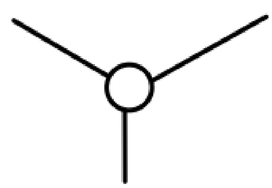
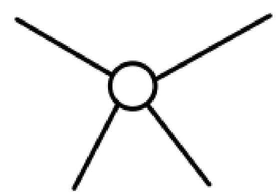
Loop

A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



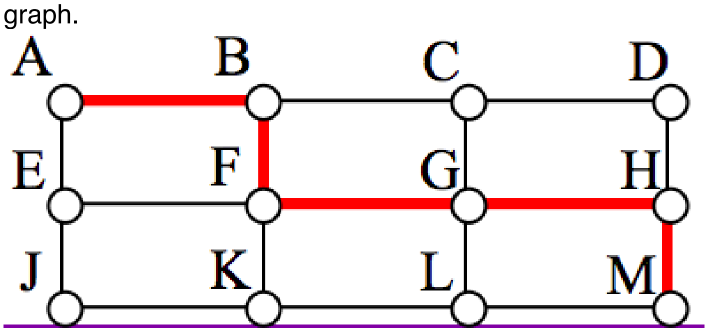
Degree of a vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.

Degree 0	Degree 1	Degree 2	Degree 3	Degree 4
				

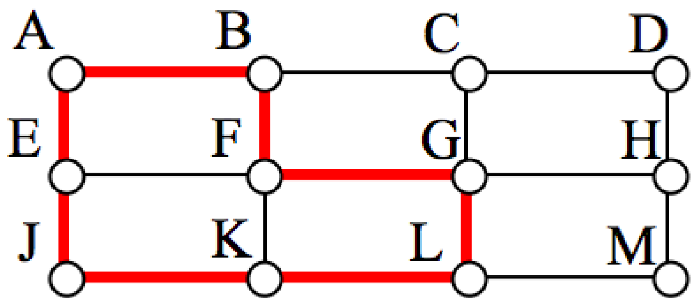
Path

A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



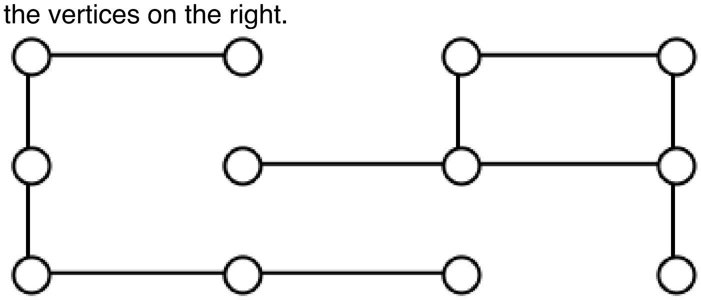
Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.



Weights

Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Euler Circuits and the Chinese Postman Problem

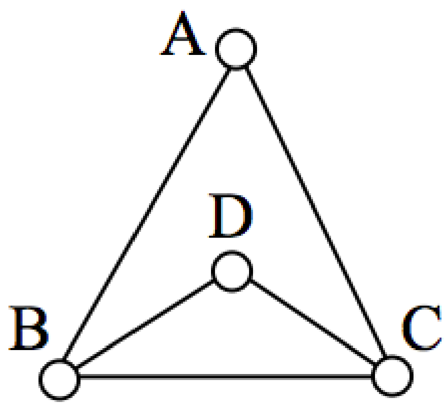
In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

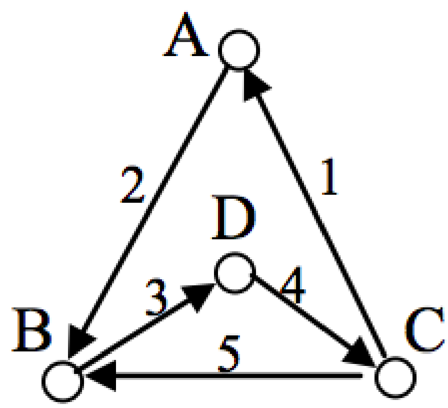
Euler Path

An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

Example 3

In the graph shown below, there are several Euler paths. One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.



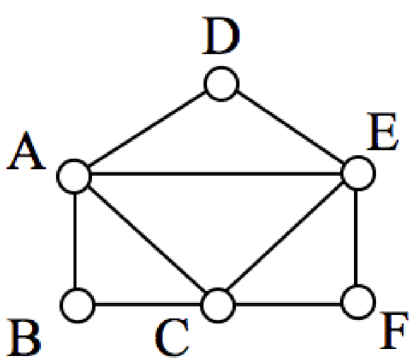


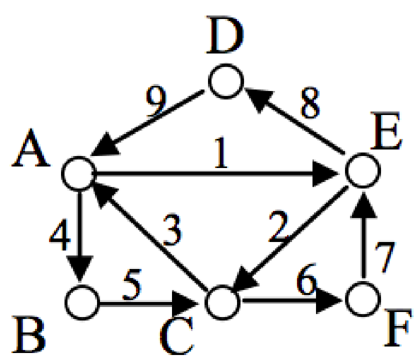
Euler Circuit

An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example 4

The graph below has several possible Euler circuits. Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.





Look back at the example used for Euler paths—does that graph have an Euler circuit? A few tries will tell you no; that graph does not have an Euler circuit. When we were working with shortest paths, we were interested in the optimal path. With Euler paths and circuits, we're primarily interested in whether an Euler path or circuit *exists*.

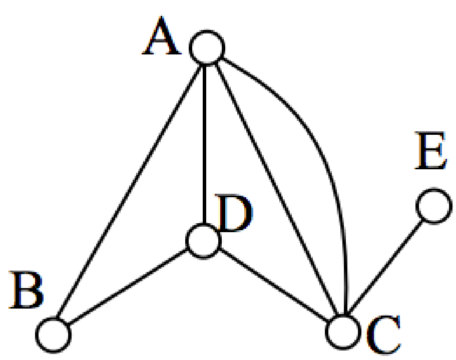
Why do we care if an Euler circuit exists? Think back to our housing development lawn inspector from the beginning of the chapter. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

Euler's Path and Circuit Theorems

A graph will contain an Euler path if it contains at most two vertices of odd degree.
A graph will contain an Euler circuit if all vertices have even degree

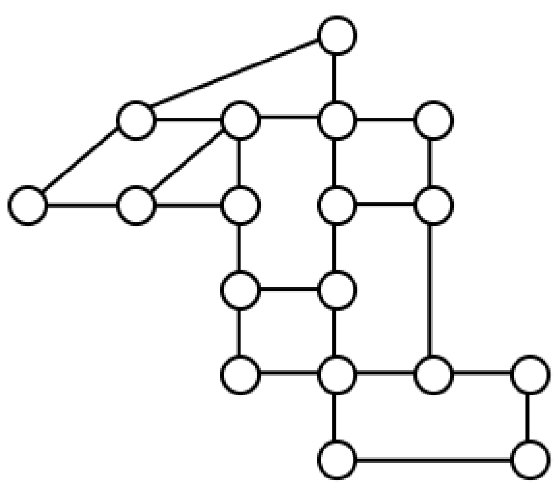
Example 5

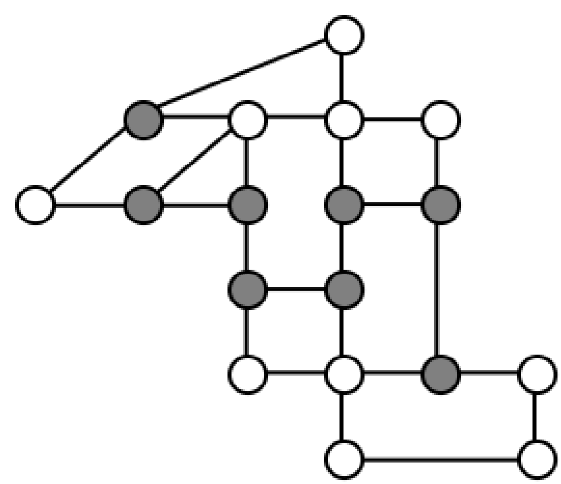
In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



Example 6

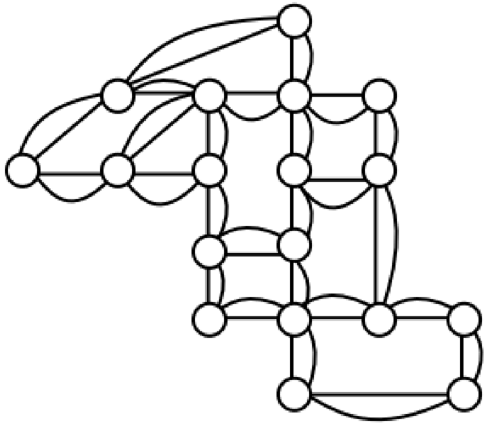
Is there an Euler circuit on the housing development lawn inspector graph we created earlier in the chapter? All the highlighted vertices have odd degree. Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately our lawn inspector will need to do some backtracking.





Example 7

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



Notice that every vertex in this graph has even degree, so this graph does have an Euler circuit.

SUPPLEMENTAL VIDEOS

This YouTube playlist contains several videos that supplement the reading on Graph Theory.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

TOPOLOGY, TILING, AND NON-EUCLIDEAN GEOMETRY

Topology

Topology is a branch of mathematics studying spaces, in which “connectiveness” of objects is a main focus. Robert Bruner has written a more detailed description: “What is Topology?”

For this course, we will determine whether two objects are topologically equivalent by comparing their genus, or number of holes. Please read and work through the lesson starting from page 1. Be sure to click the link at the bottom right hand corner of each page to get to the next page in the sequence (there are five pages in total). On the last page, you’ll find topics that can be used in the **Application of Geometry Discussion Board**.

This is a link to a *deformation*, showing how a coffee mug can be transformed into a donut; thus they are topologically equivalent.

Tiling

Please read through the following three pages for a mini-lesson on tiling:

- What is a Tiling?
- Tilings with Just a Few Shapes
- Monomorphic and K-Morphic Tilings

Non-Euclidean Geometry

Our final topic is Non-Euclidean Geometry. This website will give you an introduction to the topic.

DISCUSS: APPLICATION OF GEOMETRY

Pick a real problem and try to solve it using the geometry you learned in this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Geometry Application Directions for detailed directions.

Use the **Geometry Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate’s posting?	5
The strategies:	

• Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

SET THEORY

MODULE 3 OVERVIEW

What You'll Learn To Do: Describe set theory and perform operations.

Learning Objectives

- Describe memberships of sets, including the empty set, using proper notation, and decide whether given items are members and determine the cardinality of a given set.
- Describe the relations between sets regarding membership, equality, subset, and proper subset, using proper notation.
- Perform the operations of union, intersection, complement, and difference on sets using proper notation.
- Be able to draw and interpret Venn diagrams of set relations and operations and use Venn diagrams to solve problems.
- Recognize when set theory is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Set Theory
- **Read:** Supplemental Videos for Set Theory
- **Read:** Verbal, Roster, and Set-Builder Notation for a Set
- **Read:** Consider a Set
- **Review:** Helpful Links

Homework Assignments

- **Submit:** Set Theory Homework #1 (16 points)
- **Submit:** Set Theory Homework #2 (14 points)
- **Discuss:** Application of Set Theory (20 points)

SET THEORY

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set
A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

1. The set of all even numbers
2. The set of all books written about travel to Chile

Answers

Some examples of sets defined by listing the elements of the set:

1. {1, 3, 9, 12}
2. {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.
The symbol $\in$ means “is an element of”.
A set that contains no elements, { }, is called the **empty set** and is notated $\varnothing$

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.
If B is a subset of A, we write $B \subseteq A$
A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.
If B is a proper subset of A, we write $B \subset A$

Example 3

Consider these three sets:

A = the set of all even numbers
$B = \{2, 4, 6\}$
$C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays "Much Ado About Nothing," "MacBeth," and "A Midsummer's Night Dream." What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try It Now

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$. More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)
The **intersection** of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$. More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$.
The **complement** of a set *A* contains everything that is *not* in the set *A*. The complement is notated *A'*, or A^c, *or sometimes ~A.*

Example 5

Consider the sets:

A = {red, green, blue}
B = {red, yellow, orange}
C = {red, orange, yellow, green, blue, purple}

Find the following:

1. Find $A \cup B$
2. Find $A \cap B$
3. Find $A^c \cap C$

Answers

1. The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange} Notice we only list red once.
2. The intersection contains all the elements in both sets: $A \cap B$ = {red}
3. Here we're looking for all the elements that are *not* in set *A* and are also in *C*. $A^c \cap C$ = {orange, yellow, purple}

Try It Now

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Notice that in the example above, it would be hard to just ask for A^c, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in *A*.

Example 6

1. If we were discussing searching for books, the universal set might be all the books in the library.
2. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
3. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is U = all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then $A^c = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8

Suppose H = {cat, dog, rabbit, mouse}, F = {dog, cow, duck, pig, rabbit}, and W = {duck, rabbit, deer, frog, mouse}

1. Find $(H \cap F) \cup W$
2. Find $H \cap (F \cup W)$
3. Find $(H \cap F)^c \cap W$

Solutions

1. We start with the intersection: $H \cap F$ = {dog, rabbit}. Now we union that result with W: $(H \cap F) \cup W$ = {dog, duck, rabbit, deer, frog, mouse}
2. We start with the union: $F \cup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}. Now we intersect that result with H: $H \cap (F \cup W)$ = {dog, rabbit, mouse}
3. We start with the intersection: $H \cap F$ = {dog, rabbit}. Now we want to find the elements of W that are *not* in $H \cap F$. $(H \cap F)^c \cap W$ = {duck, deer, frog, mouse}

Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the eighteenth century. These illustrations now called **Venn Diagrams**.

Venn Diagram

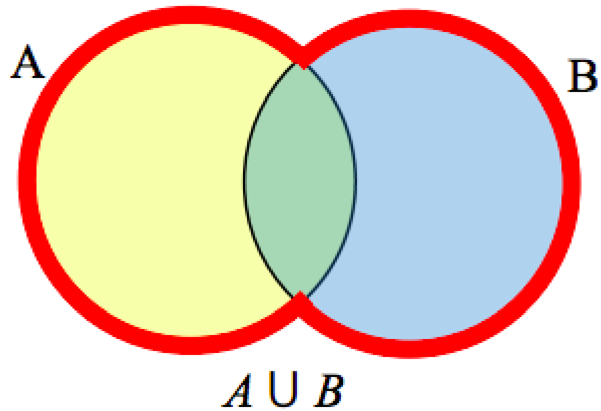
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

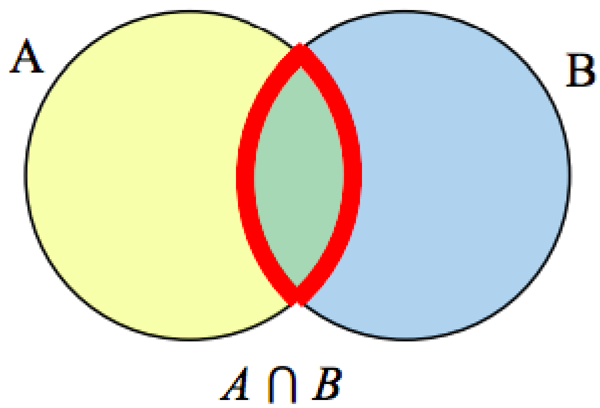
Example 9

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

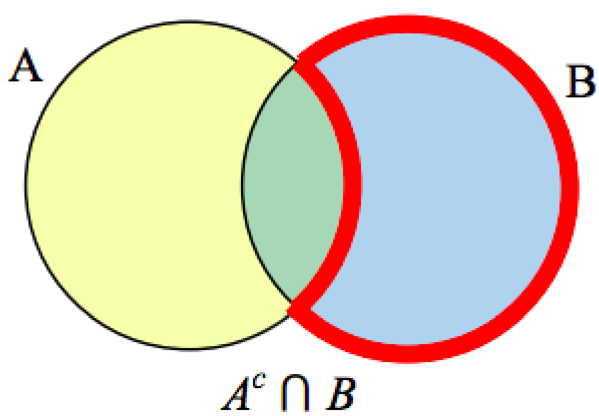
$A \cup B$ contains all elements in *either* set.



$A \cap B$ contains only those elements in both sets—in the overlap of the circles.



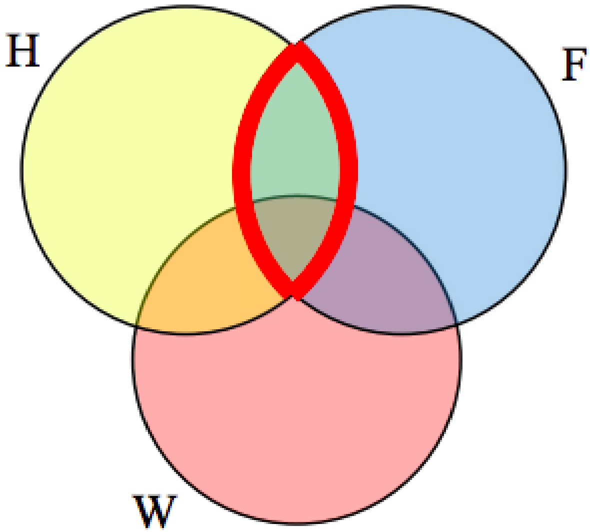
A^c will contain all elements *not* in the set A. $A^c \cap B$ will contain the elements in set B that are not in set A.



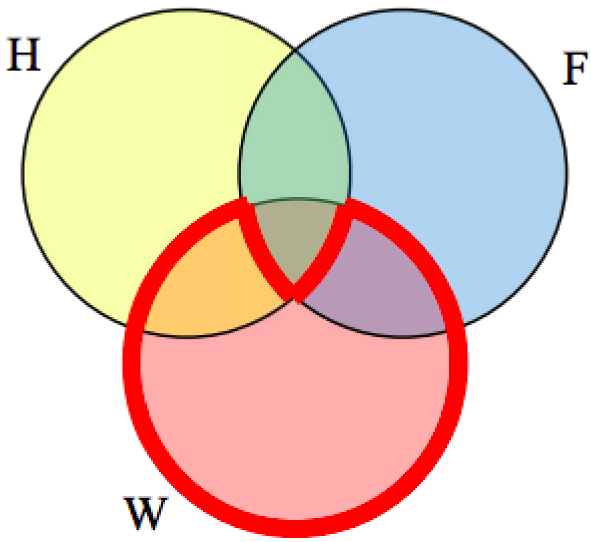
Example 10

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

We'll start by identifying everything in the set $H \cap F$

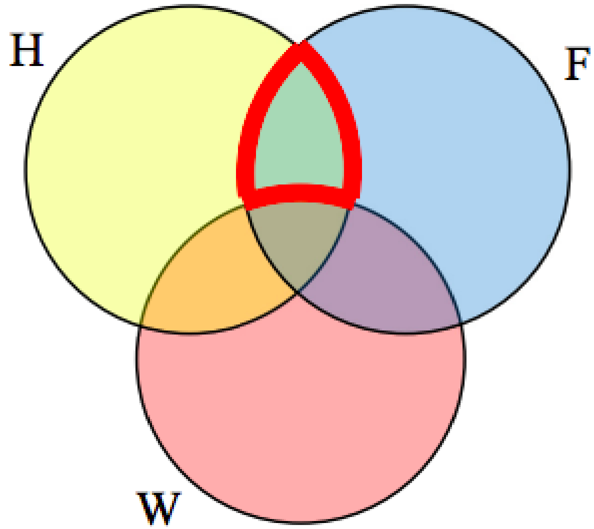


Now, $(H \cap F)^c \cap W$ will contain everything *not* in the set identified above that is also in set W.



Example 11

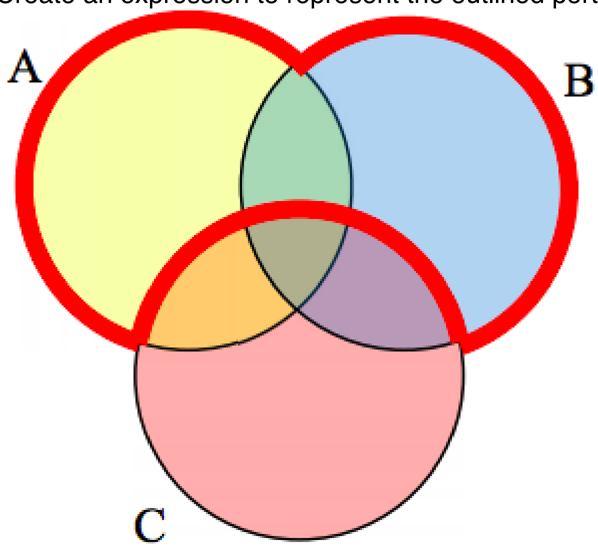
Create an expression to represent the outlined part of the Venn diagram shown.



The elements in the outlined set *are* in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W^c$

Try It Now

Create an expression to represent the outlined portion of the Venn diagram shown



Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

What is the cardinality of B? $A \cup B$, $A \cap B$?

Answers

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of P = the set of English names for the months of the year?

Answers

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Example 14

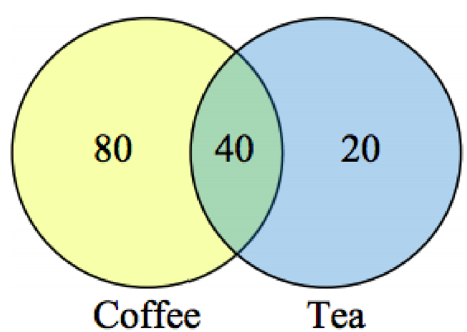
A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Answers

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.



We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

200 – 20 – 80 – 40 = 60 people who drink neither.

Example 15

A survey asks: “Which online services have you used in the last month?”

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Answers

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we’ve counted twice. In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$
$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Now, to find how many people have not used either service, we’re looking for the cardinality of $(F \cup T)^c$. Since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)^c$ must be the other 10%.

The previous example illustrated two important properties

Cardinality properties

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$n(A^c) = n(U) - n(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$n(A \cap B) = n(A) + n(B) - n(A \cup B)$

Example 16

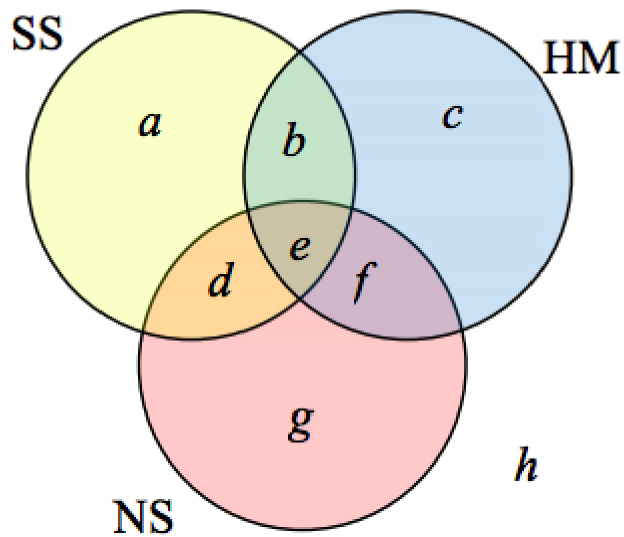
Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?

Answers

It might help to look at a Venn diagram. From the given data, we know that there are 3 students in region *e* and 7 students in region *h*.



Since 7 students were taking a SS and NS course, we know that n(*d*) + n(*e*) = 7. Since we know there are 3 students in region 3, there must be 7 – 3 = 4 students in region *d*.

Similarly, since there are 10 students taking HM and NS, which includes regions *e* and *f*, there must be 10 – 3 = 7 students in region *f*.

Since 9 students were taking SS and HM, there must be 9 – 3 = 6 students in region *b*.

Now, we know that 21 students were taking a SS course. This includes students from regions *a, b, d,* and *e*. Since we know the number of students in all but region *a*, we can determine that 21 – 6 – 4 – 3 = 8 students are in region *a*.

8 students are taking only a SS course.

Try It Now

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

SUPPLEMENTAL VIDEOS

This YouTube playlist contains several videos that supplement the reading on Set Theory.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

VERBAL, ROSTER, AND SET-BUILDER NOTATION FOR A SET

This site includes a written description and a video describing the three ways to name a set:

- By name or verbal description
- By roster (list or listing) form
- By set-builder notation. Note that the colon : may be replaced by a vertical line I

Click on the link below to view the article “Verbal, Roster, and Set-builder Notation for a Set” developed by Minnesota State University–Moorhead.

- Verbal, Roster, and Set-builder Notation for a Set

CONSIDER A SET

Consider this set: $A = \{a, b, c, d\}$.

As defined in the first reading assignment, a subset of *A* is another set that contains only elements from the set *A*, but many not contain all the elements of *A*. A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.

We can list all of the subsets of *A*:

{} (or Ø), {*a*}, {*b*}, {*c*}, {*d*}, {*a*,*b*}, {*a*,*c*}, {*a*,*d*}, {*b*,*c*}, {*b*,*d*}, {*c*,*d*}, {*a*,*b*,*c*}, {*a*,*b*,*d*}, {*a*,*c*,*d*}, {*b*,*c*,*d*}, {*a*,*b*,*c*,*d*}

You can see that there are 16 subsets, 15 of which are proper subsets.

Listing the sets is fine if you have only a few elements. However, if we were to list all of the subsets of a set containing many elements, it would be quite tedious. Instead, let's consider each element of the set separately.

In our example, there are four elements. For the first element, *a*, either it's in the set or it's not. Thus there are 2 choices for that first element. Similarly, there are two choices for *b*—either it's in the set or it's not. Using just those two elements, we see that the subsets are as follows:

{}—both elements are not in the set
{*a*}—*a* is in; *b* is not in the set
{*b*}—*a* is not in the set; *b* is in
{*a*,*b*}—*a* is in; *b* is in

Two choices for *a* times the two for *b* gives us $2^2 = 4$ subsets. You can draw a tree diagram to see this as well.

Now let's include *c*. Again, either *c* is included or it isn't, which gives us two choices. The outcomes are {}, {*a*}, {*b*}, {*c*}, {*a*,*b*}, {*a*,*c*}, {*b*,*c*}, {*a*,*b*,*c*}. Note that there are $2^3 = 8$ subsets.

Including all four elements, there are $2^4 = 16$ subsets. 15 of those subsets are proper, 1 subset, namely {*a*,*b*,*c*,*d*}, is not.

In general, if you have *n* elements in your set, then there are 2^n subsets and $2^n - 1$ proper subsets.

HELPFUL LINKS

These pages on Purplemath.com will provide further background and understanding of both Set Notation and Venn Diagrams. They provide instruction and examples.

- Set Notation
- Venn Diagrams (You'll notice the words "Page 1 of 4" at the top of this page. Be sure to look at all four pages of the Venn Diagram section.)

DISCUSS: APPLICATION OF SET THEORY

Pick a real problem and try to solve it using the general problem solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Sets Application Directions for detailed directions.

Use the **Set Theory Application** forum in the Discussion Board to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

LOGIC

MODULE 4 OVERVIEW

What You'll Learn To Do: Identify, make, and interpret logic.

Learning Objectives

- Identify, make, and interpret logical statements in English and symbolic notation.
- Express and interpret connectives (including negation) in English and symbolic notation.
- Express and interpret logical quantifiers, and their negations, in English.
- Express and interpret the propositions of conditional, converse, inverse, and contrapositive in English and symbolic notation.
- Construct and interpret truth tables to evaluate the truth value of a compound statement, determine whether or not two statements are logically equivalent, and determine the validity of an argument.
- Recognize when logic is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Logic
- **Read:** Truth Tables and Analyzing Arguments: Examples
- **Watch:** Truth Tables: Conjunction and Disjunction
- **Watch:** Truth Tables: Implication
- **Watch:** Analyzing Arguments with Truth Tables
- **Read:** Negating "all," "some," or "no" statements

Homework Assignments

- **Submit:** Logic Homework Assignment #1 (16 points)
- **Submit:** Logic Homework Assignment #2 (14 points)
- **Discuss:** Truth Table Practice (10 points)
- **Discuss:** Logic Application (20 points)
- **Complete:** Exam 2

TEST PAGE

LOGIC

Logic is a systematic way of thinking that allows us to deduce new information from old information and to parse the meanings of sentences. You use logic informally in everyday life and certainly also in doing mathematics. For example, suppose you are working with a certain circle, call it "Circle X," and you have available the following two pieces of information.

1. Circle X has radius equal to 3.
2. If any circle has radius r, then its area is πr^2 square units.

You have no trouble putting these two facts together to get:

3. Circle X has area 9π square units.

In doing this you are using logic to combine existing information to produce new information. Since a major objective in mathematics is to deduce new information, logic must play a fundamental role. This chapter is intended to give you a sufficient mastery of logic.

It is important to realize that logic is a process of deducing information correctly, *not* just deducing correct information. For example, suppose we were mistaken and Circle X actually had a radius of 4, not 3. Let's look at our exact same argument again.

1. Circle X has radius equal to 3.
2. If any circle has radius r, then its area is πr^2 square units.
3. Circle X has area 9π square units.

The sentence "*Circle X has radius equal to 3.*" is now untrue, and so is our conclusion "*Circle X has area 9π square units.*" But the logic is perfectly correct; the information was combined correctly, even if some of it was false. This distinction between correct logic and correct information is significant because it is often important to follow the consequences of an incorrect assumption. Ideally, we want both our logic *and* our information to be correct, but the point is that they are different things.

In proving theorems, we apply logic to information that is considered obviously true (such as "*Any two points determine exactly one line.*") or is already known to be true (e.g., the Pythagorean theorem). If our logic is correct, then anything we deduce from such information will also be true (or at least as true as the "obviously true" information we began with).

Statements

The study of logic begins with statements. A **statement** is a sentence or a mathematical expression that is either definitely true or definitely false. You can think of statements as pieces of information that are either correct or incorrect. Thus statements are pieces of information that we might apply logic to in order to produce other pieces of information (which are also statements).

Example 1

Here are some examples of statements. They are all true.

If a circle has radius r, then its area is πr^2 square units.

Every even number is divisible by 2.

$2 \in \mathbb{Z}$ (2 is an element of the set of integers (or more simply, 2 is an integer).)

$\sqrt{2} \notin \mathbb{Z}$ (The square root of 2 is not an integer.)

$\mathbb{N} \subseteq \mathbb{Z}$ (The set of natural numbers is a subset of the set of integers.)

The set {0,1,2} has three elements.

Some right triangles are isosceles.

Example 2

Here are some additional statements. They are all false.

All right triangles are isosceles.

5 = 2

$\sqrt{2} \notin \mathbb{R}$ (The square root of 2 is not a real number.)

$\mathbb{Z} \subseteq \mathbb{N}$ (The set of integers is a subset of the set of natural numbers.)

$0, 1, 2 \cap \mathbb{N} = \varnothing$ (The intersection of the set {0,1,2} and the natural numbers is the empty set.)

Example 3

Here we pair sentences or expressions that are not statements with similar expressions that are statements.

NOT Statements	**Statements**
Add 5 to both sides.	Adding 5 to both sides of $x - 5 = 37$ gives $x = 42$.
$\mathbb{Z}$ (The set of integers)	$42 \in \mathbb{Z}$ (42 is an element of the set of integers.)
42	42 is not a number.
What is the solution of $2x = 84$?	The solution of $2x = 84$ is 42.

Example 4

We will often use the letters P, Q, R, and S to stand for specific statements. When more letters are needed we can use subscripts. Here are more statements, designated with letters. You decide which of them are true and which are false.

P : For every integer $n > 1$, the number $2^n - 1$ is prime.
Q : Every polynomial of degree n has at most n roots.
R : The function $f(x) = x^2$ is continuous.
S_1 : $\mathbb{N} \subseteq \varnothing$

S_2 : $0, -1, -2 \cap \mathbb{N} = \varnothing$

Designating statements with letters (as was done above) is a very useful shorthand. In discussing a particular statement, such as "The function $f(x) = x^2$ is continuous," it is convenient to just refer to it as R to avoid having to write or say it many times.

Statements can contain variables. Here is an example.

P : If an integer x is a multiple of 6, then x is even.

This is a sentence that is true. (All multiples of 6 are even, so no matter which multiple of 6 the integer x happens to be, it is even.) Since the sentence P is definitely true, it is a statement. When a sentence or statement P contains a variable such as x, we sometimes denote it as $P(x)$ to indicate that it is saying something about x. Thus the above statement can be denoted as

$P(x)$: If an integer x is a multiple of 6, then x is even.

A statement or sentence involving two variables might be denoted $P(x, y)$, and so on.

It is quite possible for a sentence containing variables to not be a statement. Consider the following example.

$Q(x)$: The integer x is even.

Is this a statement? Whether it is true or false depends on just which integer x is. It is true if $x = 4$ and false if $x = 7$, etc. But without any stipulations on the value of x it is impossible to say whether $Q(x)$ is true or false. Since it is neither definitely true nor definitely false, $Q(x)$ cannot be a statement. A sentence such as this, whose truth depends on the value of one or more variables, is called an **open sentence**. The variables in an open sentence (or statement) can represent any type of entity, not just numbers. Here is an open sentence where the variables are functions:

$R(f, g)$: The function f is the derivative of the function g.

This open sentence is true if $f(x) = 2x$ and $g(x) = x^2$. It is false if $f(x) = x^3$ and $g(x) = x^2$, etc. We point out that a sentence such as $R(f, g)$ (that involves variables) can be denoted either as $R(f, g)$ or just R. We use the expression $R(f, g)$ when we want to emphasize that the sentence involves variables.

We will have more to say about open sentences later, but for now let's return to statements.

Statements are everywhere in mathematics. Any result or theorem that has been proved true is a statement. The quadratic formula and the Pythagorean theorem are both statements:

P : The solutions of the equation $ax^2 + bx + c = 0 \text{ are } x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Q : If a right triangle has legs of lengths a and b and hypotenuse of length $a^2 + b^2 = c^2$..

Here is a very famous statement, so famous, in fact, that it has a name. It is called **Fermat's last theorem** after Pierre Fermat, a seventeenth-century French mathematician who scribbled it in the margin of a notebook.

R : For all numbers *a*, *b*, *c*, $n \in \mathbb{N}$ with $n > 2$, it is the case that $a^n + b^n \neq c^n$.

Fermat believed this statement was true. He noted that he could prove it was true, except his notebook's margin was too narrow to contain his proof. It is doubtful that he really had a correct proof in mind, for after his death generations of brilliant mathematicians tried unsuccessfully to prove that his statement was true (or false). Finally, in 1993, Andrew Wiles of Princeton University announced that he had devised a proof. Wiles had worked on the problem for over seven years, and his proof runs through hundreds of pages. The moral of this story is that some true statements are not obviously true.

Here is another statement famous enough to be named. It was first posed in the eighteenth century by the German mathematician Christian Goldbach, and thus is called the Goldbach conjecture:

S : Every even integer greater than 2 is a sum of two prime numbers.

You must agree that *S* is either true or false. It appears to be true, because when you examine even numbers that are bigger than 2, they seem to be sums of two primes: 4 = 2+2, 6 = 3+3, 8 = 3+5, 10 = 5+5, 12 = 5+7, 100 = 17+83 and so on. But that's not to say there isn't some large even number that's not the sum of two primes. If such a number exists, then *S* is false. The thing is, in the over 260 years since Goldbach first posed this problem, no one has been able to determine whether it's true or false. But since it is clearly either true or false, *S* is a statement.

This book is about the methods that can be used to prove that *S* (or any other statement) is true or false. To prove that a statement is true, we start with obvious statements (or other statements that have been proven true) and use logic to deduce more and more complex statements until finally we obtain a statement such as *S*. Of course some statements are more difficult to prove than others, and *S* appears to be notoriously difficult; we will concentrate on statements that are easier to prove.

But the point is this: In proving that statements are true, we use logic to help us understand statements and to combine pieces of information to produce new pieces of information. In the next several sections we explore some standard ways that statements can be combined to form new statements, or broken down into simpler statements.

And, Or, Not

The word "and" can be used to combine two statements to form a new statement. Consider for example the following sentence.

R_1 : The number 2 is even **and** the number 3 is odd.

We recognize this as a true statement, based on our common-sense understanding of the meaning of the word "and." Notice that R_1 is made up of two simpler statements:

P : The number 2 is even.
Q : The number 3 is odd.

These are joined together by the word "and" to form the more complex statement R_1. The statement R_1 asserts that *P* and *Q* are both true. Since both *P* and *Q* are in fact true, the statement R_1 is also true.

Had one or both of *P* and *Q* been false, then R_1 would be false. For instance, each of the following statements is false.

R_2 : The number 1 is even **and** the number 3 is odd.
R_3 : The number 2 is even **and** the number 4 is odd.
R_4 : The number 3 is even **and** the number 2 is odd.

From these examples we see that any two statements P and Q can be combined to form a new statement "P **and** Q." In the spirit of using letters to denote statements, we now introduce the special symbol $\wedge$ to stand for the word "and." Thus if P and Q are statements, $P \wedge Q$ stands for the statement "P **and** Q." The statement $P \wedge Q$ is true if both P and Q are true; otherwise it is false. This is summarized in the following table, called a **truth table**.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

In this table, T stands for "True," and F stands for "False." (T and F are called **truth values**.) Each line lists one of the four possible combinations or truth values for P and Q, and the column headed by $P \wedge Q$ tells whether the statement $P \wedge Q$ is true or false in each case.

Statements can also be combined using the word "or." Consider the following four statements.

S_1 : The number 2 is even or the number 3 is odd.
S_2 : The number 1 is even or the number 3 is odd.
S_3 : The number 2 is even or the number 4 is odd.
S_4 : The number 3 is even or the number 2 is odd.

In mathematics, the assertion "P **or** Q" is always understood to mean that one or both of P and Q is true. Thus statements S_1, S_2, S_3 are all true, while S_4 is false. The symbol $\vee$ is used to stand for the word "or." So if P and Q are statements, $P \vee Q$ represents the statement "P **or** Q." Here is the truth table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

It is important to be aware that the meaning of "or" expressed in the above table differs from the way it is sometimes used in everyday conversation. For example, suppose a university official makes the following threat:

You pay your tuition **or** you will be withdrawn from school.

You understand that this means that either you pay your tuition or you will be withdrawn from school, *but not both*. In mathematics we never use the word "or" in such a sense. For us "or" means exactly what is stated in the table for $\vee$. Thus $P \vee Q$ being true means *one* or *both* of P and Q is true. If we ever need to express the fact that exactly one of P and Q is true, we use one of the following constructions:

P **or** Q, **but not both.**

Either P **or** Q.

If the university official were a mathematician, he might have qualified his statement in one of the following ways.

Pay your tuition **or** you will be withdrawn from school, **but not both**.
Either you pay your tuition **or** you will be withdrawn from school.

To conclude this section, we mention another way of obtaining new statements from old ones. Given any statement *P*, we can form the new statement "**It is not true that** *P*." For example, consider the following statement.

The number 2 is even.

This statement is true. Now change it by inserting the words "It is not true that" at the beginning:

It is not true that the number 2 is even.

This new statement is false.

For another example, starting with the false statement "$2 \in \varnothing$" we get the true statement "It is not true that

$2 \in \varnothing$."

We use the symbol ~ to stand for the words "It's not true that," so ~ *P* means "**It's not true that** *P*." We often read ~ *P* simply as "not *P*." Unlike ∧ and ∨, which combine two statements, the symbol ~ just alters a single statement. Thus its truth table has just two lines, one for each possible truth value of *P*.

P	**~ *P***
T	F
F	T

The statement ~ *P* is called the **negation** of *P*. The negation of a specific statement can be expressed in numerous ways. Consider

P : The number 2 is even.

Here are several ways of expressing its negation.

~ *P* : It's not true that the number 2 is even.
~ *P* : It is false that the number 2 is even.
~ *P* : The number 2 is not even.

In this section we've learned how to combine or modify statements with the operations ∧, ∨ and ~. Of course we can also apply these operations to open sentences or a mixture of open sentences and statements. For example, (*x* is an even integer)∧(3 is an odd integer) is an open sentence that is a combination of an open sentence and a statement.

Conditional Statements

There is yet another way to combine two statements. Suppose we have in mind a specific integer *a*. Consider the following statement about *a*.

R : If the integer a is a multiple of 6, then a is divisible by 2.

We immediately spot this as a true statement based on our knowledge of integers and the meanings of the words "if" and "then." If integer a is a multiple of 6, then a is even, so therefore a is divisible by 2. Notice that *R* is built up from two simpler statements:

P : The integer a is a multiple of 6.
Q : The integer a is divisible by 2.
R : If P, then Q.

In general, given any two statements *P* and *Q* whatsoever, we can form the new statement "*If P, then Q.*" This is written symbolically as $P \Rightarrow Q$ which we read as "*If P, then Q*," or "*P implies Q.*" Like $\wedge$ and $\vee$, the symbol $\Rightarrow$ has a very specific meaning. When we assert that the statement $P \Rightarrow Q$ is true, we mean that *if P* is true *then Q* must also be true. (In other words we mean that the condition *P* being true forces *Q* to be true.) A statement of form $P \Rightarrow Q$ is called a **conditional** statement because it means *Q* will be true *under the condition* that *P* is true.

You can think of $P \Rightarrow Q$ as being a promise that whenever *P* is true, *Q* will be true also. There is only one way this promise can be broken (i.e. be false) and that is if *P* is true but *Q* is false. Thus the truth table for the promise $P \Rightarrow Q$ is as follows:

P	***Q***	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Perhaps you are bothered by the fact that $P \Rightarrow Q$ is true in the last two lines of this table. Here's an example to convince you that the table is correct. Suppose your professor makes the following promise:

If you pass the final exam, **then** you will pass the course.

Your professor is making the promise

(You pass the exam) $\Rightarrow$ (You pass the course).

Under what circumstances did she lie? There are four possible scenarios, depending on whether or not you passed the exam and whether or not you passed the course. These scenarios are tallied in the following table.

You pass exam	**You pass course**	**(You pass exam) ⇒ (You pass course)**
T	T	T
T	F	F
F	T	T
F	F	T

The first line describes the scenario where you pass the exam and you pass the course. Clearly the professor kept her promise, so we put a *T* in the third column to indicate that she told the truth. In the second line, you passed the exam, but your professor gave you a failing grade in the course. In this case she broke her promise, and the *F* in the third column indicates that what she said was untrue.

Now consider the third row. In this scenario you failed the exam but still passed the course. How could that happen? Maybe your professor felt sorry for you. But that doesn't make her a liar. Her only promise was that if you passed the exam then you would pass the course. She did not say passing the exam was the **only way** to pass the course. Since she didn't lie, then she told the truth, so there is a *T* in the third column.

Finally look at the fourth row. In that scenario you failed the exam and you failed the course. Your professor did not lie; she did exactly what she said she would do. Hence the *T* in the third column.

In mathematics, whenever we encounter the construction "*If P, then Q*" it means exactly what the truth table for $\Rightarrow$ expresses. But of course there are other grammatical constructions that also mean $P \Rightarrow Q$. Here is a summary of the main ones.

$$P \Rightarrow Q \begin{cases} \text{If } P, \text{ then } Q. \\ Q \text{ if } P. \\ Q \text{ whenever } P. \\ Q, \text{ provided that } P. \\ \text{Whenever } P, \text{ then also } Q. \\ P \text{ is a sufficient condition for } Q. \\ \text{For } Q, \text{ it is sufficient that } P. \\ Q \text{ is a necessary condition for } P. \\ \text{For } P, \text{ it is necessary that } Q. \\ P \text{ only if } Q. \end{cases}$$

These can all be used in the place of (and mean exactly the same thing as) "*If P, then Q*." You should analyze the meaning of each one and convince yourself that it captures the meaning of $P \Rightarrow Q$. For example, $P \Rightarrow Q$ means the condition of P being true is enough (i.e., sufficient) to make Q true; hence "*P is a sufficient condition for Q*."

The wording can be tricky. Often an everyday situation involving a conditional statement can help clarify it. For example, consider your professor's promise:

(You pass the exam) $\Rightarrow$ (You pass the course)

This means that your passing the exam is a sufficient (though perhaps not necessary) condition for your passing the course. Thus your professor might just as well have phrased her promise in one of the following ways.

Passing the exam is a sufficient condition for passing the course.
For you to pass the course, it is sufficient that you pass the exam.

However, when we want to say "*If P, then Q*" in everyday conversation, we do not normally express this as "*Q is a necessary condition for P*" or "*P only if Q*." But such constructions are not uncommon in mathematics. To understand why they make sense, notice that $P \Rightarrow Q$ being true means that it's impossible that P is true but Q is false, so in order for P to be true it is necessary that Q is true; hence "*Q is a necessary condition for P*." And this means that P can only be true if Q is true, i.e., "*P only if Q*."

Biconditional Statements

It is important to understand that $P \Rightarrow Q$ is not the same as $Q \Rightarrow P$. To see why, suppose that a is some integer and consider the statements

(a is a multiple of 6) $\Rightarrow$ (a is divisible by 2),
(a is divisible by 2) $\Rightarrow$ (a is a multiple of 6).

The first statement asserts that if a is a multiple of 6 then a is divisible by 2. This is clearly true, for any multiple of 6 is even and therefore divisible by 2. The second statement asserts that if a is divisible by 2 then it is a multiple of 6. This is not necessarily true, for a = 4 (for instance) is divisible by 2, yet not a multiple of 6. Therefore the meanings of $P \Rightarrow Q$ and $Q \Rightarrow P$ are in general quite different. The conditional statement $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$, so a conditional statement and its converse express entirely different things.

However, the **contrapositive** of $P \Rightarrow Q$, $\sim Q \Rightarrow \sim P$, is equivalent to $P \Rightarrow Q$. Similarly, the **inverse** of $P \Rightarrow Q$, which is $\sim P \Rightarrow \sim Q$, is equivalent to the converse $Q \Rightarrow P$. In "Truth Tables for Statements," we will learn how to show these equivalences using a truth table.

But sometimes, if P and Q are just the right statements, it can happen that $P \Rightarrow Q$ and $Q \Rightarrow P$ are both necessarily true. For example, consider the statements

(a is even) $\Rightarrow$ (a is divisible by 2),
(a is divisible by 2) $\Rightarrow$ (a is even).

No matter what value a has, both of these statements are true. Since both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, it follows that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true.

We now introduce a new symbol $\Leftrightarrow$ to express the meaning of the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. The expression $P \Leftrightarrow Q$ is understood to have exactly the same meaning as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. According to the previous section, $Q \Rightarrow P$ is read as "*P if Q*," and $P \Rightarrow Q$ can be read as "*P only if Q*." Therefore we pronounce $P \Leftrightarrow Q$ as "*P if and only if Q*." For example, given an integer a, we have the true statement

(a is even) $\Leftrightarrow$ (a is divisible by 2),

which we can read as "*Integer a is even if and only if a is divisible by 2.*"

The truth table for $\Leftrightarrow$ is shown below. Notice that in the first and last rows, both P $\Rightarrow$ Q and Q $\Rightarrow$ P are true (according to the truth table for $\Rightarrow$), so (P $\Rightarrow$ Q) $\wedge$ (Q $\Rightarrow$ P) is true, and hence P $\Leftrightarrow$ Q is true. However, in the middle two rows one of P $\Rightarrow$ Q or Q $\Rightarrow$ P is false, so (P $\Rightarrow$ Q)$\wedge$(Q $\Rightarrow$ P) is false, making P $\Leftrightarrow$ Q false.

P	**Q**	**P $\Leftrightarrow$ Q**
T	T	T
T	F	F
F	T	F
F	F	T

Compare the statement R : (a is even) $\Leftrightarrow$ (a is divisible by 2) with this truth table. If a is even then the two statements on either side of $\Leftrightarrow$ are true, so according to the table R is true. If a is odd then the two statements on either side of $\Leftrightarrow$ are false, and again according to the table R is true. Thus R is true no matter what value a has. In general, $P \Leftrightarrow Q$ being true means P and Q are both true or both false.

Not surprisingly, there are many ways of saying $P \Leftrightarrow Q$ in English. The following constructions all mean $P \Leftrightarrow Q$:

$$P \iff Q \begin{cases} P \text{ if and only if } Q. \\ P \text{ is a necessary and sufficient condition for } Q. \\ \text{For } P \text{ it is necessary and sufficient that } Q. \\ \text{If } P, \text{ then } Q, \text{ and conversely.} \end{cases}$$

The first three of these just combine constructions from the previous section to express that $P \Rightarrow Q$ and $Q \Rightarrow P$. In the last one, the words "*and conversely*" mean that in addition to "*If P, then Q*" being true, the converse statement "*If Q, then P*" is also true.

Truth Tables for Statements

You should now know the truth tables for $\wedge$, $\vee$, $\sim$, $\Rightarrow$ and $\Leftrightarrow$. They should be *internalized* as well as memorized. You must understand the symbols thoroughly, for we now combine them to form more complex statements.

For example, suppose we want to convey that one or the other of P and Q is true but they are not both true. No single symbol expresses this, but we could combine them as

$(P \vee Q) \wedge \sim (P \wedge Q)$,

which literally means:

P or Q is true, and it is not the case that both P and Q are true.

This statement will be true or false depending on the truth values of P and Q. In fact we can make a truth table for the entire statement. Begin as usual by listing the possible true/false combinations of P and Q on four lines. The statement $(P \vee Q) \wedge \sim (P \wedge Q)$ contains the individual statements $(P \vee Q)$ and $(P \wedge Q)$, so we next tally their truth values in the third and fourth columns. The fifth column lists values for $\sim (P \wedge Q)$, and these are just the opposites of the corresponding entries in the fourth column. Finally, combining the third and fifth columns with $\wedge$, we get the values for $(P \vee Q) \wedge \sim(P \wedge Q)$ in the sixth column.

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

This truth table tells us that $(P \vee Q) \wedge \sim(P \wedge Q)$ is true precisely when one but not both of P and Q are true, so it has the meaning we intended. (Notice that the middle three columns of our truth table are just "helper columns" and are not necessary parts of the table. In writing truth tables, you may choose to omit such columns if you are confident about your work.)

For another example, consider the following familiar statement concerning two real numbers x and y:

The product xy equals zero if and only if $x = 0$ or $y = 0$.

This can be modeled as $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$. If we introduce letters P, Q, and R for the statements $xy = 0$, $x = 0$ and $y = 0$, it becomes $P \Leftrightarrow (Q \vee R)$. Notice that the parentheses are necessary here, for without them we wouldn't know whether to read the statement as $P \Leftrightarrow (Q \vee R)$ or $(P \Leftrightarrow Q) \vee R$.

Making a truth table for $P \Leftrightarrow (Q \vee R)$ entails a line for each T/F combination for the three statements P, Q, and R. The eight possible combinations are tallied in the first three columns of the following table.

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

We fill in the fourth column using our knowledge of the truth table for $\vee$. Finally the fifth column is filled in by combining the first and fourth columns with our understanding of the truth table for $\Leftrightarrow$. The resulting table gives the true/false values of $P \Leftrightarrow (Q \vee R)$ for all values of P, Q, and R.

Notice that when we plug in various values for x and y, the statements $P : xy = 0$, $Q : x = 0$ and $R : y = 0$ have various truth values, but the statement $P \Leftrightarrow (Q \vee R)$ is always true. For example, if $x = 2$ and $y = 3$, then P, Q, and R are all false. This scenario is described in the last row of the table, and there we see that $P \Leftrightarrow (Q \vee R)$ is true. Likewise if $x = 0$ and $y = 7$, then P and Q are true and R is false, a scenario described in the second line of the table, where again $P \Leftrightarrow (Q \vee R)$ is true. There is a simple reason why $P \Leftrightarrow (Q \vee R)$ is true for any values of x and y: It is that $P \Leftrightarrow (Q \vee R)$ represents $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$, which is a *true mathematical statement*. It is absolutely impossible for it to be false.

This may make you wonder about the lines in the table where $P \Leftrightarrow (Q \vee R)$ is false. Why are they there? The reason is that $P \Leftrightarrow (Q \vee R)$ can also represent a false statement. To see how, imagine that at the end of the semester your professor makes the following promise.

> You pass the class if and only if you get an “A” on the final or you get a “B” on the final.

This promise has the form $P \Leftrightarrow (Q \vee R)$, so its truth values are tabulated in the above table. Imagine it turned out that you got an “A” on the exam but failed the course. Then surely your professor lied to you. In fact, P is false, Q is true and R is false. This scenario is reflected in the sixth line of the table, and indeed $P \Leftrightarrow (Q \vee R)$ is false (i.e., it is a lie).

The moral of this example is that people can lie, but true mathematical statements **never** lie.

We close this section with a word about the use of parentheses. The symbol ~ is analogous to the minus sign in algebra. It negates the expression it precedes. Thus $\sim P \vee Q$ means $(\sim P) \vee Q$, not $\sim(P \vee Q)$. In $\sim(P \vee Q)$, the value of the entire expression $P \vee Q$ is negated.

Logical Equivalence

In contemplating the truth table for $P \Leftrightarrow Q$, you probably noticed that $P \Leftrightarrow Q$ is true exactly when P and Q are both true or both false. In other words, $P \Leftrightarrow Q$ is true precisely when at least one of the statements $P \wedge Q$ or $\sim P \wedge \sim Q$ is true. This may tempt us to say that $P \Leftrightarrow Q$ means the same thing as $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.

To see if this is really so, we can write truth tables for $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$. In doing this, it is more efficient to put these two statements into the same table, as follows. (This table has helper columns for the intermediate expressions $\sim P$, $\sim Q$, $(P \wedge Q)$, and $(\sim P \wedge \sim Q)$.)

P	Q	$\sim P$	$\sim Q$	$(P \wedge Q)$	$(\sim P \wedge \sim Q)$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$	$P \Leftrightarrow Q$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

The table shows that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ have the same truth value, no matter the values P and Q. It is as if $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ are algebraic expressions that are equal no matter what is “plugged into” variables P and Q. We express this state of affairs by writing

$$P \Leftrightarrow Q = (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

and saying that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ are **logically equivalent**.

In general, two statements are **logically equivalent** if their truth values match up line-for-line in a truth table.

Logical equivalence is important because it can give us different (and potentially useful) ways of looking at the same thing. As an example, the following table shows that $P \Rightarrow Q$ is logically equivalent to $(\sim Q) \Rightarrow (\sim P)$.

P	***Q***	***~P***	***~Q***	**(~*Q*) ⇒ (~*P*)**	***P* ⇒ *Q***
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The fact that $P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P)$ is useful because so many theorems have the form $P \Rightarrow Q$. As we will see in Chapter 5, proving such a theorem may be easier if we express it in the logically equivalent form $(\sim Q) \Rightarrow (\sim P)$.

There are two pairs of logically equivalent statements that come up again and again throughout this book and beyond. They are prevalent enough to be dignified by a special name: **DeMorgan's laws.**

Fact: DeMorgan's Laws

1. $\sim(P \wedge Q) = (\sim P) \vee (\sim Q)$
2. $\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$

The first of DeMorgan's laws is verified by the following table. You are asked to verify the second in one of the exercises.

P	***Q***	***~P***	***~Q***	***P* ∧ *Q***	**~(*P* ∧ *Q*)**	**(~*P*)∨(~*Q*)**
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

DeMorgan's laws are actually very natural and intuitive. Consider the statement $\sim(P \wedge Q)$, which we can interpret as meaning that it is not the case that both P and Q are true. If it is not the case that both P and Q are true, then at least one of P or Q is false, in which case $(\sim P) \vee (\sim Q)$ is true. Thus $\sim(P \wedge Q)$ means the same thing as $(\sim P) \vee (\sim Q)$.

DeMorgan's laws can be very useful. Suppose we happen to know that some statement having form $\sim(P \vee Q)$ is true. The second of DeMorgan's laws tells us that $(\sim Q) \wedge (\sim P)$ is also true, hence $\sim P$ and $\sim Q$ are both true as well. Being able to quickly obtain such additional pieces of information can be extremely useful.

Here is a summary of some significant logical equivalences. Those that are not immediately obvious can be verified with a truth table.

$$\text{Contrapositive law}\left\{ P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P) \right.$$

$$\text{DeMorgan's laws}\begin{cases} \sim (P \wedge Q) = \sim P \vee \sim Q \\ \sim (P \vee Q) = \sim P \wedge \sim Q \end{cases}$$

$$\text{Commutative laws}\begin{cases}(P\wedge Q)=P\wedge Q\\(P\vee Q)=P\vee Q\end{cases}$$

$$\text{Distributive laws}\begin{cases}P\wedge(Q\vee R)=(P\wedge Q)\vee(P\wedge R)\\P\vee(Q\wedge R)=(P\vee Q)\wedge(P\vee R)\end{cases}$$

$$\text{Associative laws}\begin{cases}P\wedge(Q\wedge R)=(P\wedge Q)\wedge R\\P\vee(Q\vee R)=(P\vee Q)\vee R\end{cases}$$

Notice how the distributive law $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$ has the same structure as the distributive law $p(q + r) = p \cdot q + p \cdot r$ from algebra. Concerning the associative laws, the fact that $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$ means that the position of the parentheses is irrelevant, and we can write this as $P \wedge Q \wedge R$ without ambiguity. Similarly, we may drop the parentheses in an expression such as $P \vee (Q \vee R)$.

But parentheses are essential when there is a mix of $\wedge$ and $\vee$, as in $P \vee(Q \wedge R)$. Indeed, $P \vee(Q \wedge R)$ and $(P \vee Q) \wedge R$ are **not** logically equivalent.

Negating Statements

Given a statement R, the statement $\sim R$ is called the **negation** of R. If R is a complex statement, then it is often the case that its negation $\sim R$ can be written in a simpler or more useful form. The process of finding this form is called **negating** R. In proving theorems it is often necessary to negate certain statements. We now investigate how to do this.

We have already examined part of this topic. **DeMorgan's laws**

$\sim (P \wedge Q) = (\sim P)\vee(\sim Q)$
$\sim (P \vee Q) = (\sim P)\wedge(\sim Q)$

(from "Logical Equivalence") can be viewed as rules that tell us how to negate the statements $P \wedge Q$ and $P \vee Q$. Here are some examples that illustrate how DeMorgan's laws are used to negate statements involving "and" or "or."

Example 5

Consider negating the following statement.

R : You can solve it by factoring or with the quadratic formula.

Now, R means (You can solve it by factoring) $\vee$ (You can solve it with Q.F.), which we will denote as $P \vee Q$. The negation of this is

$\sim(P \vee Q) = (\sim P)\wedge(\sim Q)$.

Therefore, in words, the negation of R is

$\sim R$: You can't solve it by factoring and you can't solve it with the quadratic formula.

Maybe you can find $\sim R$ without invoking DeMorgan's laws. That is good; you have internalized DeMorgan's laws and are using them unconsciously.

Example 6

We will negate the following sentence.

R : The numbers x and y are both odd.

This statement means (x is odd) $\wedge$ (y is odd), so its negation is

~[(x is odd) $\wedge$ (y is odd)] = ~(x is odd) $\vee$ ~(y is odd)
(x is odd) $\wedge$ (y is odd) = (x is even) $\vee$ (y is even).

Therefore the negation of R can be expressed in the following ways:

~ R : The number x is even or the number y is even.
~ R : At least one of x and y is even.

Now let's move on to a slightly different kind of problem. It's often necessary to find the negations of quantified statements. For example, consider $\sim(\forall x \in \mathbb{N}, P(x))$. Reading this in words, we have the following:

It is not the case that $P(x)$ is true for all natural numbers x.

This means $P(x)$ is false for at least one x. In symbols, this is $\exists\, x \in \mathbb{N}, \sim P(x)$. Thus $\sim(\forall x \in \mathbb{N}, P(x)) = \exists\, x \in \mathbb{N}, \sim P(x)$. Similarly, you can reason out that $\sim(\exists\, x \in \mathbb{N}, P(x)) = \forall x \in \mathbb{N}, \sim P(x)$. In general:

$\sim(\forall x \in S, P(x)) = \exists\, x \in S, \sim P(x)$
$\sim(\exists\, x \in S, P(x)) = \forall x \in S, \sim P(x)$

Logical Inference

Suppose we know that a statement of form $P \Rightarrow Q$ is true. This tells us that whenever P is true, Q will also be true. By itself, $P \Rightarrow Q$ being true does not tell us that either P or Q is true (they could both be false, or P could be false and Q true). However if in addition we happen to know that P is true then it must be that Q is true. This is called a **logical inference**: Given two true statements we can infer that a third statement is true. In this instance true statements $P \Rightarrow Q$ and P are "added together" to get Q. This is described below with $P \Rightarrow Q$ and P stacked one atop the other with a line separating them from Q. The intended meaning is that $P \Rightarrow Q$ combined with P produces Q.

$$\begin{array}{l} P \Rightarrow Q \\ P \\ \hline Q \end{array} \qquad \begin{array}{l} P \Rightarrow Q \\ \sim Q \\ \hline \sim P \end{array} \qquad \begin{array}{l} P \vee Q \\ \sim P \\ \hline Q \end{array}$$

Two other logical inferences are listed above. In each case you should convince yourself (based on your knowledge of the relevant truth tables) that the truth of the statements above the line forces the statement below the line to be true.

Following are some additional useful logical inferences. The first expresses the obvious fact that if P and Q are both true then the statement $P \wedge Q$ will be true. On the other hand, $P \wedge Q$ being true forces P (also Q) to be true. Finally, if P is true, then $P \vee Q$ must be true, no matter what statement Q is.

$$\begin{array}{l} P \\ Q \\ \hline P \wedge Q \end{array} \qquad \begin{array}{l} P \wedge Q \\ \hline P \end{array} \qquad \begin{array}{l} P \\ \hline P \vee Q \end{array}$$

These inferences are so intuitively obvious that they scarcely need to be mentioned. However, they represent certain patterns of reasoning that we will frequently apply to sentences in proofs, so we should be cognizant of the fact that we are using them.

The first two statements in each case are called "premises" and the final statement is the "conclusion." We combine premises with $\wedge$ ("and"). The premises together imply the conclusion. Thus, the first argument would have $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ as its symbolic statement.

An Important Note

It is important to be aware of the reasons that we study logic. There are three very significant reasons. First, the truth tables we studied tell us the exact meanings of the words such as "and," "or," "not," and so on. For instance, whenever we use or read the "If…, then" construction in a mathematical context, logic tells us exactly what is meant. Second, the rules of inference provide a system in which we can produce new information (statements) from known information. Finally, logical rules such as DeMorgan's laws help us correctly change certain statements into (potentially more useful) statements with the same meaning. Thus logic helps us understand the meanings of statements and it also produces new meaningful statements.

Logic is the glue that holds strings of statements together and pins down the exact meaning of certain key phrases such as the "If…, then" or "For all" constructions. Logic is the common language that all mathematicians use, so we must have a firm grip on it in order to write and understand mathematics.

But despite its fundamental role, logic's place is in the background of what we do, not the forefront. From here on, the beautiful symbols $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\sim$, $\forall$ and $\exists$ are rarely written. But we are aware of their meanings constantly. When reading or writing a sentence involving mathematics we parse it with these symbols, either mentally or on scratch paper, so as to understand the true and unambiguous meaning.

TRUTH TABLES AND ANALYZING ARGUMENTS: EXAMPLES

Truth Tables

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false

Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 1

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise."

This is a complex statement made of two simpler conditions: "is a sectional," and "has a chaise." For simplicity, let's use S to designate "is a sectional," and C to designate "has a chaise." The condition S is true if the couch is a sectional.

A truth table for this would look like this:

S	***C***	***S* or *C***
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if *S* is true and *C* is also true, then the complex statement "*S* or *C*" is true. This would be a sectional that also has a chaise, which meets our desire.

Remember also that *or* in logic is not exclusive; if the couch has both features, it does meet the condition.

To shorthand our notation further, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\wedge$ is used for *and*: *A* and *B* is notated $A \wedge B$.
The symbol $\vee$ is used for *or*: *A* or *B* is notated $A \vee B$
The symbol ~ is used for *not*: not *A* is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$.

In the previous example, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic Truth Tables

A	**B**	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	**B**	$A \vee B$

T	T	T
T	F	T
F	T	T
F	F	F

A	**~A**
T	F
F	T

Truth tables really become useful when analyzing more complex Boolean statements.

Example 2

Create a truth table for the statement $A \wedge \sim(B \vee C)$

It helps to work from the inside out when creating truth tables, and create tables for intermediate operations. We start by listing all the possible truth value combinations for *A*, *B*, and *C*. Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates. This pattern ensures that all combinations are considered. Along with those initial values, we'll list the truth values for the innermost expression, $B \vee C$.

A	*B*	*C*	$B \vee C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $B \vee C$, working off the $B \vee C$ column we just created.

A	*B*	*C*	$B \vee C$	$\sim(B \vee C)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F

T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of *A and* ~(*B* ∨ *C*)

A	*B*	*C*	*B* ∨ *C*	~(*B* ∨ *C*)	*A* ∧ ~(*B* ∨ *C*)
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is only true in one case: if A is true, B is false, and C is false.

When we discussed conditions earlier, we discussed the type where we take an action based on the value of the condition. We are now going to talk about a more general version of a conditional, sometimes called an **implication**.

Implications

Implications are logical conditional sentences stating that a statement *p*, called the antecedent, implies a consequence *q*.
Implications are commonly written as $p \rightarrow q$

Implications are similar to the conditional statements we looked at earlier; p → q is typically written as "if p then q," or "p therefore q." The difference between implications and conditionals is that conditionals we discussed earlier suggest an action—if the condition is true, then we take some action as a result. Implications are a logical statement that suggest that the consequence must logically follow if the antecedent is true.

Example 3

The English statement "If it is raining, then there are clouds is the sky" is a logical implication. It is a valid argument because if the antecedent "it is raining" is true, then the consequence "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining. If the antecedent is false, then the implication becomes irrelevant.

Example 4

A friend tells you that "if you upload that picture to Facebook, you'll lose your job." There are four possible outcomes:

1. You upload the picture and keep your job
2. You upload the picture and lose your job
3. You don't upload the picture and keep your job
4. You don't upload the picture and lose your job

There is only one possible case where your friend was lying—the first option where you upload the picture and keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't conclude their statement is invalid, even if you didn't upload the picture and still lost your job.

In traditional logic, an implication is considered valid (true) as long as there are no cases in which the antecedent is true and the consequence is false. It is important to keep in mind that symbolic logic cannot capture all the intricacies of the English language.

Truth Values for Implications

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 5

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

We start by constructing a truth table for the antecedent.

m	p	$\sim p$	$m \wedge \sim p$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Now we can build the truth table for the implication

m	p	$\sim p$	$m \wedge \sim p$	r	$(m \wedge \sim p) \rightarrow r$

T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	T	T
T	T	F	F	F	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	F	F	T

In this case, when m is true, p is false, and r is false, then the antecedent $m \wedge \sim p$ will be true but the consequence false, resulting in a invalid implication; every other case gives a valid implication.

For any implication, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statements

The original implication is "if p then q": $p \rightarrow q$
The converse is "if q then p": $q \rightarrow p$
The inverse is "if not p then not q": $\sim p \rightarrow \sim q$
The contrapositive is "if not q then not p": $\sim q \rightarrow \sim p$

Example 6

Consider again the valid implication "If it is raining, then there are clouds in the sky."

The converse would be "If there are clouds in the sky, it is raining." This is certainly not always true.

The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true.

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is valid, and is equivalent to the original implication.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Implication	**Converse**	**Inverse**	**Contrapositive**
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T

		Implication	Converse	Inverse	Contrapositive
F	F	T	T	T	T

Equivalence

A conditional statement and its contrapositive are logically equivalent.
The converse and inverse of a statement are logically equivalent.

Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.
A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 7

The argument "when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store" is an inductive argument.

The premises are:

> I forgot my purse last week
> I forgot my purse today

The conclusion is:

> I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 8

The argument "every day for the past year, a plane flies over my house at 2pm. A plane will fly over my house every day at 2pm" is a stronger inductive argument, since it is based on a larger set of evidence.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. The commonly known scientific theories, like Newton's theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence. For gravity, this happened when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes them easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if all the premises are true, and the conclusion follows logically from those premises. In other words, the premises are true, and the conclusion follows necessarily from those premises.

Example 9

The argument "All cats are mammals and a tiger is a cat, so a tiger is a mammal" is a valid deductive argument.

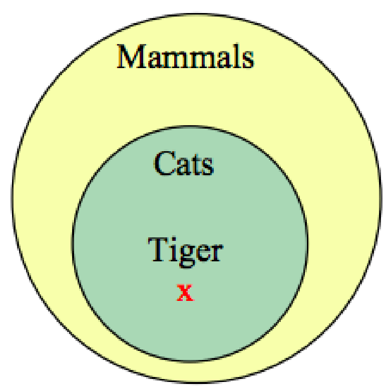
The premises are:

All cats are mammals
A tiger is a cat

The conclusion is:

A tiger is a mammal

Both the premises are true. To see that the premises must logically lead to the conclusion, one approach would be use a Venn diagram. From the first premise, we can conclude that the set of cats is a subset of the set of mammals. From the second premise, we are told that a tiger lies within the set of cats. From that, we can see in the Venn diagram that the tiger also lies inside the set of mammals, so the conclusion is valid.



Analyzing Arguments with Venn Diagrams ((Note: Technically, these are Euler circles or Euler diagrams, not Venn diagrams, but for the sake of simplicity we'll continue to call them Venn diagrams.))

To analyze an argument with a Venn diagram

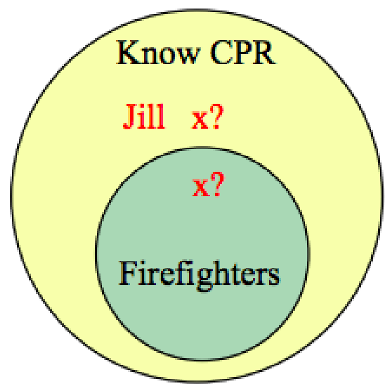
1. Draw a Venn diagram based on the premises of the argument
2. If the premises are insufficient to determine what determine the location of an element, indicate that.
3. The argument is valid if it is clear that the conclusion must be true

Example 10

Premise: All firefighters know CPR
Premise: Jill knows CPR
Conclusion: Jill is a firefighter

From the first premise, we know that firefighters all lie inside the set of those who know CPR. From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know if she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument, regardless of whether Jill actually is a firefighter.



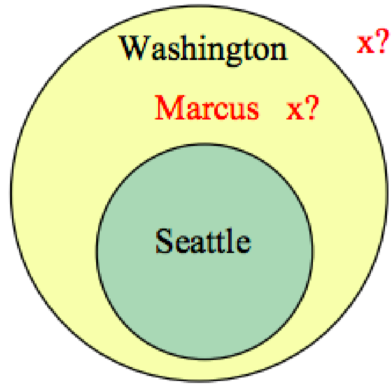
It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are only concerned with whether the premises are enough to prove the conclusion.

In addition to these categorical style premises of the form "all ___," "some ____," and "no ____," it is also common to see premises that are implications.

Example 11

Premise: If you live in Seattle, you live in Washington.
Premise: Marcus does not live in Seattle
Conclusion: Marcus does not live in Washington

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether or not Marcus lives in Washington or not. This is an invalid argument.



Example 12

Consider the argument "You are a married man, so you must have a wife."

This is an invalid argument, since there are, at least in parts of the world, men who are married to other men, so the premise not insufficient to imply the conclusion.

Some arguments are better analyzed using truth tables.

Example 13

Consider the argument:

Premise: If you bought bread, then you went to the store
Premise: You bought bread
Conclusion: You went to the store

While this example is hopefully fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then look at the implication that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll get B represent "you bought bread" and S represent "you went to the store". Then the argument becomes:

Premise: $B \rightarrow S$
Premise: B
Conclusion: S

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(B \rightarrow S) \wedge B] \rightarrow S$?

B	S	$B \rightarrow S$	$(B \rightarrow S) \wedge B$	$[(B \rightarrow S) \wedge B] \rightarrow S$
T	T	T	T	T

T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth table for $[(B\to S) \wedge B] \to S$ is always true, this is a valid argument.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises with and to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for that statement. If it is always true, then the argument is valid.

Example 14

Premise: If I go to the mall, then I'll buy new jeans
Premise: If I buy new jeans, I'll buy a shirt to go with it
Conclusion: If I got to the mall, I'll buy a shirt.

Let M = I go to the mall, J = I buy jeans, and S = I buy a shirt.

The premises and conclusion can be stated as:

Premise: $M \to J$
Premise: $J \to S$
Conclusion: $M \to S$

We can construct a truth table for $[(M\to J) \wedge (J\to S)] \to (M\to S)$

M	J	S	$M \to J$	$J \to S$	$(M\to J) \wedge (J\to S)$	$M \to S$	$[(M\to J) \wedge (J\to S)] \to (M\to S)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the truth table, we can see this is a valid argument.

TRUTH TABLES: CONJUNCTION AND DISJUNCTION

This video explores the example "It is snowing OR I am wearing my hat," and "It is snowing AND I am wearing my hat."

https://youtu.be/8rNPcI_08Ec

TRUTH TABLES: IMPLICATION

This video explores the example "If it is snowing, then I am wearing my hat."

https://youtu.be/2tIbh7I6Gi4

ANALYZING ARGUMENTS WITH TRUTH TABLES

Part I

Watch this video online: https://youtu.be/dW7US8yRaEw

Part II

Watch this video online: https://youtu.be/3PhHWT9MJew

NEGATING "ALL," "SOME," OR "NO" STATEMENTS

This reading provides information about negating “all,” “some,” or “no” statements. Click on the link below to view the page “Negation—NOT—Compound Statements and ALL/SOME” created by Donna Roberts of the Oswego City School District Regents Exam Prep Center.

- Negation—NOT—Compound Statements and ALL/SOME

For more information, look at page 229 of Logic and Sets, staring with “Statements involving the universal quantifiers *all*, *no*, *non* and *every*, or the existential quantifiers *some* and *there exists at least one* have to be negated in a different way.”

DISCUSS: TRUTH TABLE PRACTICE

You will need to become familiar with creating a Truth Table in Blackboard. You may be asked to create a Truth Table for some of your exam questions. See Creating a Truth Table for instructions.

Create a new thread in the **Truth Table Practice** forum in the **Discussion Board** to submit your assignment.

This assignment is required and worth up to 10 points.

DISCUSS: LOGIC APPLICATION

Pick a real problem and try to solve it using what you learned about logic in this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Logic Application Directions for detailed directions.

Create a new thread in the **Logic Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

NUMERATION SYSTEMS

MODULE 5 OVERVIEW

What You'll Learn To Do: Understand numbers in other bases and recognize when use of other bases is applicable to real-life situations.

Learning Objectives

- Describe historical examples of simple grouping, multiplicative grouping, and positional numeration systems in the development of modern mathematics.
- Write and interpret numbers in other bases.
- Convert directly between binary, octal, decimal, and hexadecimal bases.
- Recognize when use of other bases is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Numeration
- **Watch:** Supplemental Videos
- **Read:** Binary, Octal, and Hexadecimal

Homework Assignments

- **Submit:** Numeration Homework #1 (16 points)
- **Submit:** Numeration Homework #2 (16 points)
- **Discuss:** Numeration Application (20 points)

NUMERATION

Historical Counting Systems Introduction and Basic Number and Counting Systems

Introduction

As we begin our journey through the history of mathematics, one question to be asked is "Where do we start?" Depending on how you view mathematics or numbers, you could choose any of a number of launching points from which to begin. Howard Eves suggests the following list of possibilities. ((Note: Eves, Howard; An Introduction to the History of Mathematics, p. 9.))

Where to start the study of the history of mathematics…

- At the first logical geometric "proofs" traditionally credited to Thales of Miletus (600 BCE).
- With the formulation of methods of measurement made by the Egyptians and Mesopotamians/Babylonians.
- Where prehistoric peoples made efforts to organize the concepts of size, shape, and number.
- In pre-human times in the very simple number sense and pattern recognition that can be displayed by certain animals, birds, etc.
- Even before that in the amazing relationships of numbers and shapes found in plants.
- With the spiral nebulae, the natural course of planets, and other universe phenomena.

We can choose no starting point at all and instead agree that mathematics has *always* existed and has simply been waiting in the wings for humans to discover. Each of these positions can be defended to some degree and which one you adopt (if any) largely depends on your philosophical ideas about mathematics and numbers.

Nevertheless, we need a starting point. Without passing judgment on the validity of any of these particular possibilities, we will choose as our starting point the emergence of the idea of number and the process of counting as our launching pad. This is done primarily as a practical matter given the nature of this course. In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

Recognition of More vs. Less

The idea of number and the process of counting goes back far beyond history began to be recorded. There is some archeological evidence that suggests that humans were counting as far back as 50,000 years ago. ((Note: Eves, p. 9.)) However, we do not really know how this process started or developed over time. The best we can do is to make a good guess as to how things progressed. It is probably not hard to believe that even the earliest humans had some sense of *more* and *less*. Even some small animals have been shown to have such a sense. For example, one naturalist tells of how he would secretly remove one egg each day from a plover's nest. The mother was diligent in laying an extra egg every day to make up for the missing egg. Some research has shown that hens can be trained to distinguish between even and odd numbers of pieces of food. ((Note: McLeish, John; The Story of Numbers—How Mathematics Has Shaped Civilization, p. 7.)) With these sorts of findings in mind, it is not hard to conceive that early humans had (at least) a similar sense of more and less. However, our conjectures about how and when these ideas emerged among humans are simply that; educated guesses based on our own assumptions of what might or could have been.

The Need for Simple Counting

As societies and humankind evolved, simply having a sense of more or less, even or odd, etc., would prove to be insufficient to meet the needs of everyday living. As tribes and groups formed, it became important to be able to know how many members were in the group, and perhaps how many were in the enemy's camp. Certainly it was important for them to know if the flock of sheep or other possessed animals were increasing or decreasing in size. "Just how many of them do we have, anyway?" is a question that we do not have a hard time imagining them asking themselves (or each other).

In order to count items such as animals, it is often conjectured that one of the earliest methods of doing so would be with "tally sticks." These are objects used to track the numbers of items to be counted. With this method, each "stick" (or pebble, or whatever counting device being used) represents one animal or object. This method uses the idea of **one to one correspondence**. In a one to one correspondence, items that are being counted are uniquely linked with some counting tool.

In the picture to the right, you see each stick corresponding to one horse. By examining the collection of sticks in hand one knows how many animals should be present. You can imagine the usefulness of such a system, at least for smaller numbers of items to keep track of. If a herder wanted to "count off" his animals to make sure they were all present, he could mentally (or methodically) assign each stick to one animal and continue to do so until he was satisfied that all were accounted for.

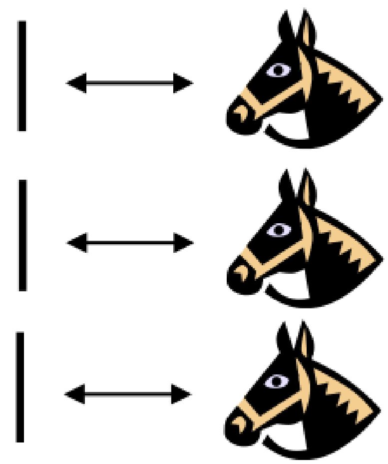


Figure 1.

Of course, in our modern system, we have replaced the sticks with more abstract objects. In particular, the top stick is replaced with our symbol "1," the second stick gets replaced by a "2" and the third stick is represented by the symbol "3," but we are getting ahead of ourselves here. These modern symbols took many centuries to emerge.

Another possible way of employing the "tally stick" counting method is by making marks or cutting notches into pieces of wood, or even tying knots in string (as we shall see later). In 1937, Karl Absolom discovered a wolf bone that goes back possibly 30,000 years. It is believed to be a counting device. ((Note: Bunt, Lucas; Jones, Phillip; Bedient, Jack; The Historical Roots of Elementary Mathematics, p. 2.)) Another example of this kind of tool is the Ishango Bone, discovered in 1960 at Ishango, and shown below. ((Note: http://www.math.buffalo.edu/mad/Ancient-Africa/mad_zaire-uganda.html)) It is reported to be between six and nine thousand years old and shows what appear to be markings used to do counting of some sort.

The markings on rows (a) and (b) each add up to 60. Row (b) contains the prime numbers between 10 and 20. Row (c) seems to illustrate for the method of doubling and multiplication used by the Egyptians. It is believed that this may also represent a lunar phase counter.

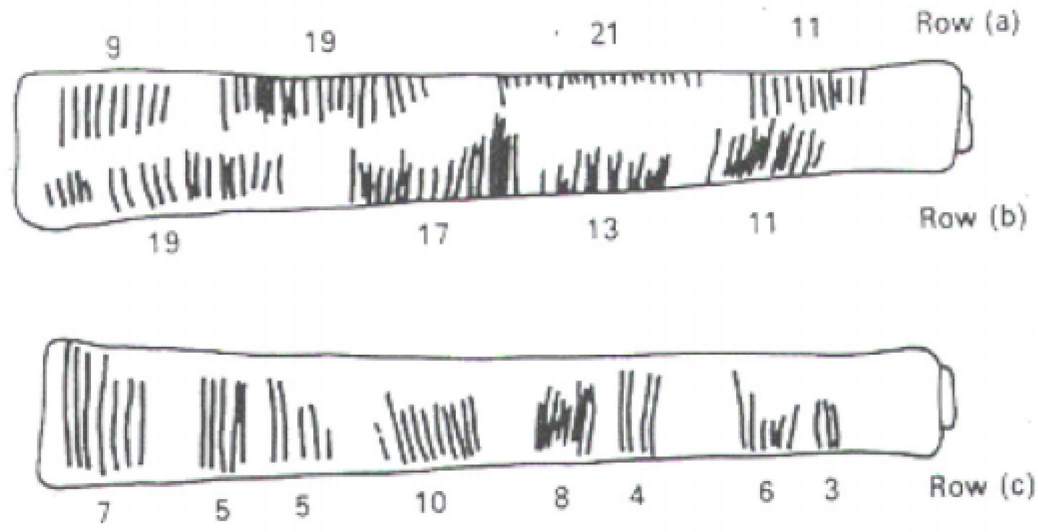


Figure 2.

Spoken Words

As methods for counting developed, and as language progressed as well, it is natural to expect that spoken words for numbers would appear. Unfortunately, the developments of these words, especially those corresponding to the numbers from one through ten, are not easy to trace. Past ten, however, we do see some patterns:

- Eleven comes from “ein lifon,” meaning “one left over.”
- Twelve comes from “twe lif,” meaning “two left over.”
- Thirteen comes from “Three and ten” as do fourteen through nineteen.
- Twenty appears to come from “twe-tig” which means “two tens.”
- Hundred probably comes from a term meaning “ten times.”

Written Numbers

When we speak of “written” numbers, we have to be careful because this could mean a variety of things. It is important to keep in mind that modern paper is only a little more than 100 years old, so “writing” in times past often took on forms that might look quite unfamiliar to us today.

As we saw earlier, some might consider wooden sticks with notches carved in them as writing as these are means of recording information on a medium that can be “read” by others. Of course, the symbols used (simple notches) certainly did not leave a lot of flexibility for communicating a wide variety of ideas or information.

Other mediums on which “writing” may have taken place include carvings in stone or clay tablets, rag paper made by hand (twelfth century in Europe, but earlier in China), papyrus (invented by the Egyptians and used up until the Greeks), and parchments from animal skins. And these are just a few of the many possibilities.

These are just a few examples of early methods of counting and simple symbols for representing numbers. Extensive books, articles and research have been done on this topic and could provide enough information to fill this entire course if we allowed it to. The range and diversity of creative thought that has been used in the past to describe numbers and to count objects and people is staggering. Unfortunately, we don’t have time to examine them all, but it is fun and interesting to look at one system in more detail to see just how ingenious people have been.

The Number and Counting System of the Inca Civilization

Background

There is generally a lack of books and research material concerning the historical foundations of the Americas. Most of the “important” information available concentrates on the eastern hemisphere, with Europe as the central focus. The reasons for this may be twofold: first, it is thought that there was a lack of specialized mathematics in the American regions; second, many of the secrets of ancient mathematics in the Americas have been closely guarded. ((Note: Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher,* Issue 60 (Oct., 1967), p. 623–28.)) The Peruvian system does not seem to be an exception here. Two researchers, Leland Locke and Erland Nordenskiold, have carried out research that has attempted to discover what mathematical knowledge was known by the Incas and how they used the Peruvian quipu, a counting system using cords and knots, in their mathematics. These researchers have come to certain beliefs about the quipu that we will summarize here.

Counting Boards

It should be noted that the Incas did not have a complicated system of computation. Where other peoples in the regions, such as the Mayans, were doing computations related to their rituals and calendars, the Incas seem to have been more concerned with the simpler task of record-keeping. To do this, they used what are called the “quipu” to record quantities of items. (We will describe them in more detail in a moment.) However, they first often needed to do computations whose results would be recorded on quipu. To do these computations, they would sometimes use a counting board constructed with a slab of stone. In the slab were cut rectangular and square compartments so that an octagonal (eight-sided) region was left in the middle. Two opposite corner rectangles were raised. Another two sections were mounted on the original surface of the slab so that there were actually three levels available. In the figure shown, the darkest shaded corner regions represent the highest, third level. The lighter shaded regions surrounding the corners are the second highest levels, while the clear white rectangles are the compartments cut into the stone slab.

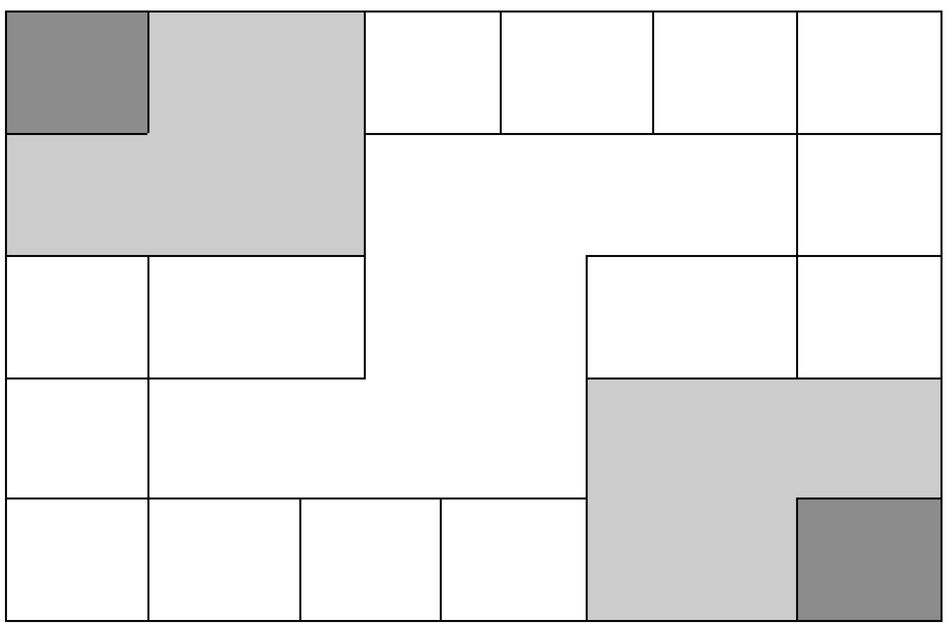


Figure 3.

Pebbles were used to keep accounts and their positions within the various levels and compartments gave totals. For example, a pebble in a smaller (white) compartment represented one unit. Note that there are 12 such squares around the outer edge of the figure. If a pebble was put into one of the two (white) larger, rectangular compartments, its value was doubled. When a pebble was put in the octagonal region in the middle of the slab, its value was tripled. If a pebble was placed on the second (shaded) level, its value was multiplied by six. And finally, if a pebble was found on one of the two highest corner levels, its value was multiplied by twelve. Different objects could be counted at the same time by representing different objects by different colored pebbles.

Example 1

Suppose you have the following counting board with two different kind of pebbles places as illustrated. Let the solid black pebble represent a dog and the striped pebble represent a cat. How many dogs are being represented?

Solution

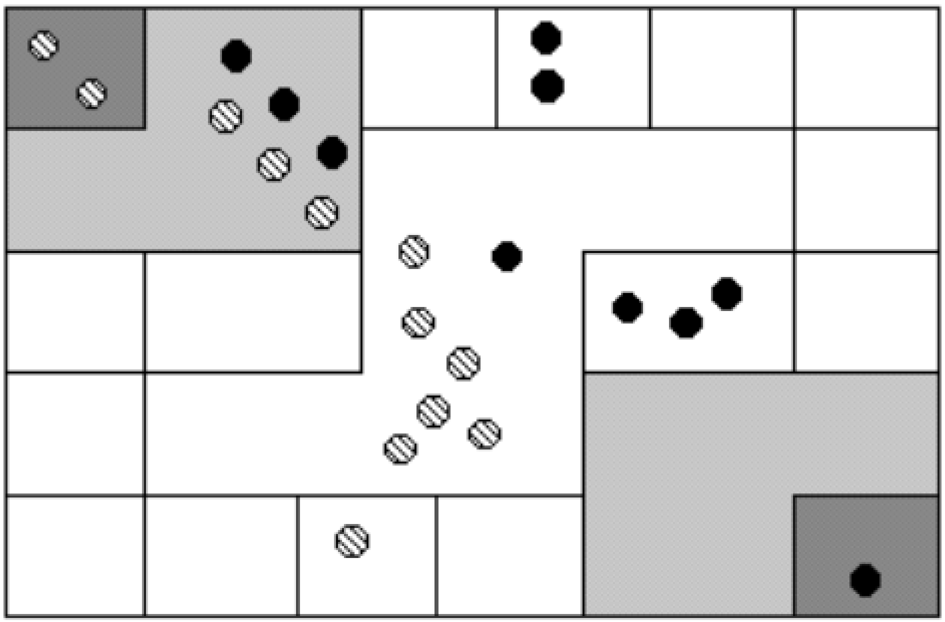


Figure 4.

There are two black pebbles in the outer square regions…these represent 2 dogs. There are three black pebbles in the larger (white) rectangular compartments. These represent 6 dogs. There is one black pebble in the middle region…this represents 3 dogs. There are three black pebbles on the second level…these represent 18 dogs. Finally, there is one black pebble on the highest corner level…this represents 12 dogs. We then have a total of 2+6+3+18+12 = 41 dogs.

Try It Now

How many cats are represented on this board?

The Quipu

This kind of board was good for doing quick computations, but it did not provide a good way to keep a permanent recording of quantities or computations. For this purpose, they used the quipu. The quipu is a collection of cords with knots in them. These cords and knots are carefully arranged so that the position and type of cord or knot gives specific information on how to decipher the cord.

A quipu is made up of a main cord which has other cords (branches) tied to it. See pictures to the right. ((Note: Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher,* Issue 60 (Oct., 1967), p. 623–28.))

Locke called the branches H cords. They are attached to the main cord. B cords, in turn, were attached to the H cords. Most of these cords would have knots on them. Rarely are knots found on the main cord, however, and tend to be mainly on the H and B cords. A quipu might also have a “totalizer” cord that summarizes all of the information on the cord group in one place.

Locke points out that there are three types of knots, each representing a different value, depending on the kind of knot used and its position on the cord. The Incas, like us, had a decimal (base-ten) system, so each kind of knot had a specific decimal value. The Single knot, pictured in the middle of figure 6 ((Note:

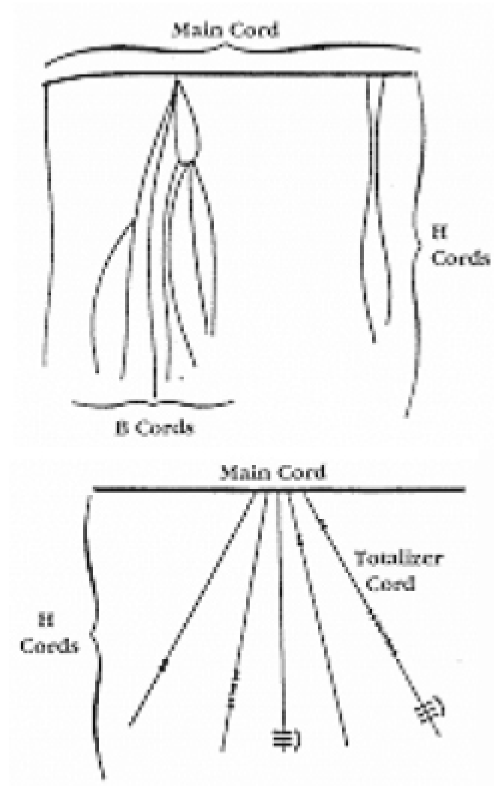


Figure 5.

http://wiscinfo.doit.wisc.edu/chaysimire/titulo2/khipus/what.htm)) was used to denote tens, hundreds, thousands, and ten thousands. They would be on the upper levels of the H cords. The figure-eight knot on the end was used to denote the integer “one.” Every other integer from 2 to 9 was represented with a long knot, shown on the left of the figure. (Sometimes long knots were used to represents tens and hundreds.) Note that the long knot has several turns in it…the number of turns indicates which integer is being represented. The units (ones) were placed closest to the bottom of the cord, then tens right above them, then the hundreds, and so on.

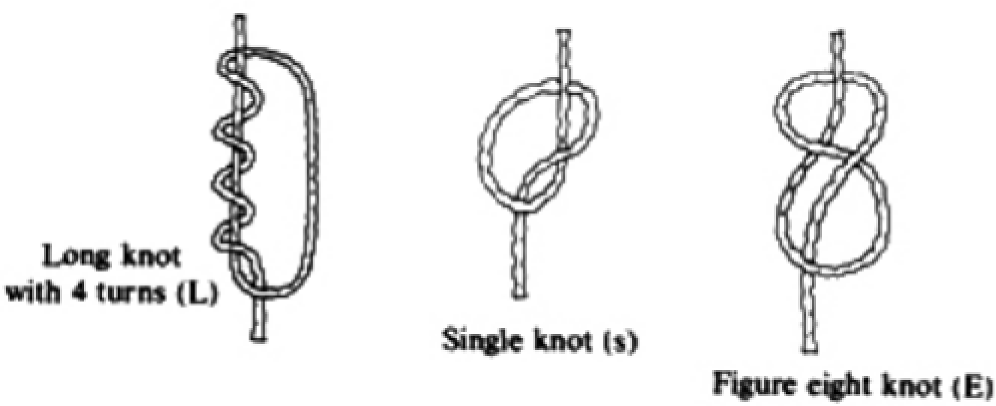


Figure 6

In order to make reading these pictures easier, we will adopt a convention that is consistent. For the long knot with turns in it (representing the numbers 2 through 9), we will use the following notation:

The four horizontal bars represent four turns and the curved arc on the right links the four turns together. This would represent the number 4.

We will represent the single knot with a large dot (·) and we will represent the figure eight knot with a sideways eight (∞).

Example 2

What number is represented on the cord shown in figure 7?

Solution

On the cord, we see a long knot with four turns in it…this represents four in the ones place. Then 5 single knots appear in the tens position immediately above that, which represents 5 tens, or 50. Finally, 4 single knots are tied in the hundreds, representing four 4 hundreds, or 400. Thus, the total shown on this cord is 454.

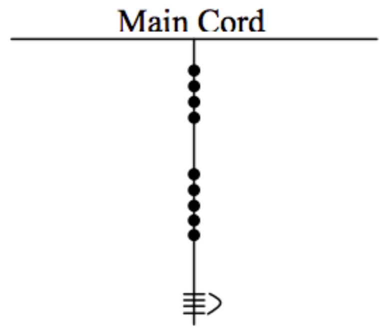


Figure 7.

Try It Now

What numbers are represented on each of the four cords hanging from the main cord?

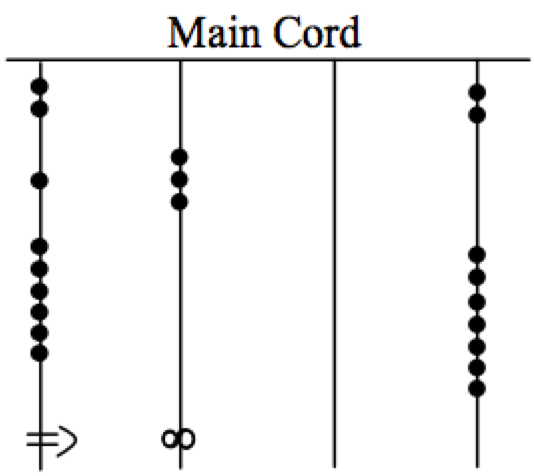


Figure 8.

The colors of the cords had meaning and could distinguish one object from another. One color could represent llamas, while a different color might represent sheep, for example. When all the colors available were exhausted, they would have to be re-used. Because of this, the ability to read the quipu became a complicated task and specially trained individuals did this job. They were called Quipucamayoc, which means keeper of the quipus. They would build, guard, and decipher quipus.

As you can see from this photograph of an actual quipu (figure 9), they could get quite complex.

There were various purposes for the quipu. Some believe that they were used to keep an account of their traditions and history, using knots to record history rather than some other formal system of writing. One writer has even suggested that the quipu replaced writing as it formed a role in the Incan postal system. ((Note: Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher,* Issue 60 (Oct., 1967), p. 623–28.)) Another proposed use of the quipu is as a translation tool. After the conquest of the Incas by the Spaniards and subsequent "conversion" to Catholicism, an Inca supposedly could use the quipu to confess their sins to a priest. Yet another proposed use of the quipu was to record numbers related to magic and astronomy, although this is not a widely accepted interpretation.

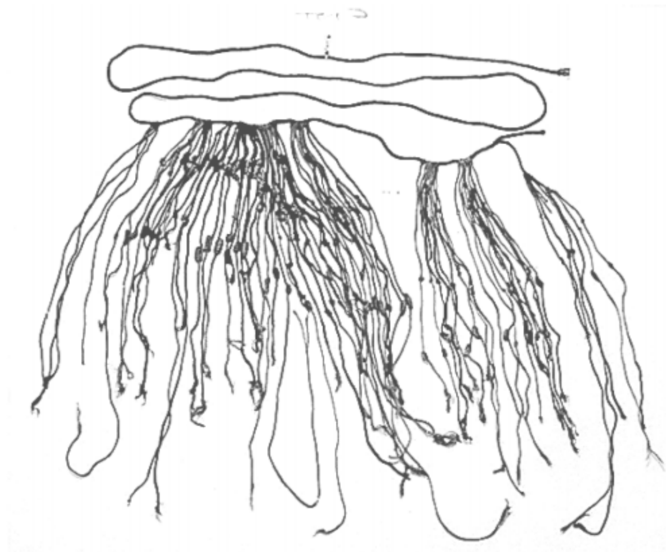


Figure 9.

The mysteries of the quipu have not been fully explored yet. Recently, Ascher and Ascher have published a book, *The Code of the Quipu: A Study in Media, Mathematics, and Culture, which is "*an extensive elaboration of the logical-numerical system of the quipu." ((Note: http://www.cs.uidaho.edu/~casey931/seminar/quipu.html)) For more information on the quipu, you may want to check out "Khipus: a unique Huarochiri legacy."

We are so used to seeing the symbols 1, 2, 3, 4, etc. that it may be somewhat surprising to see such a creative and innovative way to compute and record numbers. Unfortunately, as we proceed through our mathematical education in grade and high school, we receive very little information about the wide range of number systems that have existed and which still exist all over the world. That's not to say our own system is not important or efficient. The fact that it has survived for hundreds of years and shows no sign of going away any time soon suggests that we may have finally found a system that works well and may not need further improvement, but only time will tell that whether or not that conjecture is valid or not. We now turn to a brief historical look at how our current system developed over history.

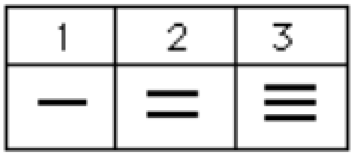
The Hindu—Arabic Number System

The Evolution of a System

Our own number system, composed of the ten symbols {0,1,2,3,4,5,6,7,8,9} is called the *Hindu-Arabic system*. This is a base-ten (decimal) system since place values increase by powers of ten. Furthermore, this system is positional, which means that the position of a symbol has bearing on the value of that symbol within the number. For example, the position of the symbol 3 in the number 435,681 gives it a value much greater than the value of the symbol 8 in that same number. We'll explore base systems more thoroughly later. The development of these ten symbols and their use in a positional system comes to us primarily from India. ((Note: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html))

It was not until the fifteenth century that the symbols that we are familiar with today first took form in Europe. However, the history of these numbers and their development goes back hundreds of years. One important source of information on this topic is the writer al-Biruni, whose picture is shown in figure 10. ((Note: http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Al-Biruni.html)) Al-Biruni, who was born in modern day Uzbekistan, had visited India on several occasions and made comments on the Indian number system. When we look at the origins of the numbers that al-Biruni encountered, we have to go back to the third century BCE. to explore their origins. It is then that the Brahmi numerals were being used.

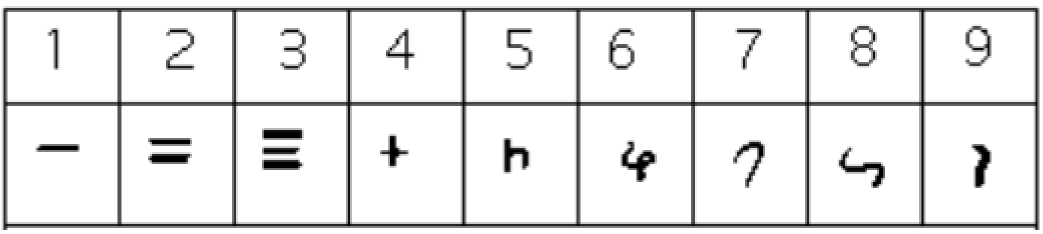
The Brahmi numerals were more complicated than those used in our own modern system. They had separate symbols for the numbers 1 through 9, as well as distinct symbols for 10, 100, 1000,…, also for 20, 30, 40,…, and others for 200, 300, 400, …, 900. The Brahmi symbols for 1, 2, and 3 are shown below. ((Note: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html))



Brahmi one, two, three

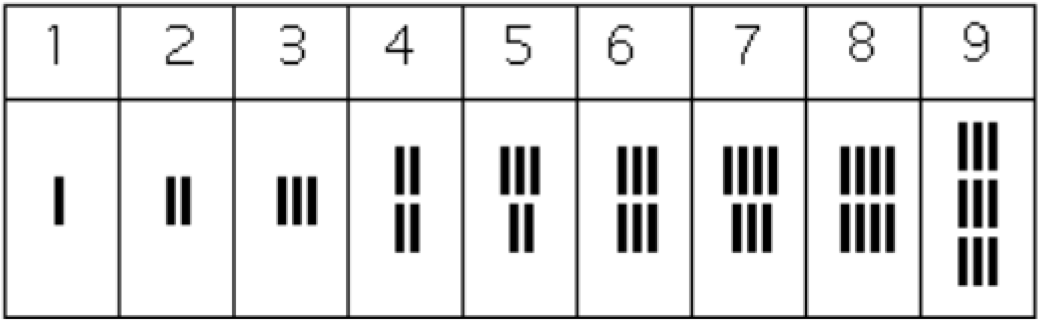
These numerals were used all the way up to the fourth century CE, with variations through time and geographic location. For example, in the first century CE, one particular set of Brahmi numerals took on the following form: ((Note: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html))

Figure 10. Al-Biruni



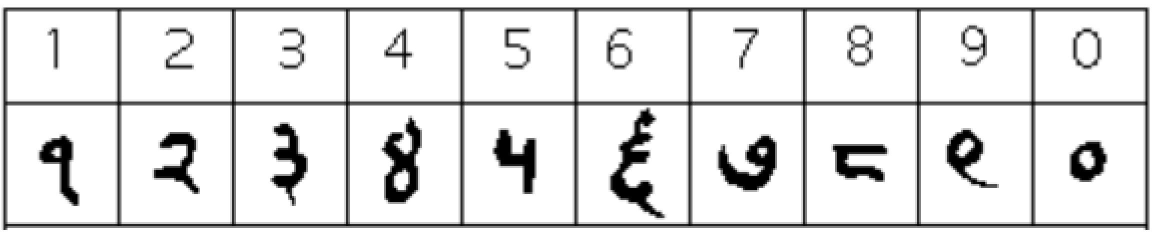
From the fourth century on, you can actually trace several different paths that the Brahmi numerals took to get to different points and incarnations. One of those paths led to our current numeral system, and went through what are called the Gupta numerals. The Gupta numerals were prominent during a time ruled by the Gupta dynasty and were spread throughout that empire as they conquered lands during the fourth through sixth centuries. They have the following form: ((Note: Ibid.))

How the numbers got to their Gupta form is open to considerable debate. Many possible hypotheses have been offered, most of which boil down to two basic types. ((Note: Ibid.)) The first type of hypothesis states that the numerals came from the initial letters of the names of the numbers. This is not uncommon . . . the Greek numerals developed in this manner. The second type of hypothesis states that they were derived from some earlier number system. However, there are other hypotheses that are offered, one of which is by the researcher Ifrah. His theory is that there were originally nine numerals, each represented by a corresponding number of vertical lines. One possibility is this: ((Note: Ibid.))



Because these symbols would have taken a lot of time to write, they eventually evolved into cursive symbols that could be written more quickly. If we compare these to the Gupta numerals above, we can try to see how that evolutionary process might have taken place, but our imagination would be just about all we would have to depend upon since we do not know exactly how the process unfolded.

The Gupta numerals eventually evolved into another form of numerals called the Nagari numerals, and these continued to evolve until the eleventh century, at which time they looked like this: ((Note: Ibid.))



Note that by this time, the symbol for 0 has appeared! The Mayans in the Americas had a symbol for zero long before this, however, as we shall see later in the chapter.

These numerals were adopted by the Arabs, most likely in the eighth century during Islamic incursions into the northern part of India. ((Note: Katz, page 230)) It is believed that the Arabs were instrumental in spreading them to other parts of the world, including Spain (see below).

Other examples of variations up to the eleventh century include: ((Note: Burton, David M., *History of Mathematics, An Introduction*, p. 254–255))

Figure 11. Devangari, eighth century

Figure 12. West Arab Gobar, tenth century

Figure 13. Spain, 976 BCE

Finally, figure 14 ((Note: Katz, page 231.)) shows various forms of these numerals as they developed and eventually converged to the fifteenth century in Europe.

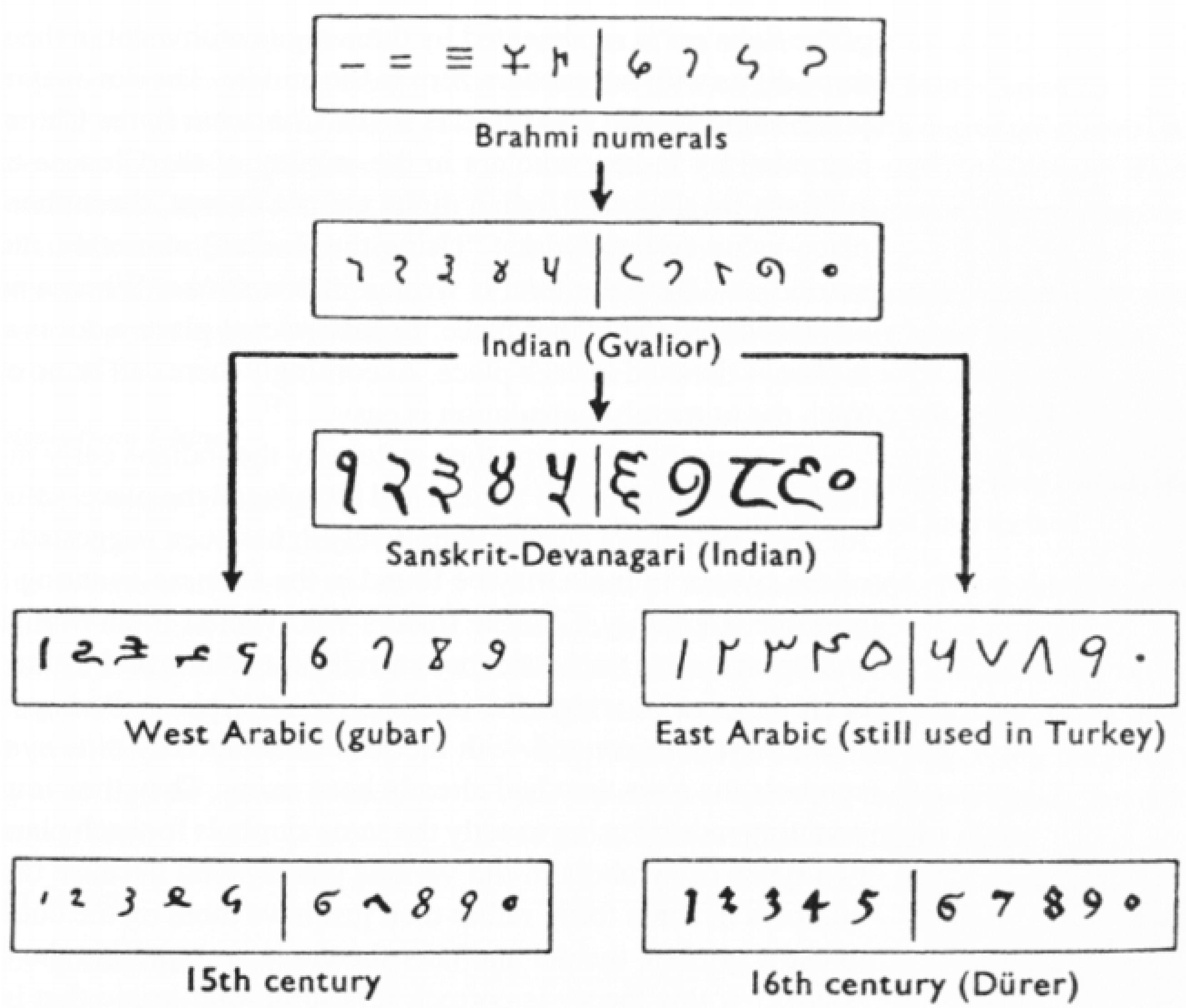


Figure 14.

The Positional System

More important than the form of the number symbols is the development of the place value system. Although it is in slight dispute, the earliest known document in which the Indian system displays a positional system dates back to 346 CE. However, some evidence suggests that they may have actually developed a positional system as far back as the first century CE.

The Indians were not the first to use a positional system. The Babylonians (as we will see in Chapter 3) used a positional system with 60 as their base. However, there is not much evidence that the Babylonian system had much impact on later numeral systems, except with the Greeks. Also, the Chinese had a base-10 system, probably derived from the use of a counting board. ((Note: Ibid, page 230)) Some believe that the positional system used in India was derived from the Chinese system.

Wherever it may have originated, it appears that around 600 CE, the Indians abandoned the use of symbols for numbers higher than nine and began to use our familiar system where the position of the symbol determines its overall value. ((Note: Ibid, page 231.)) Numerous documents from the seventh century demonstrate the use of this positional system.

Interestingly, the earliest dated inscriptions using the system with a symbol for zero come from Cambodia. In 683, the 605th year of the Saka era is written with three digits and a dot in the middle. The 608th year uses three digits with a modern 0 in the middle. ((Note: Ibid, page 232.)) The dot as a symbol for zero also appears in a Chinese work (*Chiu-chih li*). The author of this document gives a strikingly clear description of how the Indian system works:

> Using the [Indian] numerals, multiplication and division are carried out. Each numeral is written in one stroke. When a number is counted to ten, it is advanced into the higher place. In each vacant place a dot is always put. Thus the numeral is always denoted in each place. Accordingly there can be no error in determining the place. With the numerals, calculations is easy. ((Note: Ibid, page 232.))

Transmission to Europe

It is not completely known how the system got transmitted to Europe. Traders and travelers of the Mediterranean coast may have carried it there. It is found in a tenth-century Spanish manuscript and may have been introduced to Spain by the Arabs, who invaded the region in 711 CE and were there until 1492.

In many societies, a division formed between those who used numbers and calculation for practical, every day business and those who used them for ritualistic purposes or for state business. ((Note: McLeish, p. 18)) The former might often use older systems while the latter were inclined to use the newer, more elite written numbers. Competition between the two groups arose and continued for quite some time.

In a fourteenth century manuscript of Boethius' *The Consolations of Philosophy*, there appears a well-known drawing of two mathematicians. One is a merchant and is using an abacus (the "abacist"). The other is a Pythagorean philosopher (the "algorist") using his "sacred" numbers. They are in a competition that is being judged by the goddess of number. By 1500 CE, however, the newer symbols and system had won out and has persevered until today. The Seattle Times recently reported that the Hindu-Arabic numeral system has been included in the book *The Greatest Inventions of the Past 2000 Years*. ((Note: http://seattletimes.nwsource.com/news/health-science/html98/invs_20000201.html, Seattle Times, Feb. 1, 2000))

Figure 15.

One question to answer is *why* the Indians would develop such a positional notation. Unfortunately, an answer to that question is not currently known. Some suggest that the system has its origins with the Chinese counting boards. These boards were portable and it is thought that Chinese travelers who passed through India took their boards with them and ignited an idea in Indian mathematics. ((Note: Ibid, page 232.)) Others, such as G. G. Joseph propose that it is the Indian fascination with very large numbers that drove them to develop a system whereby these kinds of big numbers could easily be written down. In this theory, the system developed entirely within the Indian mathematical framework without considerable influence from other civilizations.

The Development and Use of Different Number Bases

Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is "positional." We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other possible base systems. In the next part of this section, we will journey back to Mayan civilization and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the "decimal" system, requires a total of ten different symbols/digits to write any number. They are, of course, 0, 1, 2, . . . , 9.

The decimal system is also an example of a *positional* base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven "hundred thousands," since the 7 resides in that place. Each digit has a value that is explicitly determined by its position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

5,000,000	= 5 × 1,000,000	$= 5 \times 10^6$	Five million
+700,000	= 7 × 100,000	$= 7 \times 10^5$	Seven hundred thousand
+80,000	= 8 × 10,000	$= 8 \times 10^4$	Eighty thousand
+3,000	= 3 × 1000	$= 3 \times 10^3$	Three thousand
+200	= 2 × 100	$= 2 \times 10^2$	Two hundred
+10	= 1 × 10	$= 1 \times 10^1$	Ten
+6	= 6 × 1	$= 6 \times 10^0$	Six
5,783,216	Five million, seven hundred eighty-three thousand, two hundred sixteen		

From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to "carry" over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

Other Bases

For example, let's suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We're in a different numerical world. As the base-ten system progresses from 10^0 to 10^1, so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2.

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5, where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number. Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.

	Base 5	This column coverts to base-ten	In Base-Ten
	3×5^4	$= 3 \times 625$	= 1875
+	0×5^3	$= 0 \times 125$	= 0
+	4×5^2	$= 4 \times 25$	= 100
+	1×5^1	$= 1 \times 5$	= 5
+	2×5^0	$= 2 \times 1$	= 2
		Total	1982

As you can see, the number 30412_5 is equivalent to 1,982 in base-ten. We will say $30412_5 = 1982_{10}$. All of this may seem strange to you, but that's only because you are so used to the only system that you've ever seen.

Example 3

Convert 6234_7 to a base 10 number.

Solution

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0), and then move up to the 7s, 49s (7^2), etc. Here's the breakdown:

	Base 7	Convert	Base 10
	$= 6 \times 7^3$	$= 6 \times 343$	= 2058
+	$= 2 \times 7^2$	$= 2 \times 49$	= 98
+	$= 3 \times 7$	$= 3 \times 7$	= 21
+	$= 4 \times 1$	$= 4 \times 1$	= 4
		Total	2181

Thus $6234_7 = 2181_{10}$

Try It Now

Convert 41065_7 to a base 10 number.

Converting from Base 10 to Other Bases

Converting from an unfamiliar base to the familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and then multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and you want to convert to base-five. Let's start with some simple examples before we get to a more complicated one.

Example 4

Convert twelve to a base-five number.

Solution

We can probably easily see that we can rewrite this number as follows:

$12 = (2 \times 5) + (2 \times 1)$

Hence, we have two fives and 2 ones. Hence, in base-five we would write twelve as 22_5. Thus, $12_{10} = 22_5$.

Example 5

Convert sixty-nine to a base-five number.

Solution

We can see now that we have more than 25, so we rewrite sixty-nine as follows:

$69 = (2 \times 25) + (3 \times 5) + (4 \times 1)$

Here, we have two twenty-fives, 3 fives, and 4 ones. Hence, in base five we have 234. Thus, $69_{10} = 234_5$.

Example 6

Convert the base-seven number 3261_7 to base 10.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$

$7^3 = 343$
Etc…

$3261_7 = (3 \times 343) + (2 \times 49) + (6 \times 7) + (1 \times 1) = 1170_{10}$.

Thus $3261_7 = 1170_{10}$.

Try It Now

Convert 143 to base 5

Try It Now

Convert the base-three number 21021_3 to base 10.

In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once.

In the last example, $5^2 = 25$ is the largest power of five that is present in 69, so that was our starting point. If we had moved to $5^3 = 125$, then 125 would not divide into 69 at least once.

Converting from Base 10 to Base *b*

1. Find the highest power of the base *b* that will divide into the given number at least once and then divide.
2. Write down the whole number part, then use the remainder from division in the next step.
3. Repeat step two, dividing by the next highest power of the base *b*, writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the “ones” place.
5. Collect all your whole number parts to get your number in base *b* notation.

Example 7

Convert the base-ten number 348 to base-five.

Solution

The powers of five are:

$5^0 = 1$
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$
Etc…

Since 348 is smaller than 625, but bigger than 125, we see that $5^3 = 125$ is the highest power of five present in 348. So we divide 125 into 348 to see how many of them there are:

$348 \div 125 = 2$ with remainder 98

We write down the whole part, 2, and continue with the remainder. There are 98 left over, so we see how many 25s (the next smallest power of five) there are in the remainder:

$98 \div 25 = 3$ with remainder 23

We write down the whole part, 2, and continue with the remainder. There are 23 left over, so we look at the next place, the 5s:

$23 \div 5 = 4$ with remainder 3

This leaves us with 3, which is less than our base, so this number will be in the "ones" place. We are ready to assemble our base-five number:

$348 = (2 \times 5^3) + (3 \times 5^2) + (4 \times 5^1) + (3 \times 1)$

Hence, our base-five number is 2343. We'll say that $348_{10} = 2343_5$.

Example 8

Convert the base-ten number 4,509 to base-seven.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
$7^4 = 2401$
$7^5 = 16807$
Etc…

The highest power of 7 that will divide into 4,509 is $7^4 = 2401$. With division, we see that it will go in 1 time with a remainder of 2108. So we have 1 in the 7^4 place.

The next power down is $7^3 = 343$, which goes into 2108 six times with a new remainder of 50. So we have 6 in the 7^3 place.

The next power down is $7^2 = 49$, which goes into 50 once with a new remainder of 1. So there is a 1 in the 7^2 place.

The next power down is 7^1 but there was only a remainder of 1, so that means there is a 0 in the 7s place and we still have 1 as a remainder.

That, of course, means that we have 1 in the ones place.

Putting all of this together means that $4{,}509_{10} = 16101_7$.

4,509 ÷ 7^4 = 1 R 21082108 ÷ 7^3 = 6 R 50
50 ÷ 7^2 = 1 R 1
1 ÷ 7^1 = 1
$4{,}509_{10} = 16101_7$.

Try It Now

Convert 657_{10} to a base 4 number.

Try It Now

Convert 8377_{10} to a base 8 number.

Another Method For Converting From Base 10 to Other Bases

As you read the solution to this last example and attempted the “Try it Now” problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like “How could I simplify this process?” In general, that is one of the main things that mathematicians do: they look for ways to take complicated situations and make them easier or more familiar. In this section we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that’s the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let’s start with the decimal number, 4863_{10}. We will convert this number to base 10. Yeah, I know it’s already in base 10, but if you carefully follow what we’re doing, you’ll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is 10^3 = 1000. *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

4863 ÷ 1000 = 4.863

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that’s the base we’re in) we get the following:

0.863 × 10 = 8.63

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$(0.863)(1000) = 863$
$(8.63)(100) = 863$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$0.863 \times 10 \Rightarrow 8.63$
(Parts of Thousands) $\times\ 10 \Rightarrow$ Hundreds

What we have now is 8 hundreds and a remainder of 0.63 hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$0.63 \times 10 \Rightarrow 6.3$
Hundreds $\times\ 10 \Rightarrow$ Tens

So we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$4863 \div 1000 = 4.863$$
$$0.863 \times 10 = 8.63$$
$$0.63 \times 10 = 6.3$$
$$0.3 \times 10 = 3.0$$

Converting from Base 10 to Base *b*: Another methodNote that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base, *b*, we have the following steps we can follow:

Converting from Base 10 to Base *b*: Another method

1. Find the highest power of the base *b* that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base *b*.
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base *b* notation.

We will illustrate this procedure with some examples.

Example 9

Convert the base 10 number, 348_{10}, to base 5.

Solution

This is actually a conversion that we have done in a previous example. The powers of five are:

$5^0 = 1$
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$
Etc…

The highest power of five that will go into 348 at least once is 5^3.

We divide by 125 and then proceed.

348 ÷ 5^3 = ②.784

0.784 × 5 = ③.92

0.92 × 5 = ④0.6

0.6 × 5 = ③.0

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus, $2343_5 = 348_{10}$.

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

Example 10

Convert the base 10 number, 3007_{10}, to base 5.

Solution

The highest power of 5 that divides at least once into 3007 is $5^4 = 625$. Thus, we have:

3007 ÷ 625 = ④.8112
0.8112 × 5 = ④.056
0.056 × 5 = ⓪.28
0.28 × 5 = ①0.4
0.4 × 5 = ②0.0

This gives us that $3007_{10} = 44012_5$. Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no 5^2s in this number.

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

Example 11

Convert the base 10 number, 63201_{10}, to base 7.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
$7^4 = 2401$
$7^5 = 16807$
etc…

The highest power of 7 that will divide at least once into 63201 is 7^5. When we do the initial division on a calculator, we get the following:

$63201 \div 7^5 = 3.760397453$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply subtract 3 from this to get the fractional part all by itself. **Do not round off!** Subtraction and then multiplication by seven gives:

$63201 \div 7^5 = 3.760397453$
$0.760397453 \times 7 = 5.322782174$
$0.322782174 \times 7 = 2.259475219$
$0.259475219 \times 7 = 1.816326531$
$0.816326531 \times 7 = 5.714285714$
$0.714285714 \times 7 = 5.000000000$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result: $63201_{10} = 352155_7$.

If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by 7^5, then you expect to have 6 digits in the final answer, corresponding to the places for 7^5, 7^4, and so on down to 7^0. If you find yourself with more than 6 digits due to rounding errors, you know something went wrong.

Try It Now

Convert the base 10 number, 9352_{10}, to base 5.

Try It Now

Convert the base 10 number, 1500, to base 3.
Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

The Mayan Numeral System

Background

As you might imagine, the development of a base system is an important step in making the counting process more efficient. Our own base-ten system probably arose from the fact that we have 10 fingers (including thumbs) on two hands. This is a natural development. However, other civilizations have had a variety of bases other than ten. For example, the Natives of Queensland used a base-two system, counting as follows: "one, two, two and one, two two's, much." Some Modern South American Tribes have a base-five system counting in this way: "one, two, three, four, hand, hand and one, hand and two," and so on. The Babylonians used a base-sixty (sexigesimal) system. In this chapter, we wrap up with a specific example of a civilization that actually used a base system other than 10.

The Mayan civilization is generally dated from 1500 BCE to 1700 CE. The Yucatan Peninsula (see figure 16 ((Note:

1. Chichen Itza
2. Uxmal
3. Tulum
4. Palenque
5. Bonampak, Yaxchilan
6. Tikal
7. Altun Ha
8. Copán

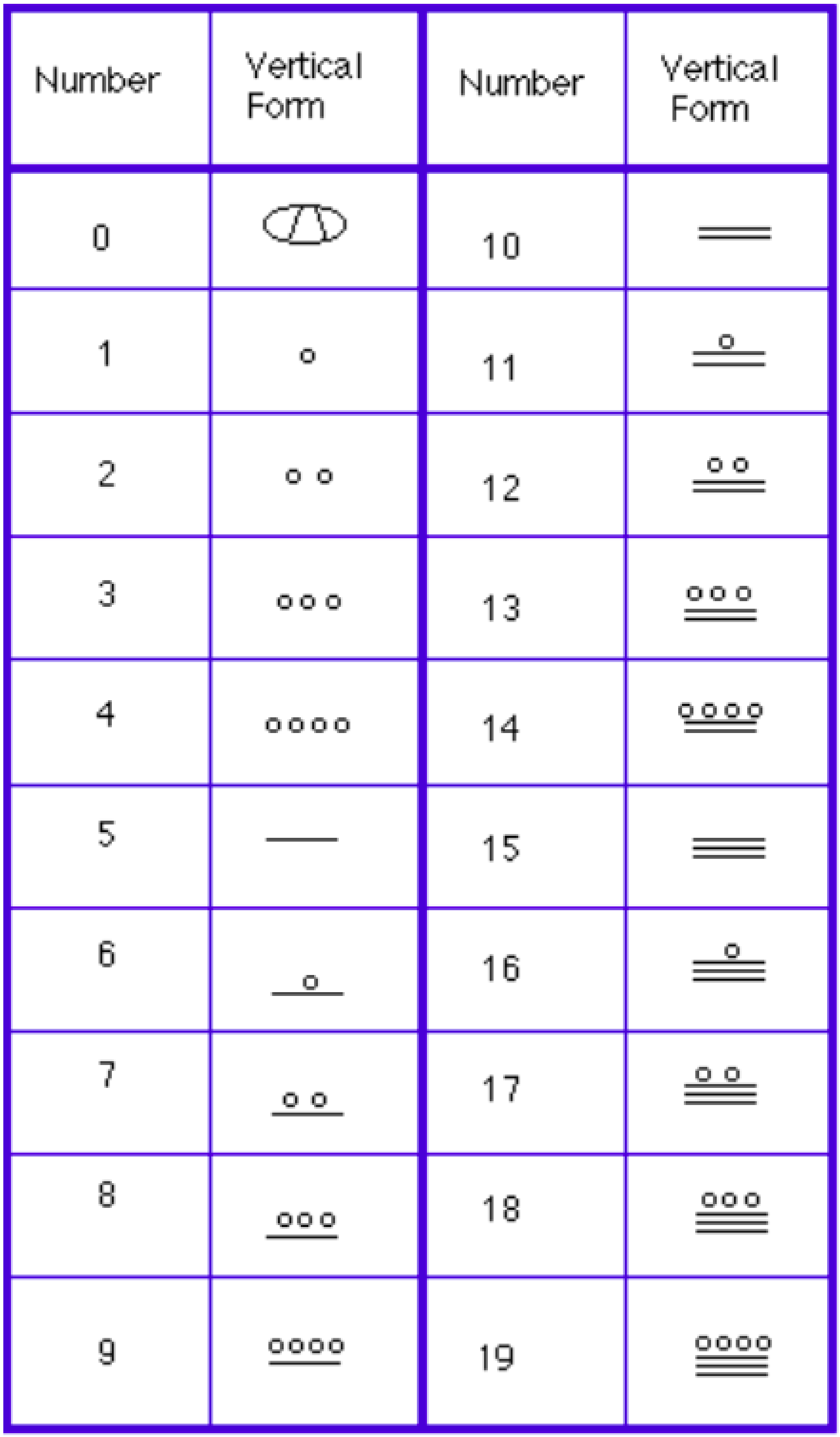
http://www.gorp.com/gorp/location/latamer/map_maya.htm))) in Mexico was the scene for the development of one of the most advanced civilizations of the ancient world. The Mayans had a sophisticated ritual system that was overseen by a priestly class. This class of priests developed a philosophy with time as divine and eternal. ((Note: Bidwell, James; Mayan Arithmetic in *Mathematics Teacher*, Issue 74 (Nov., 1967), p. 762–68.)) The calendar, and calculations related to it, were thus very important to the ritual life of the priestly class, and hence the Mayan people. In fact, much of what we know about this culture comes from their calendar records and astronomy data. Another important source of information on the Mayans is the writings of Father Diego de Landa, who went to Mexico as a missionary in 1549.

There were two numeral systems developed by the Mayans—one for the common people and one for the priests. Not only did these two systems use different symbols, they also used different base systems. For the priests, the number system was governed by ritual. The days of the year were thought to be gods, so the formal symbols for the days were decorated heads, ((Note: http://www.ukans.edu/~lctls/Mayan/numbers.html)) like the sample to the left ((Note: http://www.ukans.edu/~lctls/Mayan/numbers.html)) Since the basic calendar was based on 360 days, the priestly numeral system used a mixed base system employing multiples of 20 and 360. This makes for a confusing system, the details of which we will skip.

Powers	Base-Ten Value	Place Name
20^7	12,800,000,000	Hablat
20^6	64,000,000	Alau
20^5	3,200,000	Kinchil
20^4	160,000	Cabal
20^3	8,000	Pic
20^2	400	*Bak*
20^1	20	Kal
20^0	1	Hun

The Mayan Number System

Instead, we will focus on the numeration system of the "common" people, which used a more consistent base system. As we stated earlier, the Mayans used a base-20 system, called the "vigesimal" system. Like our system, it is positional, meaning that the position of a numeric symbol indicates its place value. In the following table you can see the place value in its vertical format. ((Note: Bidwell))



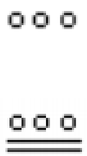
In order to write numbers down, there were only three symbols needed in this system. A horizontal bar represented the quantity 5, a dot represented the quantity 1, and a special symbol (thought to be a shell) represented zero. The Mayan system may have been the first to make use of zero as a placeholder/number. The first 20 numbers are shown in the table to the right. ((Note: http://www.vpds.wsu.edu/fair_95/gym/UM001.html))

Unlike our system, where the ones place starts on the right and then moves to the left, the Mayan systems places the ones on the **bottom** of a vertical orientation and moves up as the place value increases.

When numbers are written in vertical form, there should never be more than four dots in a single place. When writing Mayan numbers, every group of five dots becomes one bar. Also, there should never be more than three bars in a single place…four bars would be converted to one dot in the next place up. It's the same as 10 getting converted to a 1 in the next place up when we carry during addition.

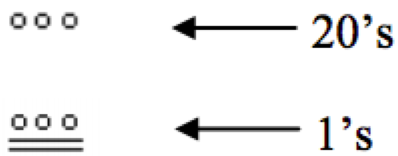
Example 12

What is the value of this number, which is shown in vertical form?



Solution

Starting from the bottom, we have the ones place. There are two bars and three dots in this place. Since each bar is worth 5, we have 13 ones when we count the three dots in the ones place. Looking to the place value above it (the twenties places), we see there are three dots so we have three twenties.

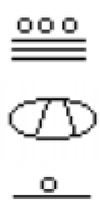


Hence we can write this number in base-ten as:

$(3 \times 20^1) + (13 \times 20^0) = (3 \times 20^1) + (13 \times 1) = 60 + 13 = 73$

Example 13

What is the value of the following Mayan number?



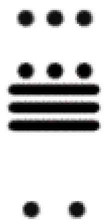
Solution

This number has 11 in the ones place, zero in the 20s place, and 18 in the $20^2 = 400$s place. Hence, the value of this number in base-ten is:

$18 \times 400 + 0 \times 20 + 11 \times 1 = 7211.$

Try It Now

Convert the Mayan number below to base 10.



Example 14

Convert the base 10 number 3575_{10} to Mayan numerals.

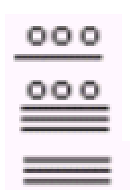
This problem is done in two stages. First we need to convert to a base 20 number. We will do so using the method provided in the last section of the text. The second step is to convert that number to Mayan symbols.

The highest power of 20 that will divide into 3575 is $20^2 = 400$, so we start by dividing that and then proceed from there:

$3575 \div 400 = 8.9375$
$0.9375 \times 20 = 18.75$
$0.75 \times 20 = 15.0$

This means that $3575_{10} = 8{,}18{,}15_{20}$

The second step is to convert this to Mayan notation. This number indicates that we have 15 in the ones position. That's three bars at the bottom of the number. We also have 18 in the 20s place, so that's three bars and three dots in the second position. Finally, we have 8 in the 400s place, so that's one bar and three dots on the top. We get the following:



Note that in the previous example a new notation was used when we wrote $8{,}18{,}15_{20}$. The commas between the three numbers 8, 18, and 15 are now separating place values for us so that we can keep them separate from each other. This use of the comma is slightly different than how they're used in the decimal system. When we write a number in base 10, such as 7,567,323, the commas are used primarily as an aide to read the number easily but they do not separate single place values from each other. We will need this notation whenever the base we use is larger than 10.

Writing numbers with bases bigger than 10

When the base of a number is larger than 10, separate each "digit" with a comma to make the separation of digits clear.
For example, in base 20, to write the number corresponding to $17 \times 20^2 + 6 \times 20^1 + 13 \times 20^0$, we'd write $17{,}6{,}13_{20}$.

Try It Now

Convert the base 10 number 10553_{10} to Mayan numerals.

Try It Now

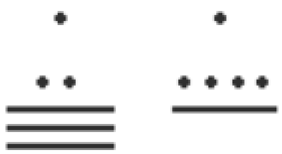
Convert the base 10 number 5617_{10} to Mayan numerals.

Adding Mayan Numbers

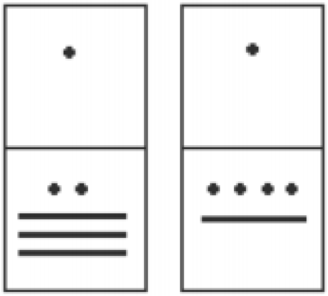
When adding Mayan numbers together, we'll adopt a scheme that the Mayans probably did not use but which will make life a little easier for us.

Example 15

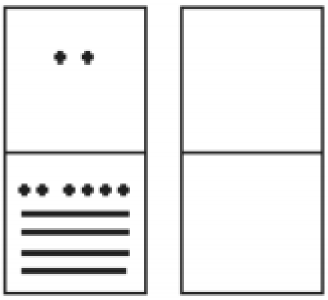
Add, in Mayan, the numbers 37 and 29: ((Note: http://forum.swarthmore.edu/k12/mayan.math/mayan2.html))



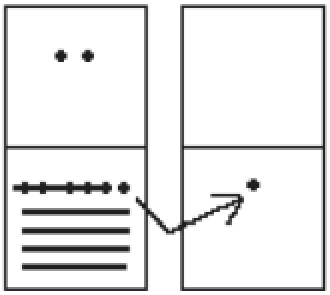
First draw a box around each of the vertical places. This will help keep the place values from being mixed up.



Next, put all of the symbols from both numbers into a single set of places (boxes), and to the right of this new number draw a set of empty boxes where you will place the final sum:

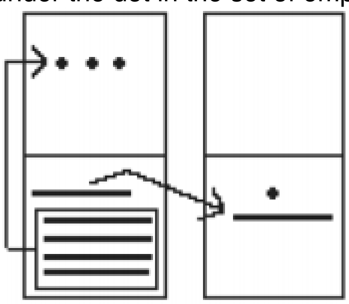


You are now ready to start carrying. Begin with the place that has the lowest value, just as you do with Arabic numbers. Start at the bottom place, where each dot is worth 1. There are six dots, but a maximum of four are allowed in any one place; once you get to five dots, you must convert to a bar. Since five dots make one bar, we draw a bar through five of the dots, leaving us with one dot which is under the four-dot limit. Put this dot into the bottom place of the empty set of boxes you just drew:

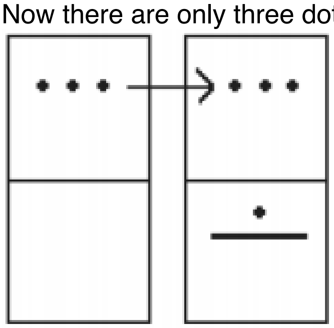


Now look at the bars in the bottom place. There are five, and the maximum number the place can hold is three. *Four bars are equal to one dot in the next highest place*.

Whenever we have four bars in a single place we will automatically convert that to a *dot* in the next place up. We draw a circle around four of the bars and an arrow up to the dots' section of the higher place. At the end of that arrow, draw a new dot. That dot represents 20 just the same as the other dots in that place. Not counting the circled bars in the bottom place, there is one bar left. One bar is under the three-bar limit; put it under the dot in the set of empty places to the right.



Now there are only three dots in the next highest place, so draw them in the corresponding empty box.



We can see here that we have 3 twenties (60), and 6 ones, for a total of 66. We check and note that 37 + 29 = 66, so we have done this addition correctly. Is it easier to just do it in base-ten? Probably, but that's only because it's more familiar to you. Your task here is to try to learn a new base system and how addition can be done in slightly different ways than what you have seen in the past. Note, however, that the concept of carrying is still used, just as it is in our own addition algorithm.

Try It Now

Try adding 174 and 78 in Mayan by first converting to Mayan numbers and then working entirely within that system. Do not add in base-ten (decimal) until the very end when you *check* your work.

Conclusion

In this reading, we have briefly sketched the development of numbers and our counting system, with the emphasis on the "brief" part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.

We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. What is important to note is that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also, we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

SUPPLEMENTAL VIDEOS

This YouTube playlist contains several videos that supplement the reading on Historical Counting.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

BINARY, OCTAL, AND HEXADECIMAL

In modern computing and digital electronics, the most commonly used bases are decimal (base 10), binary (base 2), octal (base 8), and hexadecimal (base 16). If we are converting between two bases other than decimal, we typically have to convert the number to base 10 first, and then convert that number to the second base. However, we can easily convert directly from binary to octal, and vice versa, and from binary to hexadecimal, and vice versa.

This video gives a basic introduction to these conversions:

Watch this video online: https://youtu.be/5sS7w-CMHkU

For another description, this one is more like a math lecture:

Watch this video online: https://youtu.be/2UwxdCLFW70

For further clarification, recall that the numbers 0 through 7 can be represented by up to three digits in base two. In base eight, these numbers are represented by a single digit.

Base 2 (binary) number	**Base 10 (decimal) equivalent**	**Base 8 (octal) number**
000	0	0
001	1	1
010	2	2
011	3	3
100	4	4
101	5	5
110	6	6
111	7	7

Now when we get to the number 8, we need four digits in base 2 and two digits in base 8. In fact, the numbers 8 through 63 can be represented by two digits in base 8. We need four, five, or six digits in base 2 to represent these same numbers:

Base 2 number	**Base 10 equivalent**	**Base 8 number**
1000	8	$10 = 1 \times 8 + 0 \times 1$
1001	9	$11 = 1 \times 8 + 1 \times 1$
1010	10	$12 = 1 \times 8 + 2 \times 1$
…	…	…
111100	60	$74 = 7 \times 8 + 4 \times 1$
111101	61	$75 = 7 \times 8 + 5 \times 1$
111110	62	$76 = 7 \times 8 + 6 \times 1$
111111	63	$77 = 7 \times 8 + 7 \times 1$

The number 64 in base 8 is represented by $100_8 = 1 \times 8^2 + 0 \times 8^1 + 0 \times 8^0 = 1 \times 64 + 0 \times 8 + 0 \times 1$. In base 2, this would be 1000000_2. Do you see a pattern here? For a single digit in base 8, we need up to three digits in base 2. For two digits in base 8, we need 4, 5, or 6 digits in base 2. For three digits in base 8, we need 7, 8, or 9 digits in base 2. For each additional digit in base 8, we need up to three spaces to represent it in base 2. **Here's a way to remember this: $2^3 = 8$, so we need three spaces.**

A couple of examples would help here.

1. Convert the number 6157_8 to base 2. We split each digit in base 8 to three digits in base 2, using the three digit base 2 equivalent, so $6_8 = 110_2$, $1_8 = 001_2$, etc.
2. Convert the number 10111011001010_2 to base 8. Split this number into sets of three, **starting with the right-most digit**, then convert each set of three to its equivalent in base 8.

For hexadecimal (base 16), we need up to four digits in binary to represent each single digit. Remember this by recalling that $2^4 = 16$, so we need four digits.

You may want to print out copies of these worksheets to help you with your conversions between binary and octal or hexadecimal:

- Converting from Binary to Octal
- Converting from Binary to Hexadecimal

If you would like to quiz yourself on converting the numbers 0 through 255 to binary, octal, and hexadecimal (and between those bases), here's a link to the representations of those numbers: Binary, Octal, and Hexadecimal Numbers.

DISCUSS: NUMERATION APPLICATION

Pick a real problem and try to solve it using the numeration system solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Numeration Application Directions for detailed directions.

Create a new thread in the **Numeration Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used?	5

• Are the strategies correctly identified?	
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

CONSUMER MATH

MODULE 6 OVERVIEW

What You'll Learn To Do: Understand consumer math and perform computations in daily life.

Learning Objectives

- Define interest rate
- Describe the difference between simple and compound interest
- Calculate simple and compound interest
- Describe the difference between present and future values
- Calculate present and future values
- Calculate annuities and payout annuities
- Define finance charge
- Calculate finance charges using the average daily balance method
- Describe a fixed-rate mortgage
- Prepare an amortization schedule

Learning Activities

The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Consumer Math
- **Watch:** Supplemental Videos
- **Read:** Average Daily Balance
- **Read:** How an Amortization Schedule is Calculated

Homework Assignments

- **Submit:** Consumer Math Homework #1 (22 points)
- **Submit:** Consumer Math Homework #2 (12 points)
- **Discuss:** Consumer Math Application (20 points)
- **Discuss:** Final Reflection (40 points)
- **Complete:** Exam 3

CONSUMER MATH

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

Simple Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = $5. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-Time Interest

$$I = P_0 r$$

$$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$$

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: 5% = 0.05)

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

P_0 = $300	the principal
r = 0.03	3% rate
I = $300(0.03) = $9	You will earn $9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

Each year, you would earn 5% interest: $1000(0.05) = $50 in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$I = P_0 rt$

$A = P_0 + I = P_0 + P_0 rt = P_0(1 + rt)$

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR—Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)**—the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.
For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

P_0 = $1000	the principal
r = 0.02	2% rate per half-year
t = 8	4 years = 8 half-years
I = $1000(0.02)(8) = $160	You will earn $160 interest total over the four years.

Try It Now

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR)—the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12}$ = 0.25% per month.

In the first month,

P_0 = $1000
r = 0.0025 (0.25%)
I = $1000 (0.0025) = $2.50
A = $1000 + $2.50 = $1002.50

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

P_0 = $1002.50
I = $1002.50 (0.0025) = $2.51 (rounded)
A = $1002.50 + $2.51 = $1005.01

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting balance	Interest earned	Ending Balance

1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$P_0 = \$1000$
$P_m = (1 + 0.0025)P_{m-1}$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$
$P_1 = 1.0025P_0 = 1.0025\ (1000)$
$P_2 = 1.0025P_1 = 1.0025\ (1.0025\ (1000)) = 1.0025^2(1000)$
$P_3 = 1.0025P_2 = 1.0025\ (1.0025^2(1000)) = 1.0025^3(1000)$
$P_4 = 1.0025P_3 = 1.0025\ (1.0025^3(1000)) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year. Generalizing our result, we could write

$$P_m = P_0\left(1 + \frac{r}{k}\right)^m$$

In this formula:

m is the number of compounding periods (months in our example)
r is the annual interest rate
k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = Nk$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$P_N = P_0\left(1+\frac{r}{k}\right)^N k$$

- P_N is the balance in the account after N years.
- P_0 is the starting balance of the account (also called initial deposit, or principal)
- r is the annual interest rate in decimal form
- k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account **once** and let it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution

In this example,

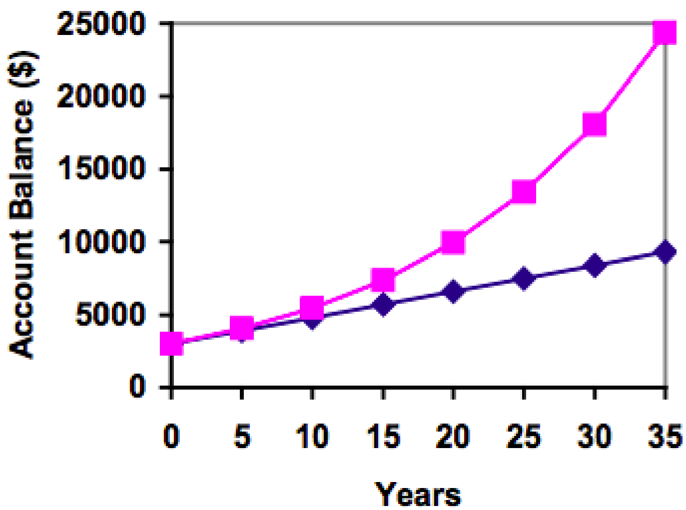
P_0 = \$3000	the initial deposit
$r = 0.06$	6% annual rate
$k = 12$	12 months in 1 year
$N = 20$	since we're looking for how much we'll have after 20 years

So $P_2 0 = 3000\left(1+\frac{0.06}{12}\right)^{20\cdot 12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest (\$15 per month)	6% compounded monthly = 0.5% each month.

5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating Exponents on the Calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators. Most scientific calculators have a button for exponents. It is typically either labeled like:

^, y^x, or x^y.

To evaluate 1.005^{240} we'd type 1.005 ^ 240, or 1.005 y^x 240. Try it out—you should get something around 3.3102044758.

Example 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

In this example,

We're looking for P_0.

$r = 0.04$	4%
$k = 4$	4 quarters in 1 year
$N = 18$	Since we know the balance in 18 years
$P_{18} = \$40,000$	The amount we have in 18 years.

In this case, we're going to have to set up the equation, and solve for P_0.

$$40000 = P_0 \left(1 + \frac{0.04}{4}\right)^{18 \cdot 4}$$

$$40000 = P_0 (2.0471)$$

$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but **keeping more digits is always better**.

Example 6

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

$P_0 = \$1000$	the initial deposit
$r = 0.05$	5%
$k = 12$	12 months in 1 year
$N = 30$	since we're looking for the amount after 30 years

Solution

If we first compute $\frac{r}{k}$, we find $\frac{0.05}{12} = 0.00416666666667$.

Here is the effect of rounding this to different values:

$\frac{r}{k}$ **rounded to**	**Gives *P*$_{30}$ to be**	**Error**
0.004	\$4208.59	\$259.15
0.0042	\$4521.45	\$53.71
0.00417	\$4473.09	\$5.35
0.004167	\$4468.28	\$0.54
0.0041667	\$4467.80	\$0.06
no rounding	\$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough—\$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using Your Calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$P_{30} = 1000\left(1+\frac{0.05}{12}\right)^{12\cdot 30}$

We can quickly calculate 12 × 30 = 360, giving $P_{30} = 1000\left(1+\frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type This	Calculator Shows
0.05 ÷ 12 **=**	0.00416666666667
+ 1 **=**	1.00416666666667
***y*^{x}** 360 **=**	4.46774431400613
× 1000 **=**	4467.74431400613

Using Your Calculator Continued

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 **×** (1 **+** 0.05 **÷** 12) ***y*^{x}** 360 **=** .

Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, *d*, to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit $100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

- r = 0.06 (6%)
- k = 12 (12 compounds/deposits per year)
- d = $100 (our deposit per month)

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$$P_0 = 0$$

$$P_1 = (1.005)P_0 + 100 = 100$$

$$P_2 = (1.005)P_1 + 100 = (1.005)100 + 100 = 100(1.005) + 100$$

$$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$$

Continuing this pattern, after *m* deposits, we'd have saved:

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100$$

In other words, after *m* months, the first deposit will have earned compound interest for *m* – 1 months. The second deposit will have earned interest for *m* – 2 months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though—it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005\left(100(1.005)^{m-1} + 100(1.005)^{m-2} + 100(1.005) + \cdots + 100(1.005) + 100\right)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \cdots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \cdots + 100(1.005)$

$P_m = 100(1.005)^{m-1} + \cdots + 100(1.005) + 100$

Almost all the terms cancel on the right hand side when we subtract, leaving

$1.005P_m - P_m = 100(1.005)^m - 100$

Solving for *Pm*

$0.005P_m = 100\left((1.005)^m - 1\right)$

$$P_m = \frac{100\left((1.005)^m - 1\right)}{0.005}$$

Replacing *m* months with 12*N*, where *N* is measured in years, gives

$$P_N = \frac{100\left((1.005)^{12N} - 1\right)}{0.005}$$

Recall 0.005 was $\frac{r}{k}$ and 100 was the deposit *d.* 12 was *k*, the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$P_N = \frac{d\left(\left(1+\frac{r}{k}\right)^{Nk} - 1\right)}{\left(\frac{r}{k}\right)}$$

- P_N is the balance in the account after *N* years.
- *d* is the regular deposit (the amount you deposit each year, each month, etc.)
- *r* is the annual interest rate in decimal form.
- *k* is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, *k* = 12.

If you make your deposits every year, use yearly compounding, *k* = 1.

If you make your deposits every quarter, use quarterly compounding, *k* = 4.

Etc.

When Do You Use This?

Annuities assume that you put money in the account **on a regular schedule (every month, year, quarter, etc.)** and let it sit there earning interest.
Compound interest assumes that you put money in the account once and let it sit there earning interest.
Compound interest: **One** deposit
Annuity: **Many** deposits.

Example 7

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

d = $100	the monthly deposit
r = 0.06	6% annual rate
k = 12	since we're doing monthly deposits, we'll compound monthly
N = 20	we want the amount after 20 years

Putting this into the equation:

$$P_{20} = \frac{100\left(\left(1+\frac{0.06}{12}\right)^{20(12)}-1\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_{20} = \frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$

$$P_{20} = \frac{100\,(3.310-1)}{(0.005)}$$

$$P_{20} = \frac{100\,(2.310)}{(0.005)} = \$46200$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $46,200 – $24,000 = $22,200.

Example 8

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example, we're looking for *d*.

$r = 0.08$	8% annual rate
$k = 12$	since we're depositing monthly
$N = 30$	30 years
$P_{30} = \$200{,}000$	The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for *d*.

$$200,000 = \frac{d\left(\left(1+\frac{0.08}{12}\right)^{30(12)}-1\right)}{\left(\frac{0.08}{12}\right)}$$

$$200,000 = \frac{d\left((1.00667)^{360}-1\right)}{(0.00667)}$$

$$200,000 = d\,(1491.57)$$

$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit $134.09 each month to have $200,000 in 30 years if your account earns 8% interest

Try It Now

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d\left(1-\left(1+\frac{r}{k}\right)^{-Nk}\right)}{\left(\frac{r}{k}\right)}$$

- P_0 is the balance in the account at the beginning (starting amount, or principal).
- d is the regular withdrawal (the amount you take out each year, each month, etc.)
- r is the annual interest rate (in decimal form. Example: 5% = 0.05)
- k is the number of compounding periods in one year.
- N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When Do You Use This

Payout annuities assume that you **take** money from the account **on a regular schedule** (**every month, year, quarter, etc.**) and let the rest sit there earning interest.

- Compound interest: **One** deposit
- Annuity: **Many** deposits.
- Payout Annuity: **Many withdrawals**

Example 9

After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

d = $1000	the monthly withdrawal
r = 0.06	6% annual rate
k = 12	since we're doing monthly withdrawals, we'll compound monthly
N = 20	since were taking withdrawals for 20 years

We're looking for P_0; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-20(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_0 = \frac{1000\left(1-(1.005)^{-240}\right)}{(0.005)}$$

$$P_0 = \frac{1000\,(1-0.302)}{(0.005)} = \$139,600$$

You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the **interest earned**. In this case it is $240,000 – $139,600 = $100,400 in interest.

Evaluating Negative Exponents on Your Calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this **(-)**, some with **+/-**. The button is often near the **=** key or the decimal point.
If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: 1.005 **^** **(-)** 240
If your calculator only shows one value at a time, then usually you hit the **(-)** key after a number to negate it, so you'd hit: 1.005 $\boldsymbol{y^x}$ 240 **(-)** =
Give it a try—you should get $1.005^{-240} = 0.302096$

Example 10

You know you will have $500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

In this example, we're looking for *d*.

$r = 0.08$	8% annual rate
$k = 12$	since we're withdrawing monthly
$N = 30$	30 years
$P_0 = \$500,000$	we are beginning with $500,000

In this case, we're going to have to set up the equation, and solve for *d*.

$$500,000 = \frac{d\left(1-\left(1+\frac{0.08}{12}\right)^{-30(12)}\right)}{\left(\frac{0.08}{12}\right)}$$

$$500,000 = \frac{d\left(1-(1.00667)^{-360}\right)}{(0.00667)}$$

$$500,000 = d\,(136.232)$$

$$d = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw $3,670.21 each month for 30 years.

Try It Now

A donor gives $100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d\left(1-\left(1+\frac{r}{k}\right)^{-Nk}\right)}{\left(\frac{r}{k}\right)}$$

- P_0 is the balance in the account at the beginning (the principal, or amount of the loan).
- d is your loan payment (your monthly payment, annual payment, etc)
- r is the annual interest rate in decimal form.
- k is the number of compounding periods in one year.
- N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments **on a regular schedule** (**every month, year, quarter, etc.**) and are paying interest on the loan.

- Compound interest: **One** deposit
- Annuity: **Many** deposits.
- Payout Annuity: **Many withdrawals**
- Loans: **Many payments**

Example 11

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

Solution

In this example,

d = $200	the monthly loan payment
r = 0.03	3% annual rate
k = 12	since we're doing monthly payments, we'll compound monthly
N = 5	since we're making monthly payments for 5 years

We're looking for P_0, the starting amount of the loan.

$$P_0 = \frac{200\left(1-\left(1+\frac{0.03}{12}\right)^{-5(12)}\right)}{\left(\frac{0.03}{12}\right)}$$

$$P_0 = \frac{200\left(1-(1.0025)^{-60}\right)}{(0.0025)}$$

$$P_0 = \frac{200\,(1-0.861)}{(0.0025)} = \$11,120$$

You can afford a $11,120 loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the **interest paid**. In this case, you're paying $12,000 – $11,120 = $880 interest total.

Example 12

You want to take out a $140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example, we're looking for *d*.

$r = 0.06$	6% annual rate
$k = 12$	since we're paying monthly
$N = 30$	30 years
$P_0 = \$140{,}000$	the starting loan amount

In this case, we're going to have to set up the equation, and solve for *d*.

$$140,000 = \frac{d\left(1-\left(1+\frac{0.06}{12}\right)^{-30(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$140,000 = \frac{d\left(1-(1.005)^{-360}\right)}{(0.005)}$$

$$140,000 = d\,(166.792)$$

$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of $839.37 per month for 30 years.

You're paying a total of $302,173.20 to the loan company: $839.37 per month for 360 months. You are paying a total of $302,173.20 – $140,000 = $162,173.20 in interest over the life of the loan.

Try It Now

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were $1,000 a month, after a year you will *not* have paid off $12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?"

Example 13

If a mortgage at a 6% interest rate has payments of $1,000 a month, how much will the loan balance be 10 years from the end the loan?

Solution

To determine this, we are looking for the amount of the loan that can be paid off by $1,000 a month payments in 10 years. In other words, we're looking for P_0 when

d = $1,000	the monthly loan payment
r = 0.06	6% annual rate
k = 12	since we're doing monthly payments, we'll compound monthly
N = 10	since we're making monthly payments for 10 more years

$$P_0=\frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-10(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_0=\frac{1000\left(1-(1.005)^{-120}\right)}{(0.005)}$$

$$P_0=\frac{200\,(1-0.5496)}{(0.005)}=\$90,073.45$$

The loan balance with 10 years remaining on the loan will be $90,073.45

Often times answering remaining balance questions requires two steps:

1. Calculating the monthly payments on the loan
2. Calculating the remaining loan balance based on the *remaining time* on the loan

Example 14

A couple purchases a home with a $180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

Solution

First we will calculate their monthly payments. We're looking for *d*.

$r = 0.04$	4% annual rate
$k = 12$	since they're paying monthly
$N = 30$	30 years
$P_0 = \$180{,}000$	the starting loan amount

We set up the equation and solve for *d*.

$$180,000 = \frac{d\left(1-\left(1+\frac{0.04}{12}\right)^{-30(12)}\right)}{\left(\frac{0.04}{12}\right)}$$

$$180,000 = \frac{d\left(1-(1.00333)^{-360}\right)}{(.00333)}$$

$$180,000 = d\,(209.562)$$

$$d = \frac{180,000}{209.562} = \$858.93$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$d = \$858.93$	the monthly loan payment we calculated above
$r = 0.04$	4% annual rate
$k = 12$	since they're doing monthly payments
$N = 25$	since they'd be making monthly payments for 25 more years

$$P_0 = \frac{858.93\left(1-\left(1+\frac{0.04}{12}\right)^{-25(12)}\right)}{\left(\frac{0.04}{12}\right)}$$

$$P_0 = \frac{858.93\left(1-(1.00333)^{-300}\right)}{(0.00333)}$$

$$P_0 = \frac{858.93\,(1-0.369)}{(0.00333)} = \$162,758.26$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $162,758.26.

Over that 5 years, the couple has paid off \$180,000 – \$162.758.26 = \$17,241.74 of the loan balance. They have paid a total of \$858.93 a month for 5 years (60 months), for a total of \$51,535.80, so \$51,535.80 – \$17,241.74 = \$34,294.06 of what they have paid so far has been interest.

Which Equation to Use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan," "amortize" (the fancy word for loans), "finance (a car)," or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try It Now

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

1. Marcy received an inheritance of \$20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?
2. Paul wants to buy a new car. Rather than take out a loan, he decides to save \$200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
3. Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have \$30,000 in the account in 6 years. How much should Keisha deposit into the account?
4. Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of \$100, how much new equipment can she buy?
5. How much would you need to save every month in an account earning 4% interest to have \$5,000 saved up in two years?

Solving for Time

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Example 15

If you invest \$2000 at 6% compounded monthly, how long will it take the account to double in value?

Solution

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

P_0 = \$2000	the initial deposit
r = 0.06	6% annual rate
k = 12	12 months in 1 year

So our general equation is:

$$P_N = 2000\left(1+\frac{0.06}{12}\right)^{N\cdot 12}$$

We also know that we want our ending amount to be double of \$2000, which is \$4000, so we're looking for N so that P_N = 4000. To solve this, we set our equation for P_N equal to 4000.

$4000 = 2000\left(1+\frac{0.06}{12}\right)^{N\cdot 12}$	Divide both sides by 2000
$2 = (1.005)^{12N}$	To solve for the exponent, take the log of both sides
$\log(2) = \log\left((1.005)^{12N}\right)$	Use the exponent property of logs on the right side
$\log(2) = 12N\log(1.005)$	Now we can divide both sides by 12log(1.005)
$\frac{\log(2)}{12\log(1.005)} = N$	Approximating this to a decimal
N = 11.581	

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded log(2) to 0.301 and log(1.005) to 0.00217, then your final answer would have been about 11.577 years.

Example 16

If you invest \$100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to \$10,000?

Solution

This is a savings annuity problem since we are making regular deposits into the account.

d = \$100	the monthly deposit
r = 0.03	3% annual rate
k = 12	since we're doing monthly deposits, we'll compound monthly

We don't know *N*, but we want *PN* to be \$10,000.

Putting this into the equation:

$$10,000 = \frac{100\left(\left(1+\frac{0.03}{12}\right)^{N(12)}-1\right)}{\left(\frac{0.03}{12}\right)}$$

Simplifying the fractions a bit

$$10,000 = \frac{100\left((1.0025)^{12N}-1\right)}{0.0025}$$

We want to isolate the exponential term, 1.0025^{12N}, so multiply both sides by 0.0025

$25 = 100\left((1.0025)^{12N}-1\right)$	Divide both sides by 100
$0.25 = (1.0025)^{12N}-1$	Add 1 to both sides
$1.25 = (1.0025)^{12N}$	Now take the log of both sides
$\log(1.25) = \log\left((1.0025)^{12N}\right)$	Use the exponent property of logs
$\log(1.25) = 12N\log(1.0025)$	Divide by 12log(1.0025)
$\frac{\log(1.25)}{12\log(1.0025)} = N$	Approximating to a decimal
N = 7.447 years	

It will take about 7.447 years to grow the account to $10,000.

Try It Now

Joel is considering putting a $1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of $30 a month?

SUPPLEMENTAL VIDEOS

This YouTube playlist contains several videos that supplement the reading on Finance.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

AVERAGE DAILY BALANCE

Click on the link below to download a document: “Credit Cards and Installment Buying” by Erie Community College. This reading shows an example of an Average Daily Balance Method of Determining Interest Rate using credit cards and installment buying to illustrate its points.

- Credit Cards and Installment Buying

HOW AN AMORTIZATION SCHEDULE IS CALCULATED

Click on the link below to view the website "How is an Amortization Schedule Calculated?" by MyAmortizationChart.com. This website provides a brief overlook Amortization Schedules.

- How is an Amortization Schedule Calculated?

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DISCUSS: CONSUMER MATH APPLICATION

Pick a real problem and try to solve it using the consumer math solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Consumer Math Application Directions for detailed directions.

Create a new thread in the **Consumer Math Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

DISCUSS: FINAL REFLECTION

Reflect on what you have learned in this course, which will help you better understand and remember it. Read the Final Reflection Directions for detailed directions. Create a new thread in the **Final Reflection** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 40 points.

Grading Criteria	Points Possible
The presentation: • Is it a substantive, two or three paragraph essay? • Is it clear and well-written with proper grammar and correct spelling?	15

The content:
 	
• For Option 1:
    ◦ Does it apply to material covered in class?
    ◦ Is the posting unique to the postings of other students and to your own application postings?
 	
• For Option 2:
    ◦ Is (are) the selected question(s) answered in depth?
    ◦ Are the facts accurate?
	25

STATISTICS

MODULE OVERVIEW

Like most people, you probably feel that it is important to “take control of your life.” But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, Statistics is one of the most important things that you can study.

To be more specific, here are some claims that we have heard on several occasions. (We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one’s life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) These chapters will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, “There are three kinds of lies—lies, damned lies, and statistics.” This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950s and 60s to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

POPULATIONS AND SAMPLES

Before we begin gathering and analyzing data we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of Washington.
- All first-year students at public colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

Population
The **population** of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the **target population**, since if we design our study badly, the collected data might not actually be representative of the intended population.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at the University of Washington might take slightly more diverse courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from the University of Washington. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 1

A newspaper website contains a poll asking people their opinion on a recent news article.

What is the population?

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define "average" more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Parameter
A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive, unless the population is very small or we already have the data collected.

Census
A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study;

Sample
A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

Statistic
A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

Example 2

A researcher wanted to know how citizens of Tacoma felt about a voter initiative. To study this, she goes to the Tacoma Mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was Tacoma citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all Tacoma citizens.

The 60% value was based on the sample, so it is a statistic.

Try it Now 1

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

Try it Now 2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

CATEGORIZING DATA

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as categorical data or quantitative data.

> Quantitative and categorical data
> **Categorical (qualitative) data** are pieces of information that allow us to classify the objects under investigation into various categories.
> **Quantitative data** are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

Example 3

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Hulk*, or *Terminator 3: Rise of the Machines*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Finding Nemo*." This would be categorical data.

Example 4

A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, …)

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, …) or the amount of money you paid for a movie ticket the last time you went to a movie theater ($5.50, $7.75, $9, …).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

Example 5

Suppose we gather respondents' ZIP codes in a survey to track their geographical location.

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example 6

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

1 – it was awful
2 – it was just OK
3 – I liked it

4 – it was great
5 – best movie ever!

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *Terminator 3*. And 33-37 year-olds might prefer…well, *Finding Nemo*.

Try it Now 3

Classify each measurement as categorical or quantitative

a. Eye color of a group of people

b. Daily high temperature of a city over several weeks

c. Annual income

SAMPLING METHODS

As we mentioned in a previous section, the first thing we should do before conducting a survey is to identify the population that we want to study. Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote, and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?

In November 1998, former professional wrestler Jesse "The Body" Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

> Sampling bias
> A sampling method is biased if every member of the population doesn't have equal likelihood of being in the sample.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

Returning to our hypothetical job as a political pollster, we would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Elks club would provide a useful picture of district-wide support for our candidate.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is simple random sampling.

> Simple random sample
> A **random sample** is one in which each member of the population has an equal probability of being chosen. A **simple random sample** is one in which every member of the population and any group of members has an equal probability of being chosen.

Example 7

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample.

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the state of Washington, some of these samples might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

> Sampling variability
> The natural variation of samples is called **sampling variability**.
> This is unavoidable and expected in random sampling, and in most cases is not an issue.

To help account for variability, pollsters might instead use a **stratified sample**.

> Stratified sampling
> In **stratified sampling**, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example 8

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans and 24% independents. In a sample of 1000 people, they would then expect to get about 390 Democrats, 370 Republicans and 240 independents. To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called **quota sampling**.

> Quota sampling
> **Quota sampling** is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

Example 9

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat.

You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any "real" questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling, but also does not ensure the same level of randomness.

Another sampling method is **cluster sampling**, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

> Cluster sampling
> In **cluster sampling**, the population is divided into subgroups (clusters), and a set of subgroups are selected to be in the sample

Example 10

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Other sampling methods include **systematic sampling**.

> Systematic sampling
> In **systematic sampling**, every *nth* member of the population is selected to be in the sample.

Example 11

To select a sample using systematic sampling, a pollster calls every 100th name in the phone book.

Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample) but it can yield acceptable samples.

Perhaps the worst types of sampling methods are **convenience samples** and **voluntary response samples**.

Convenience sampling and voluntary response sampling

Convenience sampling is samples chosen by selecting whoever is convenient.

Voluntary response sampling is allowing the sample to volunteer.

Example 12

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is a convenience sample.

Example 13

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

Try it Now 4

In each case, indicate what sampling method was used

a. Every 4th person in the class was selected

b. A sample was selected to contain 25 men and 35 women

c. Viewers of a new show are asked to vote on the show's website

d. A website randomly selects 50 of their customers to send a satisfaction survey to

e. To survey voters in a town, a polling company randomly selects 10 city blocks, and interviews everyone who lives on those blocks.

HOW TO MESS THINGS UP BEFORE YOU START

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – **sampling** or **selection bias**, which is when the sample is not representative of the population. One example of this is **voluntary response bias**, which is bias introduced by only collecting data from those who volunteer to participate. This is not the only potential source of bias.

Sources of bias
Sampling bias – when the sample is not representative of the population
Voluntary response bias – the sampling bias that often occurs when the sample is volunteers
Self-interest study – bias that can occur when the researchers have an interest in the outcome
Response bias – when the responder gives inaccurate responses for any reason
Perceived lack of anonymity – when the responder fears giving an honest answer might negatively affect them
Loaded questions – when the question wording influences the responses
Non-response bias – when people refusing to participate in the study can influence the validity of the outcome

Example 14

Consider a recent study which found that chewing gum may raise math grades in teenagers ((Note: Reuters. http://news.yahoo.com/s/nm/20090423/od_uk_nm/oukoe_uk_gum_learning. Retrieved 4/27/09)). This study was conducted by the Wrigley Science Institute, a branch of the Wrigley chewing gum company. This is an example of a **self-interest study**; one in which the researches have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny.

Example 15

A survey asks people "when was the last time you visited your doctor?" This might suffer from **response bias**, since many people might not remember exactly when they last saw a doctor and give inaccurate responses.

Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

Example 16

A survey asks participants a question about their interactions with members of other races. Here, a **perceived lack of anonymity** could influence the outcome. The respondent might not want to be perceived as racist even if they are, and give an untruthful answer.

Example 17

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Here, answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer.

Example 18

A survey asks "do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?" This is an example of a **loaded** or **leading question** – questions whose wording leads the respondent towards an answer.

Loaded questions can occur intentionally by pollsters with an agenda, or accidentally through poor question wording. Also a concern is **question order**, where the order of questions changes the results. A psychology researcher provides an example ((Note: Swartz, Norbert. http://www.umich.edu/~newsinfo/MT/01/Fal01/mt6f01.html. Retrieved 3/31/2009)):

> "My favorite finding is this: we did a study where we asked students, 'How satisfied are you with your life? How often do you have a date?' The two answers were not statistically related – you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, 'How often do you have a date? How satisfied are you with your life?' the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student's life as dating frequency."

Example 19

A telephone poll asks the question "Do you often have time to relax and read a book?", and 50% of the people called refused to answer the survey. It is unlikely that the results will be representative of the entire population. This is an example of **non-response bias**, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

Try it Now 5

In each situation, identify a potential source of bias

a. A survey asks how many sexual partners a person has had in the last year

b. A radio station asks readers to phone in their choice in a daily poll.

c. A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.

d. High school students are asked if they have consumed alcohol in the last two weeks.

e. The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.

f. A poll asks "Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?"

EXPERIMENTS

So far, we have primarily discussed **observational studies** – studies in which conclusions would be drawn from observations of a sample or the population. In some cases these observations might be unsolicited, such as studying the percentage of cars that turn right at a red light even when there is a "no turn on red" sign. In other cases the observations are solicited, like in a survey or a poll.

In contrast, it is common to use **experiments** when exploring how subjects react to an outside influence. In an experiment, some kind of **treatment** is applied to the subjects and the results are measured and recorded.

Observational studies and experiments
An **observational study** is a study based on observations or measurements
An **experiment** is a study in which the effects of a **treatment** are measured

Here are some examples of experiments:

Example 20

a. A pharmaceutical company tests a new medicine for treating Alzheimer's disease by administering the drug to 50 elderly patients with recent diagnoses. The treatment here is the new drug.

b. A gym tests out a new weight loss program by enlisting 30 volunteers to try out the program. The treatment here is the new program.

c. You test a new kitchen cleaner by buying a bottle and cleaning your kitchen. The new cleaner is the treatment.

d. A psychology researcher explores the effect of music on temperament by measuring people's temperament while listening to different types of music. The music is the treatment.

Try it Now 6

Is each scenario describing an observational study or an experiment?

a. The weights of 30 randomly selected people are measured

b. Subjects are asked to do 20 jumping jacks, and then their heart rates are measured

c. Twenty coffee drinkers and twenty tea drinkers are given a concentration test

When conducting experiments, it is essential to isolate the treatment being tested.

Example 21

Suppose a middle school (junior high) finds that their students are not scoring well on the state's standardized math test. They decide to run an experiment to see if an alternate curriculum would improve scores. To run the test, they hire a math specialist to come in and teach a class using the new curriculum. To their delight, they see an improvement in test scores.

The difficulty with this scenario is that it is not clear whether the curriculum is responsible for the improvement, or whether the improvement is due to a math specialist teaching the class. This is called **confounding** – when it is not clear which factor or factors caused the observed effect. Confounding is the downfall of many experiments, though sometimes it is hidden.

> Confounding
> **Confounding** occurs when there are two potential variables that could have caused the outcome and it is not possible to determine which actually caused the result.

Example 22

A drug company study about a weight loss pill might report that people lost an average of 8 pounds while using their new drug. However, in the fine print you find a statement saying that participants were encouraged to also diet and exercise. It is not clear in this case whether the weight loss is due to the pill, to diet and exercise, or a combination of both. In this case confounding has occurred.

Example 23

Researchers conduct an experiment to determine whether students will perform better on an arithmetic test if they listen to music during the test. They first give the student a test without music, then give a similar test while the student listens to music. In this case, the student might perform better on the second test, regardless of the music, simply because it was the second test and they were warmed up.

There are a number of measures that can be introduced to help reduce the likelihood of confounding. The primary measure is to use a **control group**.

> Control group
> When using a control group, the participants are divided into two or more groups, typically a **control group** and a treatment group. The treatment group receives the treatment being tested; the control group does not receive the treatment.

Ideally, the groups are otherwise as similar as possible, isolating the treatment as the only potential source of difference between the groups. For this reason, the method of dividing groups is important. Some researchers attempt to ensure that the groups have similar characteristics (same number of females, same number of people over 50, etc.), but it is nearly impossible to control for every characteristic. Because of this, random assignment is very commonly used.

Example 24

To determine if a two day prep course would help high school students improve their scores on the SAT test, a group of students was randomly divided into two subgroups. The first group, the treatment group, was given a two day prep course. The second group, the control group, was not given the prep course. Afterwards, both groups were given the SAT.

Example 25

A company testing a new plant food grows two crops of plants in adjacent fields, the treatment group receiving the new plant food and the control group not. The crop yield would then be compared. By growing them at the same time in adjacent fields, they are controlling for weather and other confounding factors.

Sometimes not giving the control group anything does not completely control for confounding variables. For example, suppose a medicine study is testing a new headache pill by giving the treatment group the pill and the control group nothing. If the treatment group showed improvement, we would not know whether it was due to the medicine in the pill, or a response to have taken any pill. This is called a **placebo effect**.

> Placebo effect
> The **placebo effect** is when the effectiveness of a treatment is influenced by the patient's perception of how effective they think the treatment will be, so a result might be seen even if the treatment is ineffectual.

Example 26

A study found that when doing painful dental tooth extractions, patients told they were receiving a strong painkiller while actually receiving a saltwater injection found as much pain relief as patients receiving a dose of morphine. ((Note: Levine JD, Gordon NC, Smith R, Fields HL. (1981) Analgesic responses to morphine and placebo in individuals with postoperative pain. Pain. 10:379-89.))

To control for the placebo effect, a **placebo**, or dummy treatment, is often given to the control group. This way, both groups are truly identical except for the specific treatment given.

> Placebo and Placebo controlled experiments
> A **placebo** is a dummy treatment given to control for the placebo effect.
> An experiment that gives the control group a placebo is called a **placebo controlled experiment**.

Example 27

a. In a study for a new medicine that is dispensed in a pill form, a sugar pill could be used as a placebo.

b. In a study on the effect of alcohol on memory, a non-alcoholic beer might be given to the control group as a placebo.

c. In a study of a frozen meal diet plan, the treatment group would receive the diet food, and the control could be given standard frozen meals stripped of their original packaging.

In some cases, it is more appropriate to compare to a conventional treatment than a placebo. For example, in a cancer research study, it would not be ethical to deny any treatment to the control group or to give a placebo treatment. In this case, the currently acceptable medicine would be given to the second group, called a **comparison group** in this case. In our SAT test example, the non-treatment group would most likely be encouraged to study on their own, rather than be asked to not study at all, to provide a meaningful comparison.

When using a placebo, it would defeat the purpose if the participant knew they were receiving the placebo.

Blind studies
A **blind study** is one in which the participant does not know whether or not they are receiving the treatment or a placebo.
A **double-blind study** is one in which those interacting with the participants don't know who is in the treatment group and who is in the control group.

Example 28

In a study about anti-depression medicine, you would not want the psychological evaluator to know whether the patient is in the treatment or control group either, as it might influence their evaluation, so the experiment should be conducted as a double-blind study.

It should be noted that not every experiment needs a control group.

Example 29

If a researcher is testing whether a new fabric can withstand fire, she simply needs to torch multiple samples of the fabric – there is no need for a control group.

Try it Now 7

To test a new lie detector, two groups of subjects are given the new test. One group is asked to answer all the questions truthfully, and the second group is asked to lie on one set of questions. The person administering the lie detector test does not know what group each subject is in.

Does this experiment have a control group? Is it blind, double-blind, or neither?

Try it Now Answers

1. The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch the bait.

2. This is a parameter, since the college would have access to data on all students (the population)

3. a. Categorical. b. Quantitative c. Quantitative

4. a. Systematic

b. Stratified or Quota

c. Voluntary response

d. Simple random

e. Cluster

5. a. Response bias – historically, men are likely to over-report, and women are likely to under-report to this question.

b. Voluntary response bias – the sample is self-selected

c. Sampling bias – the sample may not be representative of the whole class

d. Lack of anonymity

e. Self-interest study

f. Loaded question

6. a. Observational study

b. Experiment; the treatment is the jumping jacks

c. Experiment; the treatments are coffee and tea

7. The truth-telling group could be considered the control group, but really both groups are treatment groups here, since it is important for the lie detector to be able to correctly identify lies, and also not identify truth telling as lying. This study is blind, since the person running the test does not know what group each subject is in.

DESCRIBING DATA

Once we have collected data from surveys or experiments, we need to summarize and present the data in a way that will be meaningful to the reader. We will begin with graphical presentations of data then explore numerical summaries of data.

Presenting Categorical Data Graphically

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table.**

Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39

Color	Frequency
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

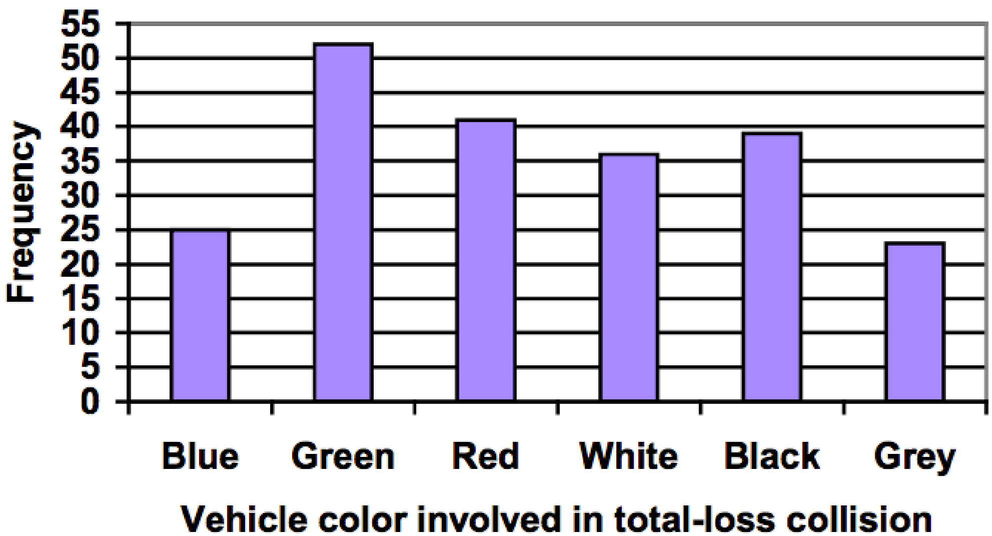
Bar Graph

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

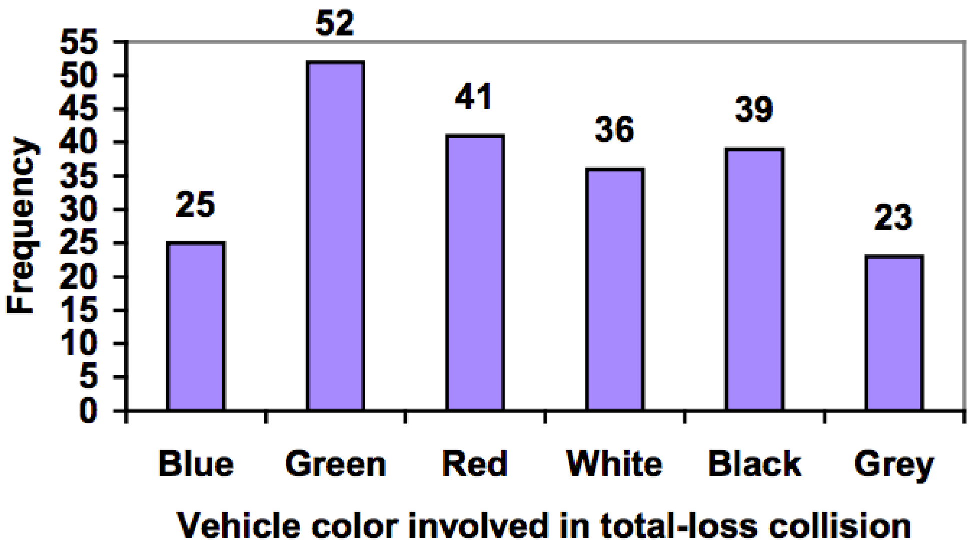
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. The construction of a bar chart is most easily described by use of an example.

Example 2

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



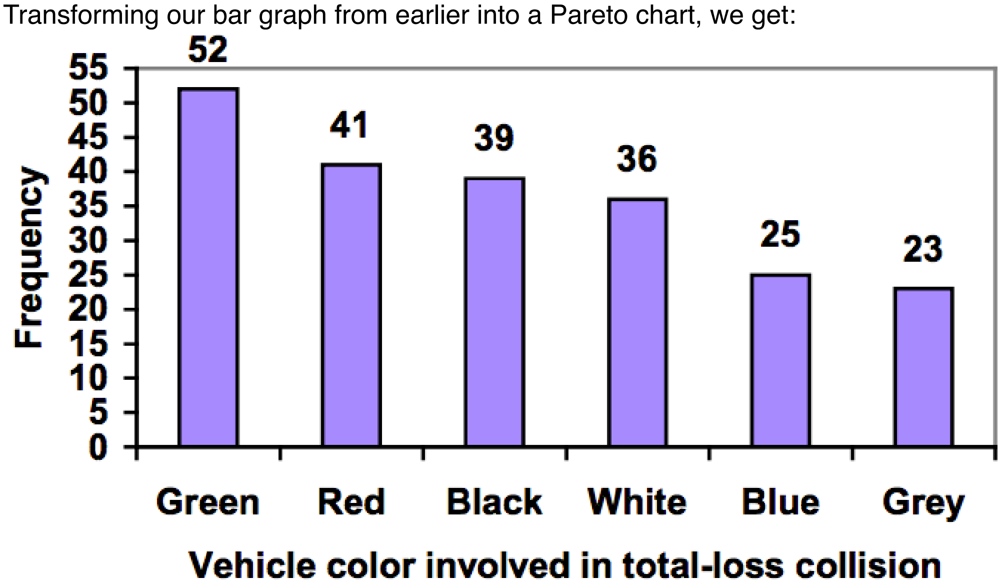
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

Pareto Chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency

Example 3

Transforming our bar graph from earlier into a Pareto chart, we get:



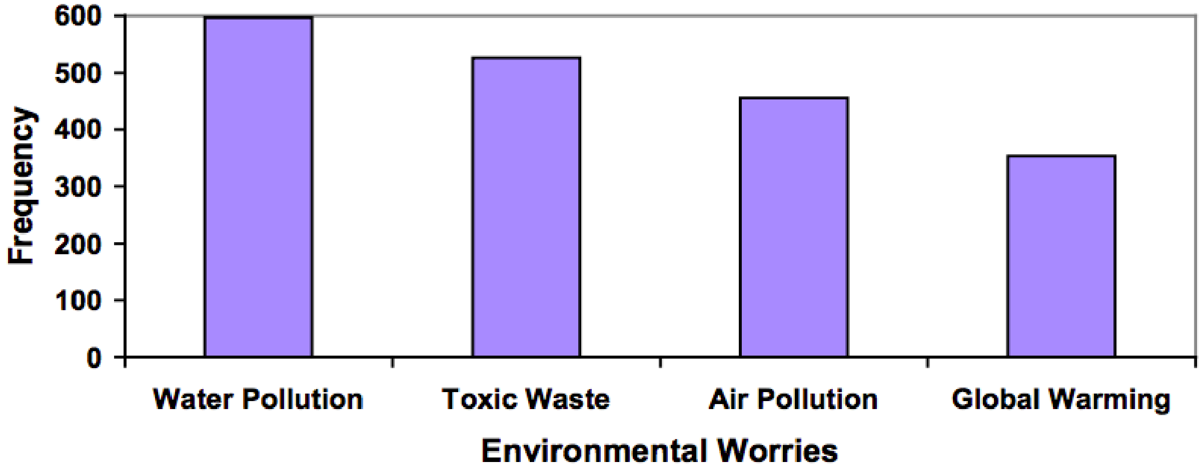
Example 4

In a survey, ((Note: Gallup Poll. March 5–8, 2009. http://www.pollingreport.com/enviro.htm)) adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of

1012 surveyed) who indicated that they worried "a great deal" about some selected concerns are summarized below.

Environmental Issue	Frequency
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

This data could be shown graphically in a bar graph:



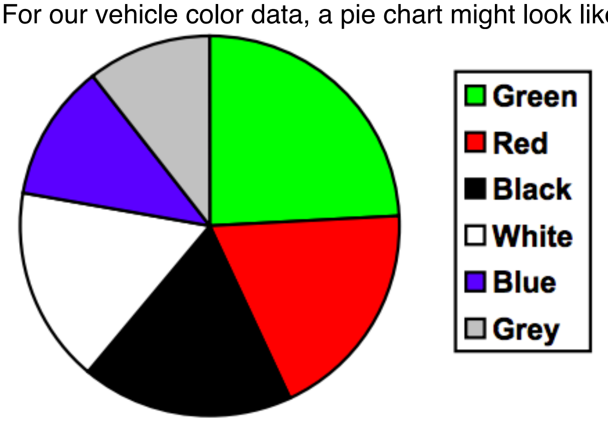
To show relative sizes, it is common to use a pie chart.

Pie Chart

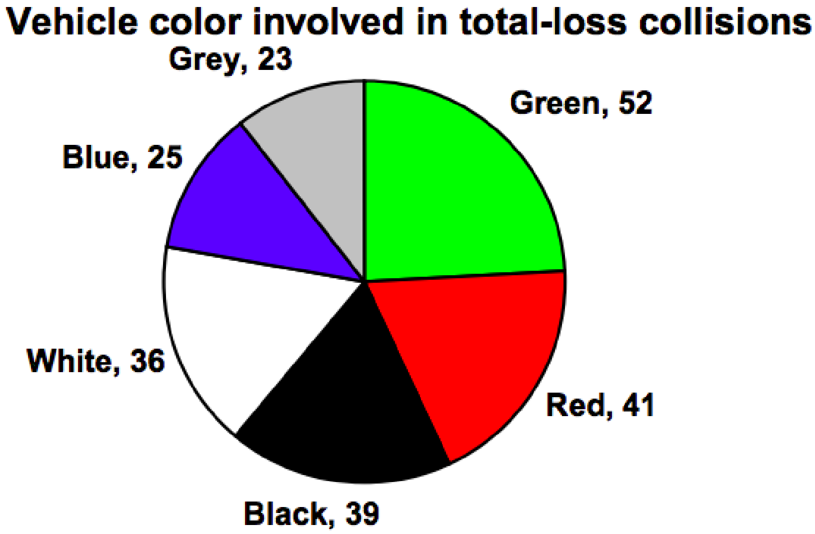
A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example 5

For our vehicle color data, a pie chart might look like this:



Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.



Example 6

The pie chart to the right shows the percentage of voters supporting each candidate running for a local senate seat.

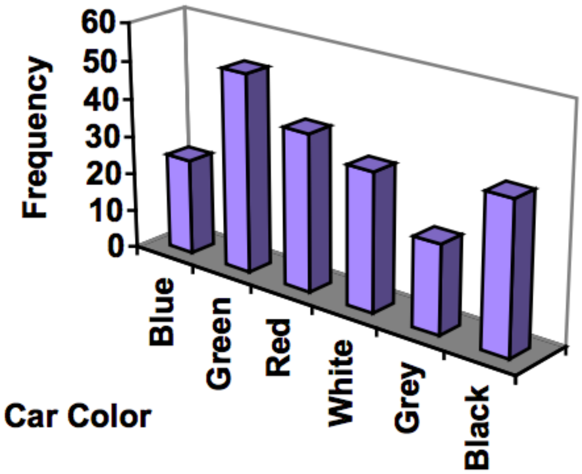
If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.

Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs. ((Note: For example: http://nces.ed.gov/nceskids/createAgraph/ or http://docs.google.com))

Try It Now

Create a bar graph and a pie chart to illustrate the grades on a history exam below.
A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.



Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.

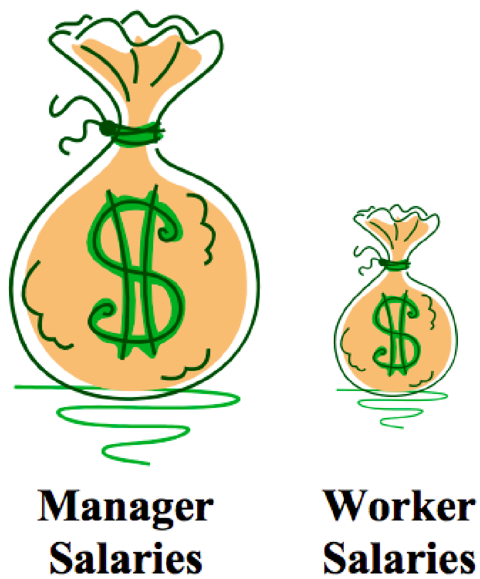
Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example 7

A labor union might produce the graph to the right to show the difference between the average manager salary and the average worker salary.

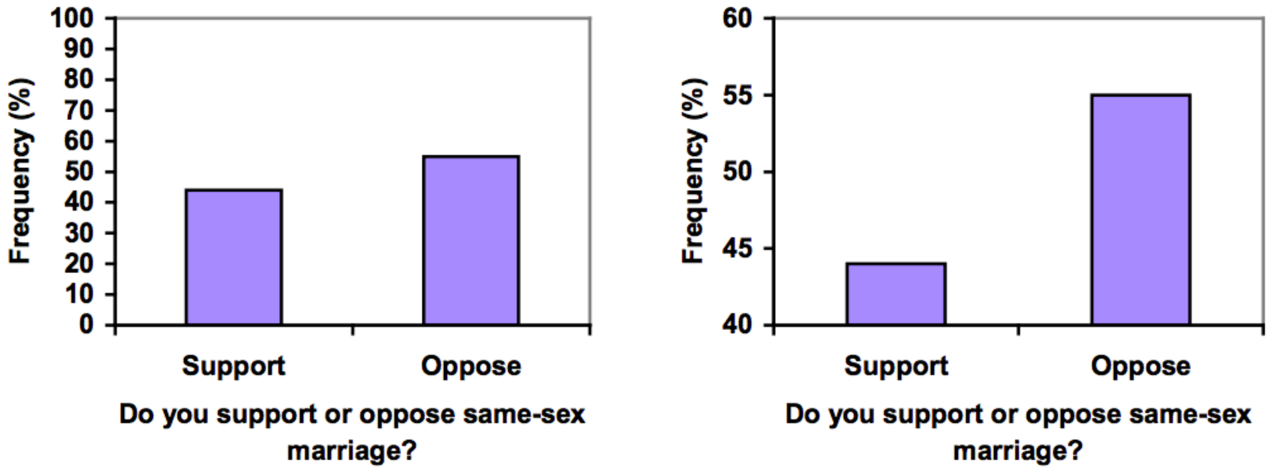
Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries —the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.



Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

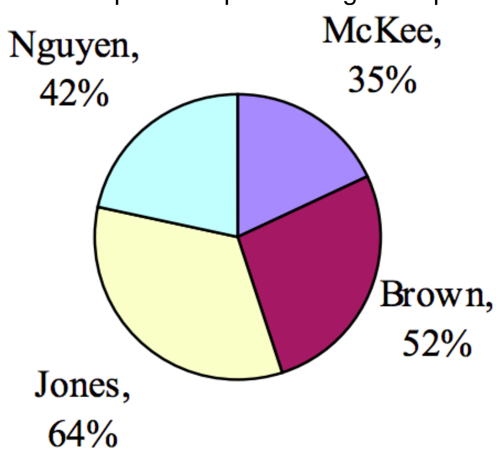
Example 8

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008. ((Note: CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from http://www.pollingreport.com/civil.htm)) The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.



Try It Now

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.



Presenting Quantitative Data Graphically

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 9

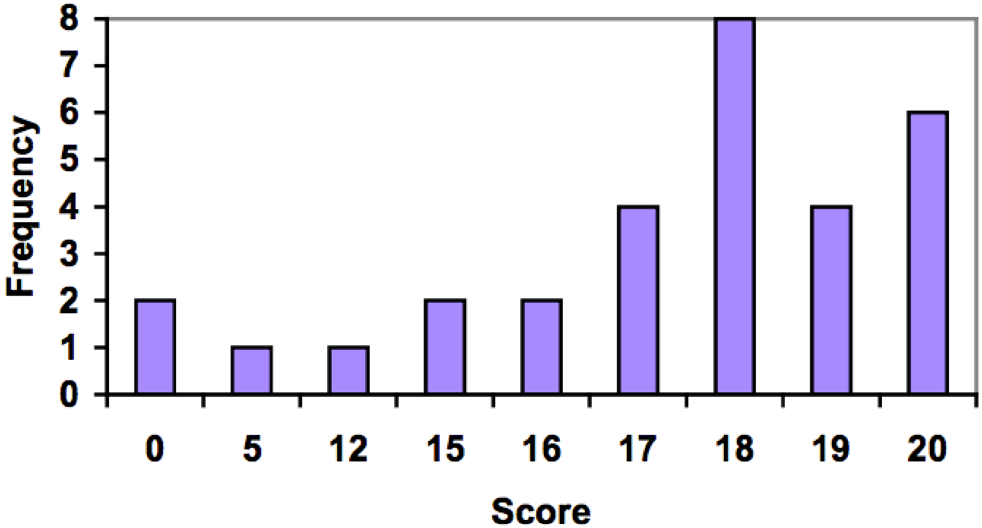
A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by grouping like values:

Score	**Frequency**
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary, like we did for categorical data:



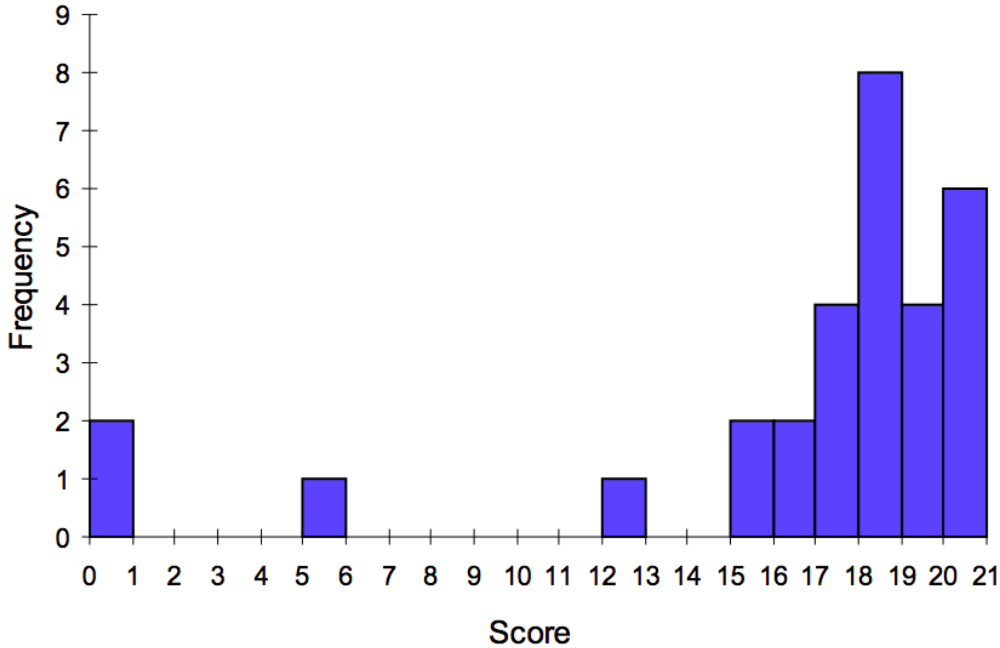
However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Histogram

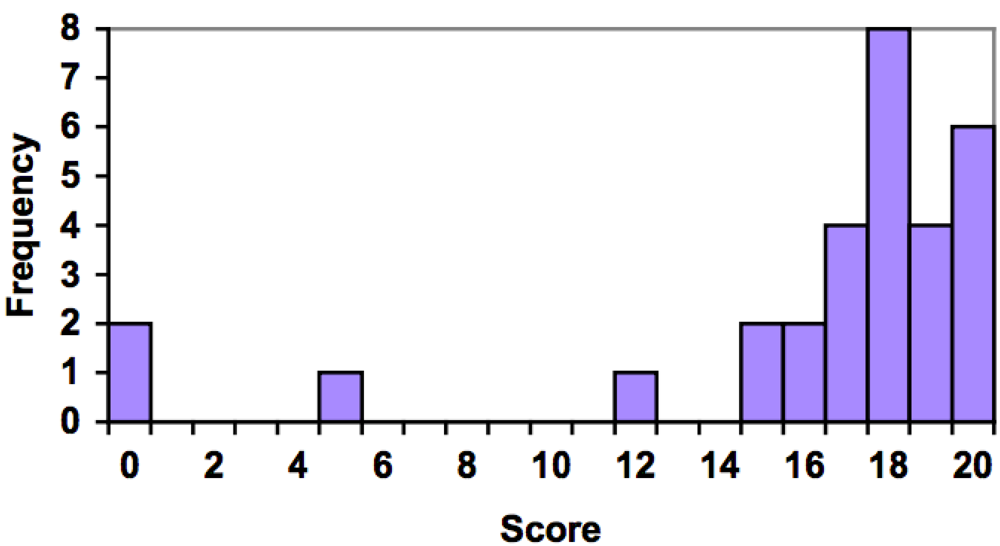
A histogram is like a bar graph, but where the horizontal axis is a number line

Example 10

For the values above, a histogram would look like:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at ½ values to avoid this ambiguity.



Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

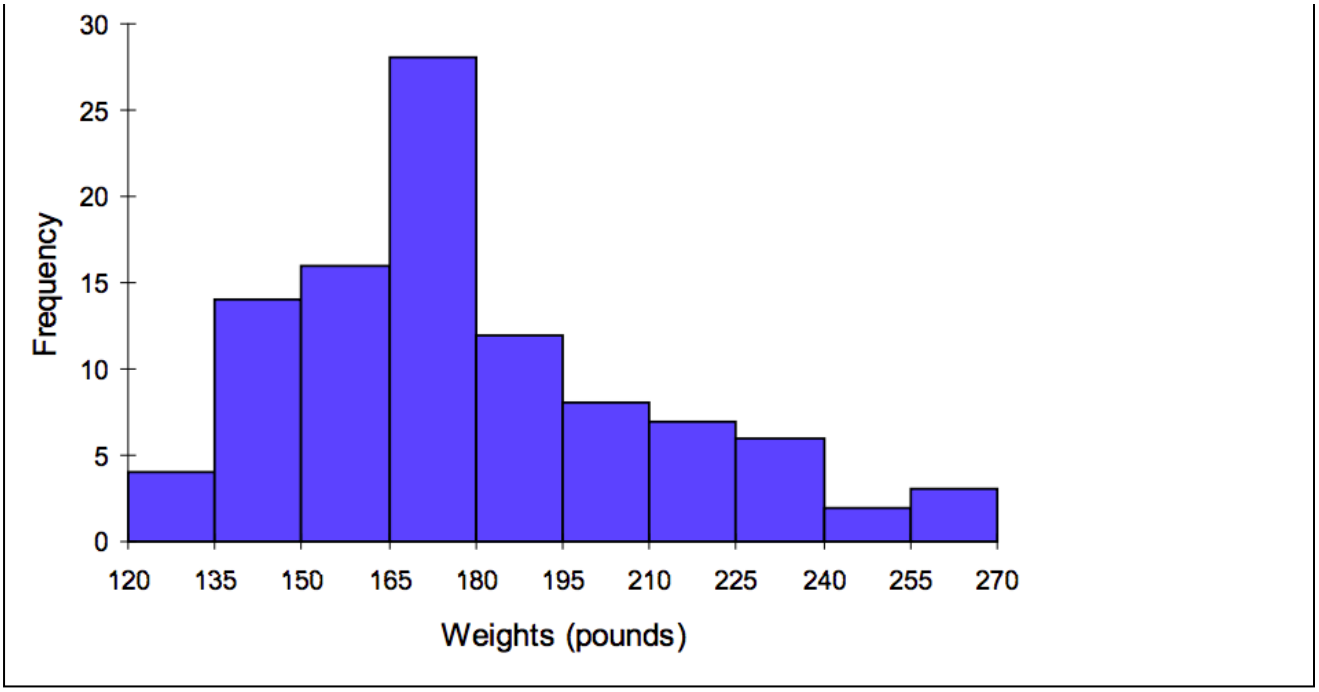
- Each interval is equal in size. For example, if the first class contains values from 120 to 129, the second class should include values from 130 to 139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 11

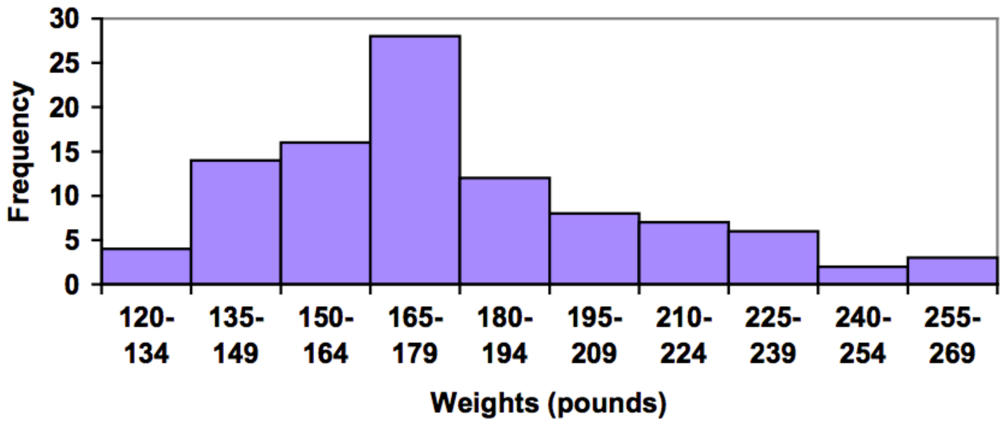
Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Interval	**Frequency**
120–134	4
135–149	14
150–164	16
165–179	28
180–194	12
195–209	8
210–224	7
225–239	6
240–254	2
255–269	3

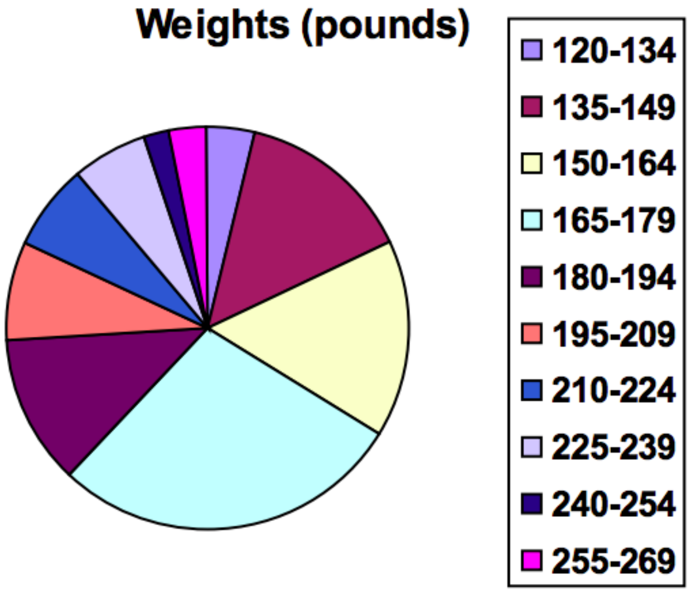
A histogram of this data would look like:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



Try It Now

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140	$160	$160	$165	$180	$220	$235	$240	$250	$260	$280	$285
$285	$285	$290	$300	$300	$305	$310	$310	$315	$315	$320	$320
$330	$340	$345	$350	$355	$360	$360	$380	$395	$420	$460	$460

When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

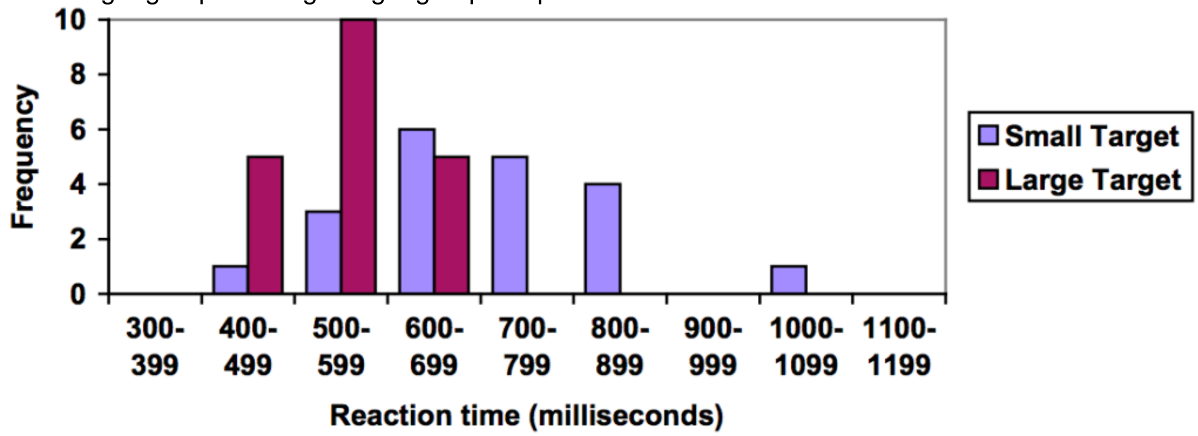
Example 12

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Interval (milliseconds)	**Frequency: small target**	**Frequency: large target**
300–399	0	0
400–499	1	5
500–599	3	10
600–699	6	5
700–799	5	0
800–899	4	0

Interval (milliseconds)	Frequency: small target	Frequency: large target
900–999	0	0
1000–1099	1	0
1100–1199	0	0

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.

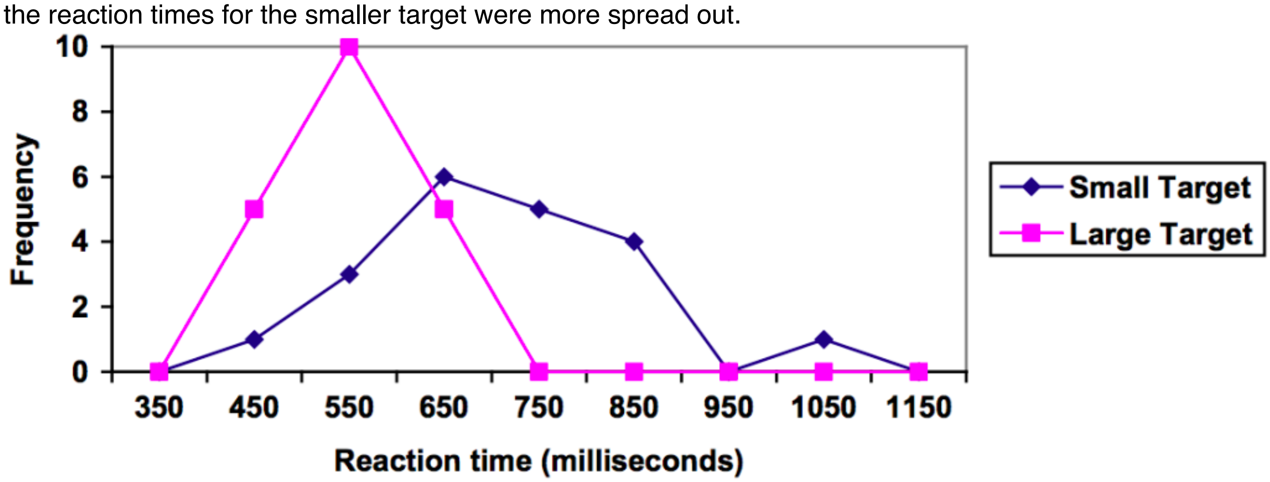


Frequency Polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 13

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



Numerical Summaries of Data

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second section describes measures of variability.

Measures of Central Tendency

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 14

Marci's exam scores for her last math class were: 79, 86, 82, 94. The mean of these values would be:

$\frac{79+86+82+94}{4} = 85.25$. Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 15

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get $\frac{634}{31} = 20.4516$. It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean."

Try It Now

The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the mean price.

Example 16

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest \$5 thousand dollars. The results are summarized in a frequency table below.

Income (thousands of dollars)	**Frequency**
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{\overbrace{15+\cdots+15}^{\text{6 terms}}+\overbrace{20+\cdots+20}^{\text{8 terms}}+\overbrace{25+\cdots+25}^{\text{11 terms}}+\cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as 15 × 6 = 90. Using this simplification, we get

$$\frac{15\cdot6+20\cdot8+25\cdot11+30\cdot17+35\cdot19+40\cdot20+\cdot12+50\cdot7}{100}=\frac{3390}{100}=33.9$$

The mean household income of our sample is 33.9 thousand dollars (\$33,900).

Example 17

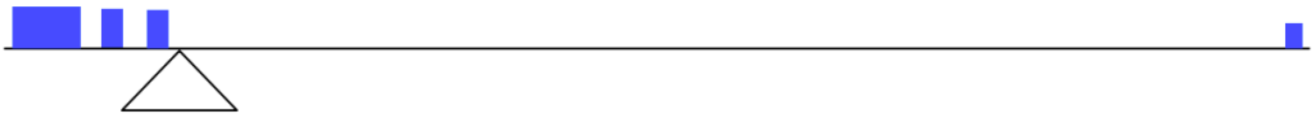
Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of \$5 million (\$5000 thousand). Adding this to our sample, our mean is now:

$$\frac{15\cdot6+20\cdot8+25\cdot11+30\cdot17+35\cdot19+40\cdot20+\cdot12+50\cdot7+5000\cdot1}{101}=\frac{3390}{101}=83.069$$

While 83.1 thousand dollars (\$83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.

If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers**—values far outside the primary grouping—it is common to use a different measure of center, the **median**.

Median

The **median** of a set of data is the value in the middle when the data is in order.
To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.
If the number of data values, *N*, is odd, then the median is the middle data value. This value can be found by rounding *N*/2 up to the next whole number.
If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values *N*/2 and *N*/2 + 1)

Example 18

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value $\frac{31}{2} = 15.5$, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the

median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 19

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get $\frac{6+7}{2} = 6.5$.

The median quiz score was 6.5.

Try It Now

The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the median price.

Example 20

Let us return now to our original household income data:

Income (thousands of dollars)	**Frequency**
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values: the 50th and 51st data values. To find these, we start counting up from the bottom:

> There are 6 data values of \$15, so values 1 to 6 are \$15 thousand
> The next 8 data values are \$20, so values 7 to (6+8)=14 are \$20 thousand
> The next 11 data values are \$25, so values 15 to (14+11)=25 are \$25 thousand
> The next 17 data values are \$30, so values 26 to (25+17)=42 are \$30 thousand
> The next 19 data values are \$35, so values 43 to (42+19)=61 are \$35 thousand

From this we can tell that values 50 and 51 will be \$35 thousand, and the mean of these two values is \$35 thousand. The median income in this neighborhood is \$35 thousand.

Example 21

If we add in the new neighbor with a \$5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is \$35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 22

In our vehicle color survey, we collected the data

Color	**Frequency**
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try It Now

Reviewers were asked to rate a product on a scale of 1 to 5. Find

1. The mean rating
2. The median rating
3. The mode rating

Rating	**Frequency**
1	4
2	8
3	7
4	3
5	1

Measures of Variation

Consider these three sets of quiz scores:

Section A: 5 5 5 5 5 5 5 5 5 5
Section B: 0 0 0 0 0 10 10 10 10 10
Section C: 4 4 4 5 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the “typical” or “middle” value, we also need a measure of how “spread out” or varied each data set is.

There are several ways to measure this “spread” of the data. The first is the simplest and is called the **range**.

Range

The range is the difference between the maximum value and the minimum value of the data set.

Example 23

Using the quiz scores from above,

- For section A, the range is 0 since both maximum and minimum are 5 and 5 – 5 = 0
- For section B, the range is 10 since 10 – 0 = 10
- For section C, the range is 2 since 6 – 4 = 2

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we’ll have to turn to more sophisticated measures of variation.

Standard Deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	**deviation: data value – mean**
0	0 – 5 = –5
5	5 – 5 = 0

data value	deviation: data value – mean
5	5 – 5 = 0
5	5 – 5 = 0
5	5 – 5 = 0
5	5 – 5 = 0
5	5 – 5 = 0
5	5 – 5 = 0
5	5 – 5 = 0
10	10 – 5 = 5

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

data value	deviation: data value – mean	deviation squared
0	0–5 = –5	$(-5)^2 = 25$
5	5 – 5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5 – 5 = 0	$0^2 = 0$
5	5 – 5 = 0	$0^2 = 0$
5	5 – 5 = 0	$0^2 = 0$
5	5 – 5 = 0	$0^2 = 0$
5	5 – 5 = 0	$0^2 = 0$
5	5 – 5 = 0	$0^2 = 0$
10	10 – 5 = 5	$(5)^2 = 25$

We then add the squared deviations up to get 25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50. Ordinarily we would then divide by the number of scores, *n*, (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by *n* – 1 (in this case, 10 – 1 = 9). ((Note: The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.))

So in our example, we would have 50/10 = 5 if section D represents a population and 50/9 = about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

population standard deviation $= \sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$

or

population standard deviation $= \sqrt{\frac{50}{9}} \approx 2.4$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To compute standard deviation:

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by *n*, the number of data values, if the data represents a whole population; divide by *n* – 1 if the data is from a sample.
5. Compute the square root of the result.

Example 24

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	**deviation: data value – mean**	**deviation squared**
0	0 – 5 = –5	$(–5)^2 = 25$
0	0 – 5 = –5	$(–5)^2 = 25$
0	0 – 5 = –5	$(–5)^2 = 25$
0	0 – 5 = –5	$(–5)^2 = 25$
0	0 – 5 = –5	$(–5)^2 = 25$
10	10 – 5 = –5	$(5)^2 = 25$
10	10 – 5 = –5	$(5)^2 = 25$
10	10 – 5 = –5	$(5)^2 = 25$

data value	deviation: data value – mean	deviation squared
10	10 – 5 = –5	$(5)^2 = 25$
10	10 – 5 = –5	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}} = \sqrt{\frac{250}{10}} = 5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Try It Now

The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, **quartiles** are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.
The first quartile (Q_1) is the value so that 25% of the data values are below it; the third quartile (Q_3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.
This divides the data into quarters; 25% of the data is between the minimum and Q_1, 25% is between Q_1 and the median, 25% is between the median and Q_3, and 25% is between Q_3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five number summary

The five number summary takes this form:
Minimum, Q_1, Median, Q_3, Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If *n* is the number of data values, we compute a locator by finding 25% of *n*. If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

To find the first quartile, Q_1

Begin by ordering the data from smallest to largest
Compute the locator: $L = 0.25n$
If L is a decimal value:

- Round up to L^+
- Use the data value in the L^+th position

If L is a whole number:

- Find the mean of the data values in the Lth and $L + 1$th positions.

To find the third quartile, Q_3

Use the same procedure as for Q_1L = 0.75*n*

Examples should help make this clearer.

Example 25

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is 0.75(9) = 6.75. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Example 26

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value *is* a whole number, we will find the mean of the 2nd and 3rd data values: (60+62)/2 = 61, so the first quartile is 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: (67+69)/2 = 68, so Q_3 is 68.

Note that the median could be computed the same way, using 50%.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 27

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary would be: 59, 61, 65, 68, 70.

Example 28

Returning to our quiz score data. In each case, the first quartile locator is 0.25(10) = 2.5, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Section and data	**5-number summary**
Section A: 5 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 5 6 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly, since the summary contains almost as many values as the original data.

Try It Now

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

$140	$160	$160	$165	$180	$220	$235	$240	$250	$260	$280	$285
$285	$285	$290	$300	$300	$305	$310	$310	$315	$315	$320	$320
$330	$340	$345	$350	$355	$360	$360	$380	$395	$420	$460	$460

Example 29

Returning to the household income data from earlier, create the five-number summary.

Income (thousands of dollars)	**Frequency**

15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was $35 thousand. We can see in the table that the minimum income is $15 thousand, and the maximum is $50 thousand.

To find Q_1, we calculate the locator: $L = 0.25(100) = 25$. This is a whole number, so Q_1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of $15, so values 1 to 6 are $15 thousand

The next 8 data values are $20, so values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so values 26 to (25+17)=42 are $30 thousand

The 25th data value is $25 thousand, and the 26th data value is $30 thousand, so Q_1 will be the mean of these: (25 + 30)/2 = $27.5 thousand.

To find Q_3, we calculate the locator: $L = 0.75(100) = 75$. This is a whole number, so Q_3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are $35, so values 43 to (42+19)=61 are $35 thousand

The next 20 data values are $40, so values 61 to (61+20)=81 are $40 thousand

Both the 75th and 76th data values lie in this group, so Q_3 will be $40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between $40 thousand and $50 thousand.

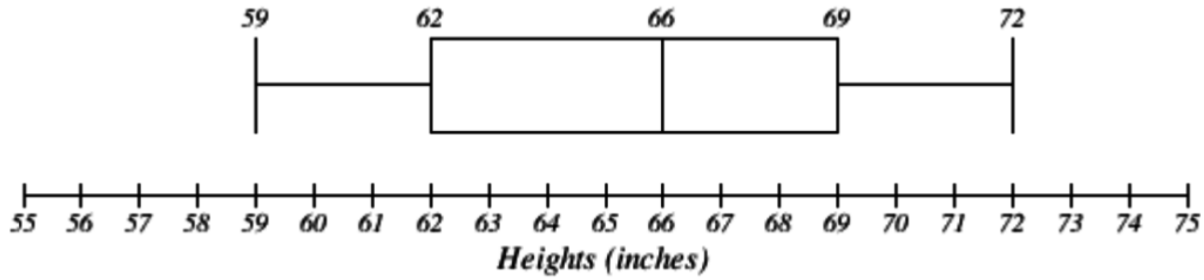
For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

Box Plot

A **box plot** is a graphical representation of a five-number summary.
To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. "Whiskers" are extended out to the minimum and maximum values.

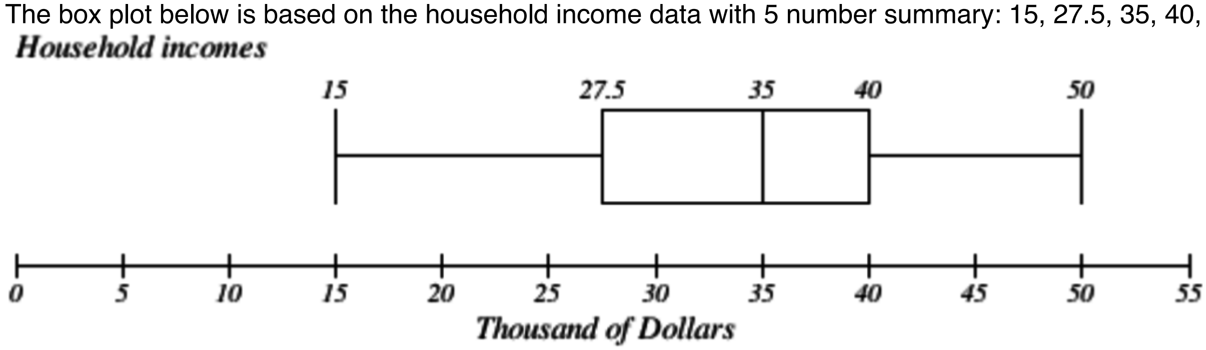
Example 30

The box plot below is based on the 9 female height data with 5 number summary: 59, 62, 66, 69, 72.



Example 31

The box plot below is based on the household income data with 5 number summary: 15, 27.5, 35, 40, 50



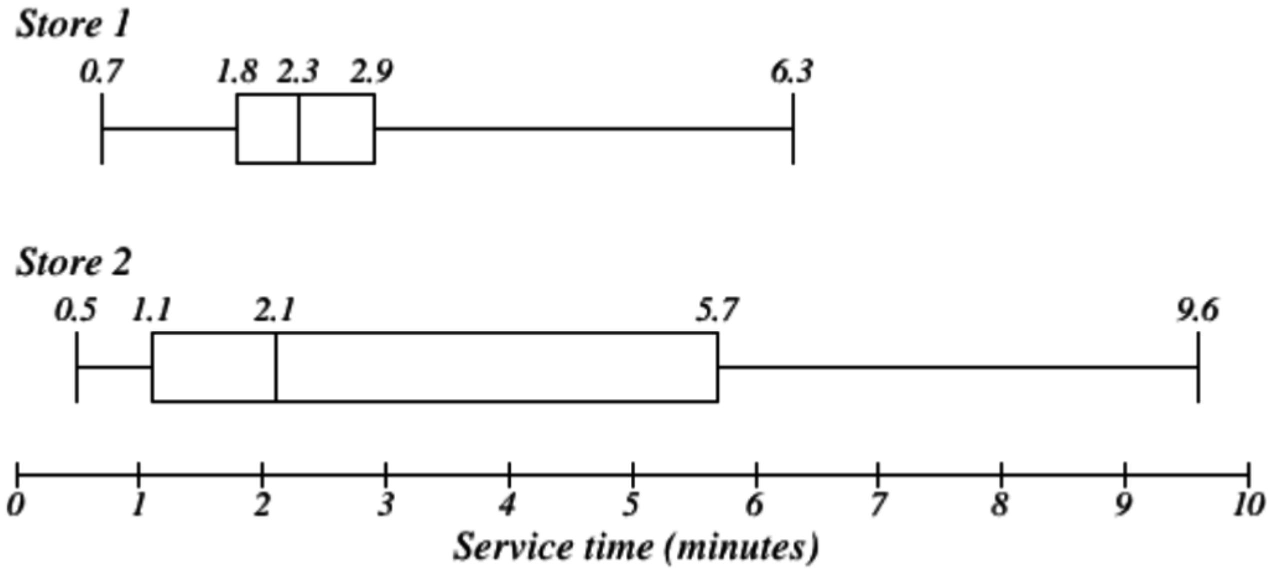
Try It Now

Create a boxplot based on the textbook price data from the last Try it Now.

Box plots are particularly useful for comparing data from two populations.

Example 32

The box plot of service times for two fast-food restaurants is shown below.



While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck: 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 33

The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS). ((Note: van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. *Arch. Disease in Childhood*, 48, 249–255. As quoted on http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398296&section=1.1.3)) The boxplot is separated to show the birth weights of infants who survived and those that did not.

Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

UNDERSTANDING NORMAL DISTRIBUTION

Objective

Here you will learn about the Normal Distribution. You will learn what it is and why it is important, and you will begin to develop an intuition for the rarity of a value in a set by comparing it to the mean and standard deviation of the data.

If you knew that the prices of t-shirts sold in an online shopping site were **normally distributed**, and had a mean cost of \$10, with a standard deviation of \$1.50, how could that information benefit you as you are looking at various t-shirt prices on the site? How could you use what you know if you were looking to make a profit by purchasing unusually inexpensive shirts to resell at prices that are more common?

Watch This: Spread of a Normal Distribution

Watch this video in your course online.

Guidance

A **distribution** is an evaluation of the way that points in a data set are clustered or spread across their **range** of values. A **normal distribution** is a very specific symmetrical distribution that indicates, among other things, that exactly $\frac{1}{2}$ of the data is below the mean, and $\frac{1}{2}$ is above, that approximately 68% of the data is within 1, approximately 96% of the data is within 2, and approximately 99.7% is within 3 **standard deviations** of the mean.

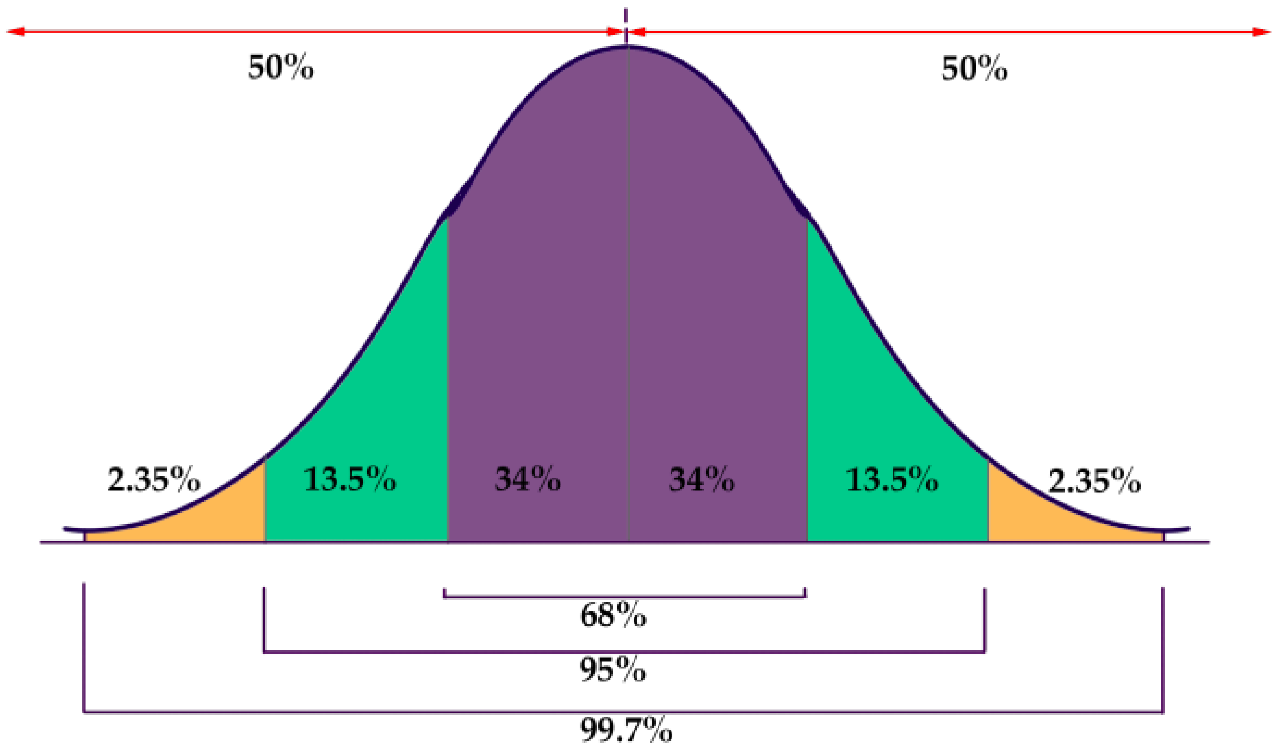
There are a number of reasons that it is important to become familiar with the normal distribution, as you will discover throughout this chapter. Examples of values associated with normal distribution:

- Physical characteristics such as height, weight, arm or leg length, etc.
- The percentile rankings of standardized testing such as the ACT and SAT
- The volume of water produced by a river on a monthly or yearly basis
- The velocity of molecules in an ideal gas

Knowing that the values in a set are exactly or approximately normally distributed allows you to get a feel for how common a particular value might be in that set. Because the values of a normal distribution are predictably clustered around the mean, you can estimate in short order the rarity of a given value in the set. In our upcoming lesson on the Empirical Rule, you will see that it is worth memorizing that normally distributed data has the characteristics mentioned above:

- 50% of all data points are above the mean and 50% are below
- Apx 68% of all data points are within 1 standard deviation of the mean
- Apx 95% of all data points are within 2 standard deviations of the mean
- Apx 99.7% of all data points are within 3 standard deviations of the mean

In this lesson, we will be practicing a 'rough estimate' of the probability that a value within a given range will occur in a particular set of data, just to develop an intuition of the use of a normal distribution. In subsequent lessons, we will become more specific with our estimates. The image below will be used in greater detail in the lesson on the Empirical Rule, but you may use it as a reference for this lesson also.



Example 1

Human height is commonly considered an approximately normally distributed measure. If the mean height of a male adult in the United States is 5′ 1″, with a standard deviation of 1.5″, how common are men with heights greater than 6′ 2″?

Solution

Since each standard deviation of this normally distributed data is 1.5″, and 6′ 2″ is 400 above the mean for the population, 6′ 2″ is nearly 3 standard deviations above the mean. That tells us that men taller than 6′ 2″ are quite rare in this population.

Example 2

If the fuel mileage of a particular model of car is normally distributed, with a mean of 26 mpg and a standard deviation of 2 mpg, how common are cars with a fuel efficiency of 24 to 25 mpg?

Solution

We know that apx 68% of the cars in the population have an efficiency of between 24 and 28 mpg, since that would be 1 SD below and 1 SD above the mean. That suggests that apx 34% have an efficiency of 24 to 26 mpg, so we can say that it is uncommon to see a car with an efficiency between 24 and 25 mpg, but not extremely so.

Example 3

If the maximum jumping height of US high school high jumpers is normally distributed with a mean of 5′ 11.5″ and a SD of 2.2″, how unusual is it to see a high school jumper clear 6′ 3″?

Solution

If the mean is 5′ 11.5″, then 1 SD above is 6′ 1.7″ and 2 SDs is 6′ 3.9″. That means that less than 2.5% of jumpers 6′ 3.9″, so it would be pretty uncommon to see a high-school competitor exceed 6′ 3″.

Intro Problem Revisited

*If you knew that the prices of t-shirts sold in an online shopping site were **normally distributed**, and had a mean cost of $10, with a standard deviation of $1.50, how could that information benefit you as you are looking at various t-shirt styles and designs on the site? How could you use what you know if you were looking to make a profit by purchasing unusually inexpensive shirts to resell at prices that are more common?*

By knowing the mean and SD of the shirt prices, and knowing that they are normally distributed, you can estimate right away if a shirt is priced at a point significantly below the norm. For instance, with this data, we can estimate that a shirt priced at $7.00 is less expensive than apx 97.5% of all shirts on the site, and could likely be resold at a profit (assuming there is not something wrong the shirt that is not obvious from the listing).

Vocabulary

Distribution: an arrangement of values of a variable showing their observed or theoretical frequency of occurrence.
Range of values of a distribution: is the difference between the least and greatest values.
Normal distribution: a very specific distribution that is symmetric about its mean. Half the values of the random variable are below the mean and half are above the mean. Approximately 68% of the data is within 1 standard deviation of the mean; aproximately 96% is within 2 SDs, and 99.7% within 3 SDs.
Standard deviation: a measure of how spread out the data is from the mean. To determine if a data value is far from the mean, determine how many standard deviations it is from the mean. The SD is calculated as the square root of the variance.

Guided Practice

Assume the data to be normally distributed, and describe the rarity of an event using the following scale:

- 0% to < 1% probability = very rare
- 1% to < 5% = rare
- 5% to < 34% = uncommon
- 34% to < 50% = common
- 50% to 100% = likely

Questions

1. If the mean (μ) of the data is 75, and the standard deviation (σ) is 5, how common is a value between 70 and 75?
2. If the μ is .02 and the σ is .005, how common is a value between .005 and .01?
3. If the μ is 1280 and the σ is 70, how common is a value between 1210 and 1350?
4. If the mean defect rate at a cellphone production plant is .1%, with a standard deviation of .03%, would it seem reasonable for a quality assurance manager to be concerned about 3 defective phones in a single 1000 unit run?

Solutions

1. A value of 70 is only 1 standard deviation below the mean, so a value between 70 and 75 would be expected approximately 34% of the time, so it would be common.
2. A value of .01 is 2 SDs below the mean, and .005 is 3 SDs below, so we would expect there to be about a 2.5% probability of a value occurring in that range. A value between 0.005 and 0.01 would be rare.
3. 1210 is 1 SD below the mean, and 1350 is 1 SD above the mean, so we would expect approximately 68% of the data to be in that range, meaning that it is likely that a value in that range would occur.
4. .1% translates into 1 per thousand, with a standard deviation of 3 per ten thousand. That means that 3 defects in the same thousand is nearly 7 SDs above the mean, well into the very rare category. While it is not impossible for random chance to result in such a value, it would certainly be prudent for the manager to investigate.

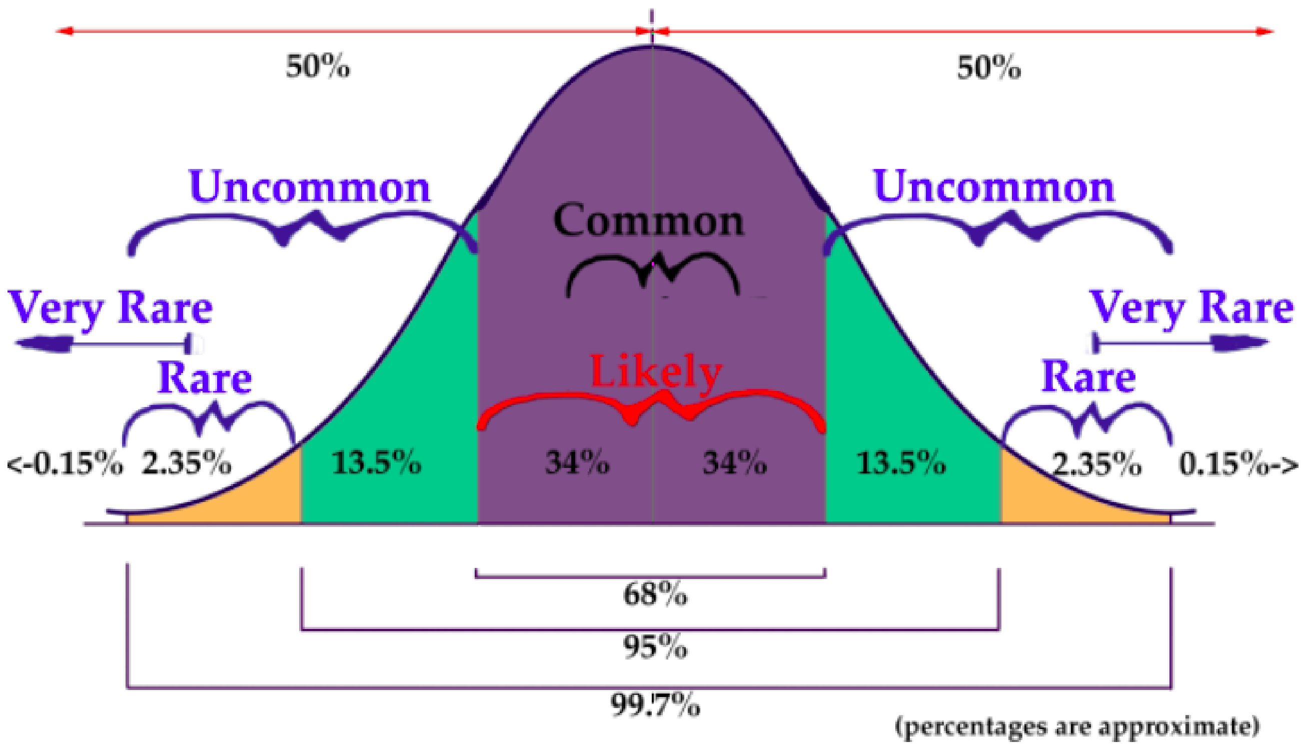
Practice Questions

Assume all sets/populations to be approximately normally distributed, and describe the rarity of an event using the following scale:

- 0% to< 1% probability = very rare

- 1% to < 5% = rare
- 5% to < 34% = uncommon
- 34% to < 50% = common
- 50% to 100% = likely.

You may reference the image below:



1. Scores on a certain standardized test have a mean of 500, and a standard deviation of 100. How common is a score between 600 and 700?
2. Considering a full-grown show-quality male Siberian Husky has a mean weight of 52.5 lbs, with SD of 7.5 lbs, how common are male huskies in the 37.5–45 lbs range?
3. A population $\mu = 125$, and $\sigma = 25$, how common are values in the 100 – 150 range?
4. Population $\mu = 0.0025$ and $\sigma = 0.0005$, how common are values between 0.0025 and 0.0030?
5. A 12 oz can of soda has a mean volume of 12 oz, with a standard deviation of .25 oz. How common are cans with between 11 and 11.5 oz of soda?
6. $\mu = 0.0025$ and $\sigma = 0.0005$, how common are values between 0.0045 and 0.005?
7. If a population $\mu = 1130$ and $\sigma = 5$, how common are values between 0 and 1100?
8. Assuming population $\mu = 1130$ and $\sigma = 5$, how common are values between 1125 and 1135?
9. The American Robin Redbreast has a mean weight of 77 g, with a standard deviation of 6 g. How common are Robins in the 59 g–71 g range?
10. Population $\mu = \frac{3}{5}$ and $\sigma = \frac{1}{10}$, how common are values between $\frac{2}{5}$ and 1?
11. Population $\mu = 0.25\%$ and $\sigma = 0.05\%$, how common are values between 0.35% and 0.45%?
12. Population $\mu = 156.5$ and $\sigma = 0.25$, how common are values between 155 and 156?

Objective

Here you will learn how to use the Empirical Rule to estimate the probability of an event.

If the price per pound of USDA Choice Beef is normally distributed with a mean of $4.85/lb and a standard deviation of $0.35/lb, what is the estimated probability that a randomly chosen sample (from a randomly chosen market) will be between $5.20 and $5.55 per pound?

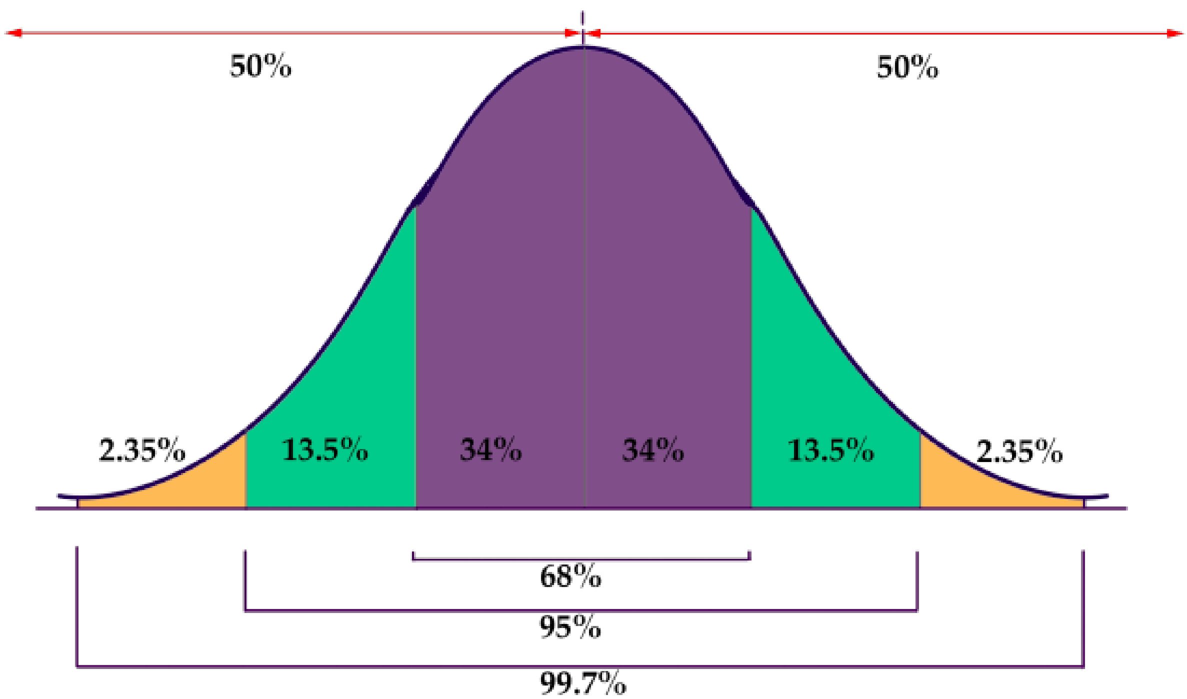
Watch This: Empirical Rule

Watch this video in your course online.

Guidance

This reading on the **Empirical Rule** is an extension of the previous reading “Understanding the Normal Distribution.” In the prior reading, the goal was to develop an intuition of the interaction between decreased probability and increased distance from the mean. In this reading, we will practice applying the Empirical Rule to estimate the specific probability of occurrence of a sample based on the range of the sample, measured in standard deviations.

The graphic below is a representation of the Empirical Rule:



The graphic is a rather concise summary of the **vital statistics** of a Normal Distribution. Note how the graph resembles a bell? Now you know why the normal distribution is also called a “ bell curve.”

- 50% of the data is above, and 50% below, the mean of the data
- Approximately 68% of the data occurs within 1 SD of the mean

- Approximately 95% occurs within 2 SD's of the mean
- Approximately 99.7% of the data occurs within 3 SDs of the mean

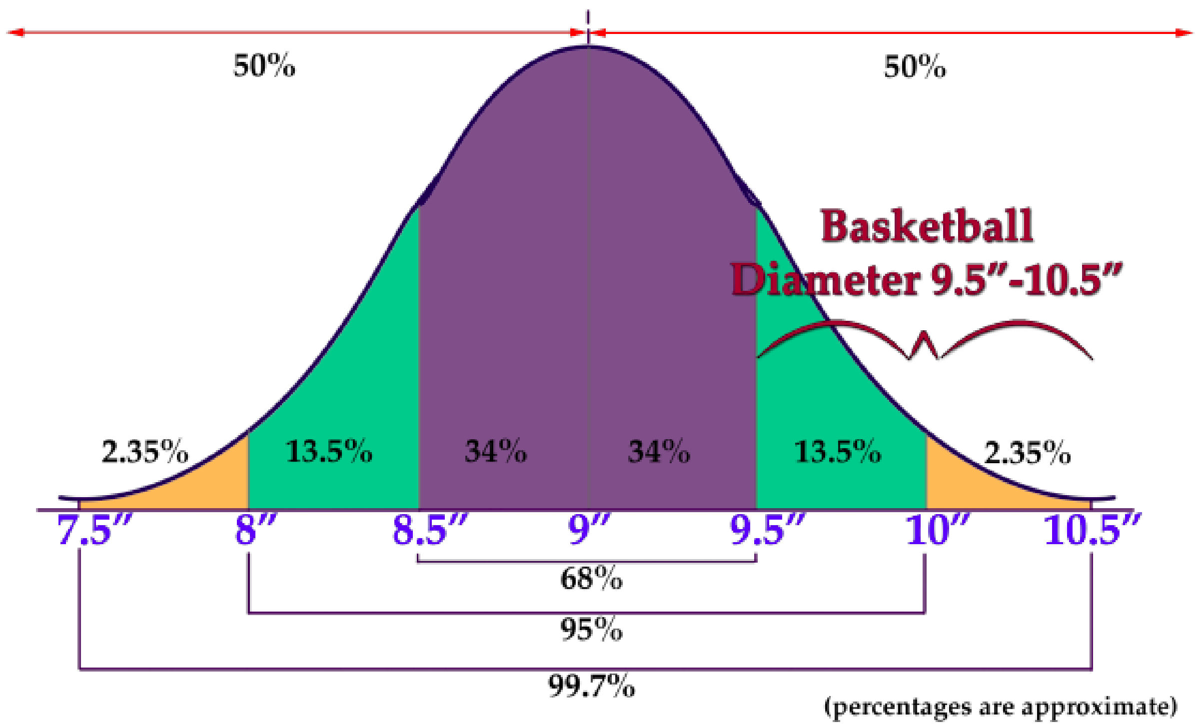
It is due to the probabilities associated with 1, 2, and 3 SDs that the Empirical Rule is also known as the **68–95–99.7 rule**.

Example 1

If the diameter of a basketball is normally distributed, with a mean (μ) of 9″, and a standard deviation (σ) of 0.5″, what is the probability that a randomly chosen basketball will have a diameter between 9.5″ and 10.5″?

Solution

Since the σ = 0.5″ and the μ = 9″, we are evaluating the probability that a randomly chosen ball will have a diameter between 1 and 3 standard deviations above the mean. The graphic below shows the portion of the normal distribution included between 1 and 3 SDs:



The percentage of the data spanning the 2nd and 3rd SDs is 13.5% + 2.35% = 15.85%

The probability that a randomly chosen basketball will have a diameter between 9.5 and 10.5 inches is 15.85%.

Example 2

If the depth of the snow in my yard is normally distributed, with μ = 2.5″ and σ = .25″, what is the probability that a randomly chosen location will have a snow depth between 2.25 and 2.75 inches?

Solution

2.25 inches is $\mu - 1\sigma$, and 2.75 inches is $\mu + 1\sigma$, so the area encompassed approximately represents 34% + 34% = 68%.

The probability that a randomly chosen location will have a depth between 2.25 and 2.75 inches is 68%.

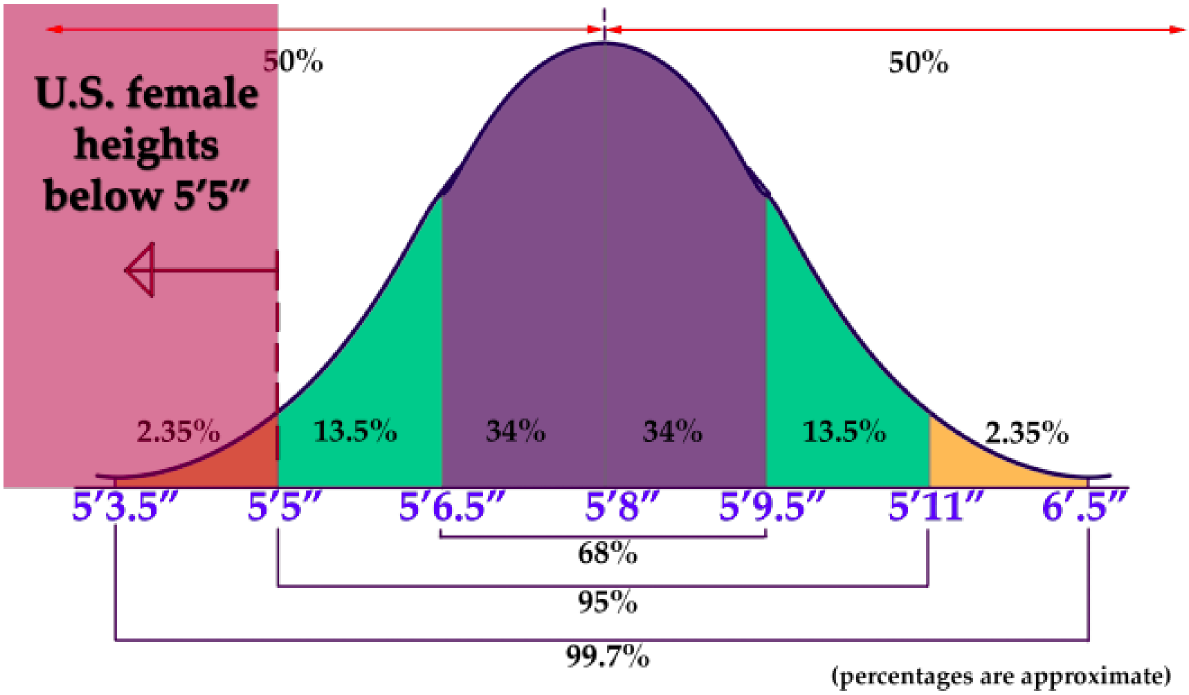
Example 3

If the height of women in the United States is normally distributed with $\mu = 5' 8''$ and $\sigma = 1.5''$, what is the probability that a randomly chosen woman in the United States is shorter than 5′ 5″?

Solution

This one is slightly different, since we aren't looking for the probability of a limited range of values. We want to evaluate the probability of a value occurring anywhere below 5′ 5″. Since the domain of a normal distribution is infinite, we can't actually state the probability of the portion of the distribution on "that end" because it has no "end"! What we need to do is add up the probabilities that we do know and subtract them from 100% to get the remainder.

Here is that normal distribution graphic again, with the height data inserted:



Recall that a normal distribution always has 50% of the data on each side of the mean. That indicates that 50% of US females are taller than 5′ 8″, and gives us a solid starting point to calculate from. There is another 34% between 5′ 6.5″ and 5′ 8″ and a final 13.5% between 5′ 5″ and 5′ 6.5″. Ultimately that totals: 50% + 34% + 13.5% = 97.5%. Since 97.5% of US females are 5′ 5″ or taller, that leaves 2.5% that are less than 5′ 5″ tall.

Intro Problem Revisited

If the price per pound of USDA Choice Beef is normally distributed with a mean of $4.85/lb and a standard deviation of $0.35/lb, what is the estimated probability that a randomly chosen sample (from a randomly chosen market) will be between $5.20 and $5.55 per pound?

$5.20 is $\mu + 1\sigma$, and $5.55 is $\mu + 2\sigma$, so the probability of a value occurring in that range is approximately 13.5%.

Vocabulary

Normal distribution: a common, but specific, distribution of data with a set of characteristics detailed in the lesson above.
Empirical Rule: a name for the way in which the normal distribution divides data by standard deviations: 68% within 1 SD, 95% within 2 SDs and 99.7 within 3 SDs of the mean
68-95-99.7 rule: another name for the Empirical Rule
Bell curve: the shape of a normal distribution

Guided Practice

1. A normally distributed data set has $\mu = 10$ and $\sigma = 2.5$, what is the probability of randomly selecting a value greater than 17.5 from the set?
2. A normally distributed data set has $\mu = .05$ and $\sigma = .01$, what is the probability of randomly choosing a value between .05 and .07 from the set?

3. A normally distributed data set has μ = 514 and an unknown standard deviation, what is the probability that a randomly selected value will be less than 514?

Solutions

1. If μ = 10 and σ = 2.5, then 17.5 = μ + 3σ. Since we are looking for all data above that point, we need to subtract the probability that a value will occur below that value from 100%: The probability that a value will be less than 10 is 50%, since 10 is the mean. There is another 34% between 10 and 12.5, another 13.5% between 12.5 and 15, and a final 2.35% between 15 and 17.5. 100% –50% –34% –13.5% –2.35% = 0.15% probability of a value greater than 17.5
2. 0.05 is the mean, and 0.07 is 2 standard deviations above the mean, so the probability of a value in that range is 34% + 13.5% = 47.5%
3. 514 is the mean, so the probability of a value less than that is 50%.

Practice Questions

Assume all distributions to be normal or approximately normal, and calculate percentages using the 68–95–99.7 rule.

1. Given mean 63 and standard deviation of 168, find the approximate percentage of the distribution that lies between –105 and 567.
2. Approximately what percent of a normal distribution is between 2 standard deviations and 3 standard deviations from the mean?
3. Given standard deviation of 74 and mean of 124, approximately what percentage of the values are greater than 198?
4. Given σ = 39 and μ = 101, approximately what percentage of the values are less than 23?
5. Given mean 92 and standard deviation 189, find the approximate percentage of the distribution that lies between –286 and 470.
6. Approximately what percent of a normal distribution lies between μ + 1σ and μ + 2σ?
7. Given standard deviation of 113 and mean 81, approximately what percentage of the values are less than –145?
8. Given mean 23 and standard deviation 157, find the approximate percentage of the distribution that lies between 23 and 337.
9. Given σ = 3 and μ = 84, approximately what percentage of the values are greater than 90?
10. Approximately what percent of a normal distribution is between μ and μ+1σ?
11. Given mean 118 and standard deviation 145, find the approximate percentage of the distribution that lies between –27 and 118.
12. Given standard deviation of 81 and mean 67, approximately what percentage of values are greater than 310?
13. Approximately what percent of a normal distribution is less than 2 standard deviations from the mean?
14. Given μ + 1σ = 247 and μ + 2σ = 428, find the approximate percentage of the distribution that lies between 66 and 428.
15. Given μ – 1σ = –131 and μ + 1σ = 233, approximately what percentage of the values are greater than –495?

Z-SCORES

Objectives

1. Here you will learn how z-scores can be used to evaluate how extreme a given value is in a particular set or
population.
2. Here you will learn to evaluate z-scores as they relate to probability.
3. Here you will learn to calculate the probability of a z-score between two others.

Part I

Using the Empirical Rule can give you a good idea of the probability of occurrence of a value that happens to be exactly one, two or three to either side of the mean, but how do you compare the probabilities of values that are in between standard deviations?

Watch This: Maths Tutorial: Z scores

The British video below is very clear and easy to follow. It is worth noting, particularly for US students, that the instructor uses the notation *x* rather than *μ* for mean, and pronounces *z* as "zed."
Watch this video online: https://youtu.be/2JjaWQZChqs

Z-scores are related to the Empirical Rule from the standpoint of being a method of evaluating how extreme a particular value is in a given set. **You can think of a z-score as the number of standard deviations there are between a given value and the mean of the set.** While the Empirical Rule allows you to associate the first three standard deviations with the percentage of data that each SD includes, the z-score allows you to state (as accurately as you like), just how many SDs a given value is above or below the mean.

Conceptually, the z-score calculation is just what you might expect, given that you are calculating the number of SDs between a value and the mean. You calculate the z-score by first calculating the difference between your value and the mean, and then dividing that amount by the standard deviation of the set. The formula looks like this:

$$\text{z-score} = \frac{(\text{value} - \text{mean})}{\text{standard deviation}} = \frac{(x = \mu)}{\sigma}$$

In this lesson, we will practice calculating the z-score for various values. In the next lesson, we will learn how to associate the z-score of a value with the probability that the value will occur.

Example 1

What is the z-score of a value of 27, given a set mean of 24, and a standard deviation of 2?

Solution

To find the z-score we need to divide the difference between the value, 27, and the mean, 24, by the standard deviation of the set, 2.

$$\text{z-score} = \frac{27 - \mu}{\sigma} = \frac{27 - 24}{2} = \frac{3}{2}$$

$$\text{z-score of } 27 = +1.5$$

This indicates that 27 is 1.5 standard deviations above the mean.

Example 2

What is the z-score of a value of 104.5, in a set with μ = 125 and σ = 6.2?

Solution

Find the difference between the given value and the mean, then divide it by the standard deviation.

$$\text{z-score} = \frac{104.5 - \mu}{\sigma} = \frac{104.5 - 125}{6.2} = \frac{-20.5}{6.2}$$

$$\text{z-score of } 104005 = -3.306$$

Note that the z-score is negative, since the measured value, 104.5, is less than (below) the mean, 125.

Example 3

Find the value represented by a z-score of 2.403, given μ = 63 and σ = 4.25.

Solution

This one requires that we solve for a missing value rather than for a missing z-score, so we just need to fill in our formula with what we know and solve for the missing value:

$$\text{z-score} = \frac{x - \mu}{\sigma}$$

$$2.403 = \frac{x - 63}{4.25}$$

$$10.213 = x - 63$$

$$73.213 = x$$

73.213 has a z-score of 2.403

Intro Problem Revisited

Using the Empirical Rule can give you a good idea of the probability of occurrence of a value that happens to be right on one of the first three standard deviations to either side of the mean, but how do you compare the probabilities of values that are in between standard deviations?

The z-score of a value is the count of the number of standard deviations between the value and the mean of the set. You can find it by subtracting the value from the mean, and dividing the result by the standard deviation.

Vocabulary

The z-score of a value is the number of standard deviations between the value and the mean of the set.

Guided Practice

1. What is the z-score of the price of a pair of skis that cost $247, if the mean ski price is $279, with a standard deviation of $16?
2. What is the z-score of a 5-scoop ice cream cone if the mean number of scoops is 3, with a standard deviation of 1 scoop?
3. What is the z-score of the weight of a cow that tips the scales at 825 lbs, if the mean weight for cows of her type is 1150 lbs, with a standard deviation of 77 lbs?
4. What is the z-score of a measured value of 0.0034, given μ = 0.0041 and σ = 0.0008?

Solutions

1. First find the difference between the measured value and the mean, then divide that difference by the standard deviation:
   $\frac{247-279}{16} = \frac{-32}{16}$
   z-score = –2
2. This one is easy: The difference between 5 scoops and 3 scoops is +2, and we divide that by the standard deviation of 1, so the z-score is +2.
3. First find the difference between the measured value and the mean, then divide that difference by the standard deviation:
   $\frac{825 \text{ lbs}-1150 \text{ lbs}}{771 \text{ lbs}} = \frac{-325}{77}$
   z-score = –4.2407
4. First find the difference between the measured value and the mean, then divide that difference by the standard deviation:
   $\frac{0.0034-0.0041}{0.0008} = \frac{-0.0007}{0.0008}$
   z-score = –0.875

Part II

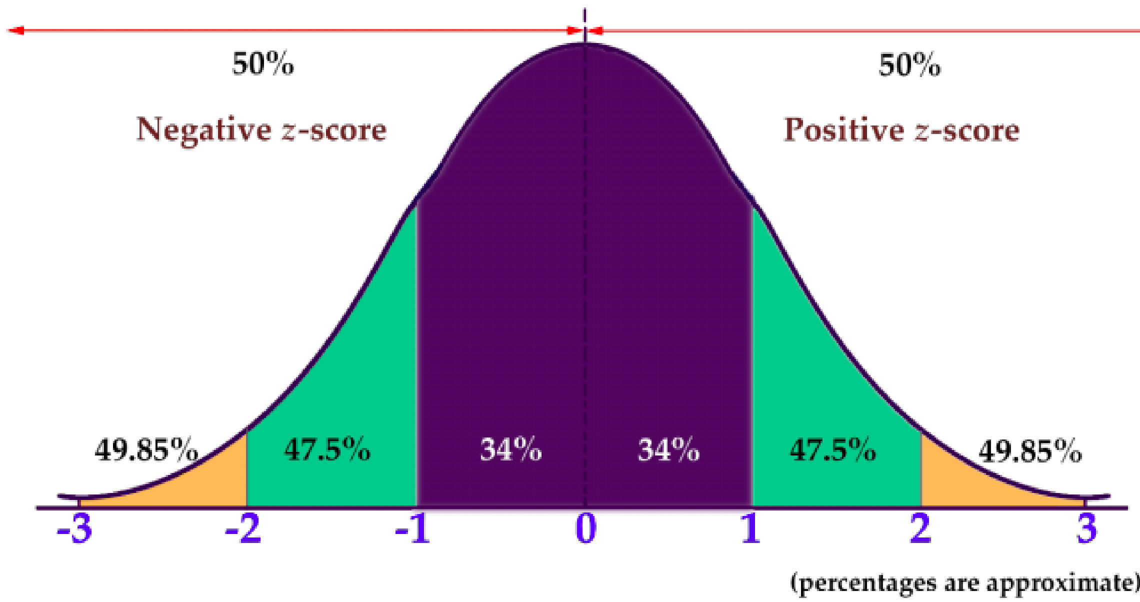
Knowing the z-score of a given value is great, but what can you do with it? How does a z-score relate to probability? What is the probability of occurrence of a z-score less than +2.47?

Watch This: The Normal Distribution

The video below provides a demonstration of how to use a z-score probability reference table, as we do in this lesson. The table he uses in the video is slightly different, but the concept is the same.
Watch this video online: https://youtu.be/rEmNUkKSpbU

Since z-scores are a measure of the number of SDs between a value and the mean, they can be used to calculate probability by comparing the location of the z-score to the area under a normal curve either to the left or right. The area can be calculated using calculus, but we will just use a table to look up the area.

I believe that the concept of comparing z-scores to probability is most easily understood with a graphic like the one we used in the lesson on the Empirical Rule, so I included one below. Be sure to review the examples to see how the scores work.



Like the graphic we viewed in the Empirical Rule lesson, this one only provides probability percentages for integer values of z-scores (standard deviations). In order to find the values for z-scores that aren't integers, you can use a table like the one below. To find the value associated with a given z-score, you find the first decimal of your z-score on the left or right side and then the 2nd decimal of your z-score across the top or bottom of the table. Where they intersect you will find the decimal expression of the percentage of values that are less than your sample (see example 4).

Table 1 (scroll to see all values)

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	Z
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359	0.0
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	0.1
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	0.2
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517	0.3
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879	0.4
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224	0.5
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	0.6
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	0.7
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	0.8
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389	0.9
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	1.0

Table 1 (scroll to see all values)

1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883	1.1
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015	1.2
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	1.3
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1.4
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1.5
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1.6
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1.7
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1.8
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767	1.9
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	2.0
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857	2.1
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989	2.2
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	2.3
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	2.4
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	2.5
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964	2.6
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974	2.7
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981	2.8
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	2.9
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999	3.0
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	3.1
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	3.2
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	3.3
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	3.4
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	3.5
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.6
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.7
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.8

Table 1 (scroll to see all values)											
3.9	1	1	1	1	1	1	1	1	1	1	3.9
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	**Z**

Z-score tables like the one above describe the probability that a given value, or any value less than it, will occur in a given set. This particular table assumes you are looking to find the probability associated with a positive z-score. You may have additional work to do if the z-score is negative.

- To find the percentage of values greater than a negative Z score, just look up the matching positive Z score value.
- To find the percentage of values less than a negative z-score, subtract the chart value from 1.
- To find the percentage of values greater than a positive z-score, subtract the chart value from 1.

Example 4

What is the probability that a value with a z-score less than 2.47 will occur in a normal distribution?

Solution

Scroll up to the table above and find “2.4” on the left or right side. Now move across the table to “0.07” on the top or bottom, and record the value in the cell: **0.9932**. That tells us that **99.32% of values in the set are at or below a z-score of 2.47**.

Example 5

What is the probability that a value with a z-score greater than 1.53 will occur in a normal distribution?

Solution

Scroll up to the table of z-score probabilities again and find the intersection between 1.5 on the left or right and 3 on the top or bottom, record the value in the cell: **0.937**.

That decimal lets us know that 93.7% of values in the set are *below* the z-score of 1.53. To find the percentage that is above that value, we subtract 0.937 from 1.0 (or 93.7% from 100%), to get **0.063** or **6.3%**.

Example 6

What is the probability of a random selection being less than 3.65, given a normal distribution with μ = 5 and σ = 2.2?

Solution

This question requires us to first find the z-score for the value 3.65, then calculate the percentage of values below
that z-score from a reference.

1. Find the z-score for 3.65, using the z-score formula: $\frac{(x-\mu)}{\sigma} = \frac{3.65-5}{2.2} = \frac{-1.35}{2.2} \approx -0.61$
2. Now we can scroll up to our z-score reference above and find the intersection of 0.6 and 0.01, which should be .7291.

3. Since this is a negative z-score, and we want the percentage of values below it, we subtract that decimal from 1.0 (reference the three steps highlighted by bullet points below the chart if you didn't recall this), to get 1 – .7291 = .2709

There is approximately a 27.09% probability that a value less than 3.65 would occur from a random selection
of a normal distribution with mean 5 and standard deviation 2. 2.

Concept Problem Revisited

Knowing the z-score of a given value is great, but what can you do with it? How does a z-score relate to probability? What is the probability of occurrence of a z-score less than 2.47?

A z-score lets you calculate the probability that a randomly selected value will be greater or less than a particular value in a set.

To find the probability of a z-score below +2.47, using a reference such as the table in the lesson above:

1. Find 2.4 on the left or right side
2. Move across to 0.07 on the top or bottom.
3. The cell you arrive at says: 0.9932, which means that apx 99.32% of the values in a normal distribution will occur below a z-score of 2.47.

Vocabulary

Z-score table: a table that associates the various common z-scores between 0 and 3.99 with the decimal probability of being less than or equal to that z-score.

Guided Practice

1. What is the probability of occurrence of a value with z-score greater than 1.24?
2. What is the probability of $z < -.23$?
3. What is $P(Z < 2.13)$?

Solutions

1. Since this is a positive z-score, we can use the value for $z = 1.24$ directly from the table, and just express it as a percentage: 0.8925 or 89.25%
2. This is a negative z-score, and we want the percentage of values greater than it, so we need to subtract the value for $z = +0.23$ from 1: 1 – 0.591 = .409 or 40.9%
3. This is a positive z-score, and we need the percentage of values below it, so we can use the percentage associated with $z = +2.13$ directly from the table: 0.9834 or 98.34%

Part III

Do z-score probabilities always need to be calculated as the chance of a value either above or below a given score? How would you calculate the probability of a z-score between –0.08 and +1.92?

Watch This: Reading Probabilities from the Z-table

Watch this video online: https://youtu.be/tnVr1Qbk4YU

To calculate the probability of getting a value with a z-score between two other z-scores, you can either use a reference table to look up the value for both scores and subtract them to find the difference, or you can use technology. In this lesson, which is an extension of Z-scores and Z-scores II, we will practice both methods.

Historically, it has been very common to use a z-score probability table like the one below to look up the probability associated with a given z-score:

Table 2 (scroll to see all values)

Z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**Z**
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359	0.0
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	0.1
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	0.2
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517	0.3
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879	0.4
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224	0.5
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	0.6
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	0.7
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	0.8
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389	0.9
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	1.0
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883	1.1
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015	1.2
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	1.3
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1.4
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1.5
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1.6
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1.7
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1.8
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767	1.9
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	2.0
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857	2.1

Table 2 (scroll to see all values)											
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989	2.2
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	2.3
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	2.4
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	2.5
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964	2.6
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974	2.7
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981	2.8
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	2.9
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999	3.0
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	3.1
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	3.2
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	3.3
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	3.4
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	3.5
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.6
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.7
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.8
3.9	1	1	1	1	1	1	1	1	1	1	3.9
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	**Z**

Since the proliferation of the Internet, however, you can also use a free online calculator such as one of these three:

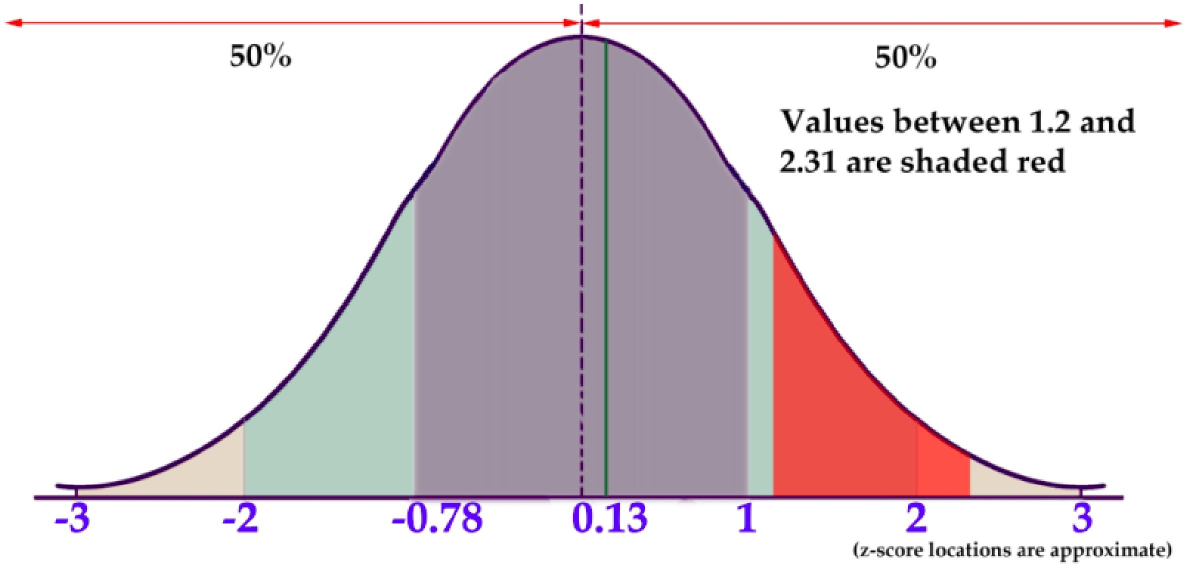
- z-Score Calculator
- z-Score to p-Value Calculator
- z-Score Calculator

Example 7

What is the probability associated with a z-score between 1.2 and 2.31?

Solution

To evaluate the probability of a value occurring within a given range, you need to find the probability of both the upper and lower values in the range, and subtract to find the difference.



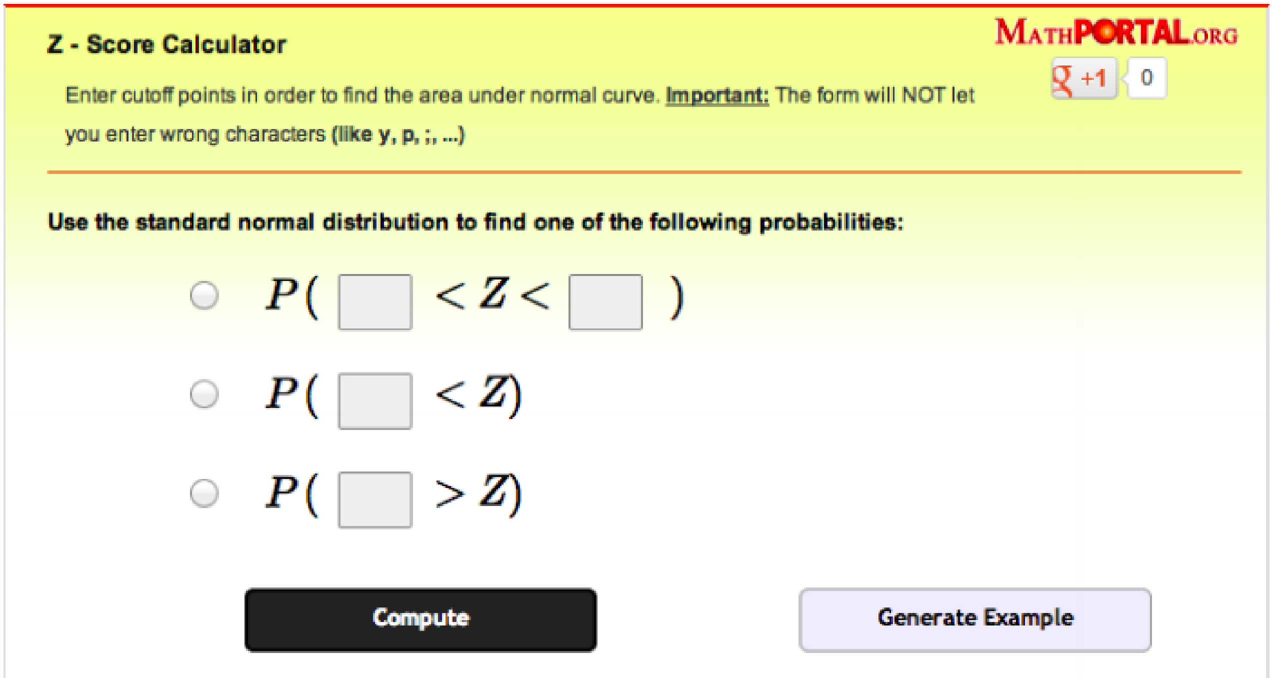
- First find z = 1.2 on the z-score probability reference above: .8849. Remember that value represents the percentage of values below 1.2.
- Next, find and record the value associated with z = 2.31: .9896
- Since approximately 88.49% of all values are below z = 1.2 and approximately 98.96% of all values are below z = 2.31, there are 98.96% –88.49% = 10.47% of values between.

Example 8

What is the probability that a value with a z-score between -1.32 and +1.49 will occur in a normal distribution?

Solution

Let's use the online calculator at mathportal.org for this one. When you open the page, you should see a window like this:



All you need to do is select the radio button to the left of the first type of probability, input "–1.32" into the first box, and 1.49 into the second. When you click **Compute**, you should get the result $P(-1.32 < Z < 1.49) = 0.8385$

Which tells us that there is approximately and 83.85% probability that a value with a z-score between 1.32 and 1.49 will occur in a normal distribution.

Notice that the calculator also details the steps involved with finding the answer:

1. Estimate the probability using a graph, so you have an idea of what your answer should be.
2. Find the probability of $z < 1.49$, using a reference. (0.9319)
3. Find the probability of $z < -1.32$, again, using a reference. (0.0934)
4. Subtract the values: 0.9319 – 0.0934 = 0.8385 or 83.85%

Example 9

What is the probability that a random selection will be between 8.45 and 10.25, if it is from a normal distribution with $\mu = 10$ and $\sigma = 2$?

Solution

This question requires us to first find the z-scores for the value 8.45 and 10.25, then calculate the percentage of value between them by using values from a z-score reference and finding the difference.

1. Find the z-score for 8.45, using the z-score formula: $\frac{(x-\mu)}{\sigma}$

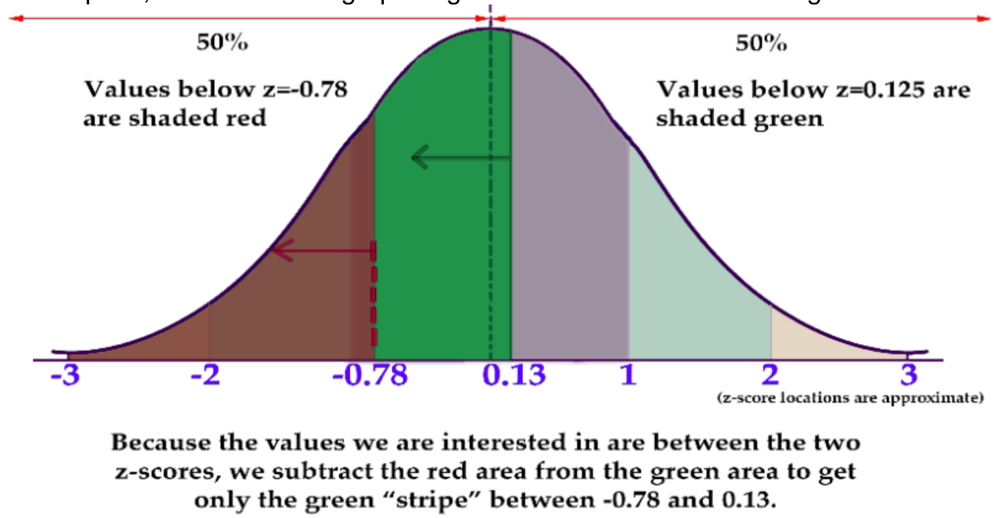
   $\frac{8.45-10}{2} = \frac{-1.55}{2} \approx -0.78$

2. Find the z-score for 10.25 the same way:
   $\frac{10.25-10}{2} = \frac{0.25}{2} \approx .13$

3. Now find the percentages for each, using a reference (don't forget we want the probability of values less than our negative score and less than our positive score, so we can find the values between):

$P(Z < -0.78) = .2177$ or 21.77%
$P(Z < .13) = .5517$ or 55.17%

4. At this point, let's sketch the graph to get an idea what we are looking for:



5. Finally, subtract the values to find the difference:
   $.5517 - .2177 = .3340$ *or about* 33.4%
   There is approximately a 33.4% probability that a value between 8.45 and 10.25 would result from a random selection of a normal distribution with mean 10 and standard deviation 2.

Concept Problem Revisited

Do z-score probabilities always need to be calculated as the chance of a value either above or below a given score? How would you calculate the probability of a z-score between –0.08 and +1.92?

After this lesson, you should know without question that z-score probabilities do not need to assume only probabilities above or below a given value, the probability between values can also be calculated.

The probability of a z-score below –0.08 is 46.81%, and the probability of a z-score below 1.92 is 97.26%, so the probability between them is 97.26%–46.81% = 50.45%.

Vocabulary

z-score: a measure of how many standard deviations there are between a data value and the mean.
z-score probability table: a table that associates z-scores to area under the normal curve. The table may be used to associate a Z-score with a percent probability.

Guided Practice

1. What is the probability of a z-score between –0.93 and 2.11?
2. What is $P(1.39 < Z < 2.03)$?
3. What is $P(-2.11 < Z < 2.11)$?

Solutions

1. Using the z-score probability table above, we can see that the probability of a value below –0.93 is .1762, and the probability of a value below 2.11 is .9826. Therefore, the probability of a value

between them is .9826–.1762 = .8064 or 80.64%
2. Using the z-score probability table, we see that the probability of a value below z = 1.39 is .9177, and a value below z = 2.03 is .9788. That means that the probability of a value between them is .9788 – .9177 = .0611 or 6.11%
3. Using the online calculator at mathportal.org, we select the top calculation with the associated radio button to the left of it, enter "–2.11" in the first box, and "2.11" in the second box. Click **Compute** to get " .9652," and convert to a percentage. The probability of a z-score between –2.11 and +2.11 is about 96.52%.

Practice Questions

1. Given a distribution with a mean of 70 and standard deviation of 62, find a value with a z-score of –1.82.
2. What does a z-score of 3.4 mean?
3. Given a distribution with a mean of 60 and standard deviation of 98, find the z-score of 120.76.
4. Given a distribution with a mean of 60 and standard deviation of 21, find a value with a z-score of 2.19.
5. Find the z-score of 187.37, given a distribution with a mean of 185 and standard deviation of 1.
6. What does a z-score of –3.8 mean?
7. Find the z-score of 125.18, given a distribution with a mean of 101 and standard deviation of 62.
8. Given a distribution with a mean of 117 and standard deviation of 42, find a value with a z-score of –0.94.
9. Given a distribution with a mean of 126 and standard deviation of 100, find a value with a z-score of –0.75.
10. Find the z-score of 264.16, given $\mu = 188$ and $\sigma = 64$.
11. Find a value with a z-score of –0.2, given $\mu = 145$ and $\sigma = 56$.
12. Find the z-score of 89.79 given $\mu = 10$ and $\sigma = 79$.

Find the probabilities, use the table from the lesson above.

1. What is the probability of a z-score less than +2.02?
2. What is the probability of a z-score greater than +2.02?
3. What is the probability of a z-score less than –1.97?
4. What is the probability of a z-score greater than –1.97?
5. What is the probability of a z-score less than +0.09?
6. What is the probability of a z-score less than –0.02?
7. What is $P(Z < 1.71)$?
8. What is $P(Z > 2.22)$?
9. What is $P(Z < -1.19)$?
10. What is $P(Z > -2.71)$?
11. What is $P(Z < 3.71)$?
12. What is the probability of the random occurrence of a value greater than 56 from a normally distributed population with mean 62 and standard deviation 4.5?
13. What is the probability of a value of 329 or greater, assuming a normally distributed set with mean 290 and standard deviation 32?
14. What is the probability of getting a value below 1.2 from the random output of a normally distributed set with $\mu = 2.6$ and $\sigma = .9$?

Find the probabilities, use the table from the lesson or an online resource.

1. What is the probability of a z-score between +1.99 and +2.02?
2. What is the probability of a z-score between –1.99 and +2.02?
3. What is the probability of a z-score between –1.20 and –1.97?
4. What is the probability of a z-score between +2.33 and –0.97?
5. What is the probability of a z-score greater than +0.09?
6. What is the probability of a z-score greater than –0.02?
7. What is $P(1.42 < Z < 2.01)$?
8. What is $P(1.77 < Z < 2.22)$?
9. What is $P(-2.33 < Z < -1.19)$?
10. What is $P(-3.01 < Z < -0.71)$?
11. What is $P(2.66 < Z < 3.71)$?
12. What is the probability of the random occurrence of a value between 56 and 61 from a normally distributed population with mean 62 and standard deviation 4.5?

13. What is the probability of a value between 301 and 329, assuming a normally distributed set with mean 290 and standard deviation 32?
14. What is the probability of getting

REGRESSION AND CORRELATION

Objective

- Here you will learn about pairs of variables that are related in a linear fashion, including those with values occurring in a slightly random manner.
- Here you will learn to calculate the linear correlation coefficient, and how to use it to describe the relationship between an explanatory and response variable.

Linear Relationships

Imagine walking through the electronics section of your local department store. On the wall are examples of dozens of television sets, from little 1900 units made to sit on a kitchen counter to 72″+ monsters meant to be the centerpiece of a home theatre. Looking at the prices, you note without surprise that the 72″ model is more expensive than the 19″, and a 42″ model is priced in between. It seems rather clear that as the TV gets larger, the price goes up. Does that mean increased screen size causes increased price?

Look to the end of the section for the answer.

Watch This: Exploring Linear Relationships

Watch this video online: https://youtu.be/qPx7i1jwXX4

When two quantities are compared, it is not uncommon to note a relationship between them that indicates both quantities increase and decrease at the same time, or that one increases as the other decreases. If both quantities are plotted on coordinate axes, the data points show a general or definite linear trend.

If the points actually form a clearly defined line, the variables may be an example of a **deterministic** relationship. A deterministic relationship indicates that the value of one variable can be reliably and accurately determined by the manipulation of the other variable. An example might be inches and centimeters: one inch is the same as 2.54 centimeters. If you know how many inches long something is, you can reliably and accurately calculate the number of centimeters long the same item is.

As you likely recall from Algebra, the **slope** describes the angle of the line created by plotting points from a linear relationship, and the point where the **explanatory variable** has a value of zero is called the y-intercept (commonly denoted b).

Often, particularly in research situations when one or both variables are measured, the plotted values are generally linear, but do not line up precisely. When two variables seem to show a linear relationship, but the values display some amount of randomness, we commonly visually describe the relationship with a **scatter plot**. As you will see throughout this chapter, the strength of the linear relationship of the variables can be described through mathematics.

Example 1

Given the equation $y = 2.3x + 5$:

1. Create an x–y table to describe the values of at least four points
2. What is the slope of the line?

3. What is the y-intercept?

Solution

1. Pick a value for x, substitute the chosen value for x in the equation, and calculate y:

   Table 1

x	calculation	y
1	$y = 2.3(1) + 5$	7.3
2	$y = 2.3(2) + 5$	9.6
0	$y = 2.3(0) + 5$	5
–1	$y = 2.3(–1) + 5$	2.7

   The equation in the problem is in $y = mx + b$ form (also known as **slope-intercept form**), where b is the y-value when $x = 0$, and m is the slope of the line.
2. $m = 2.3$
3. $b = 5$

Example 2

Given the equation $y = –3x + 3.9$:

1. Create an x–y table to describe the values of at least four points
2. What is the slope of the line?
3. What is the y-intercept?

Solution

1. Pick a value for x, substitute the chosen value for x in the equation, and calculate y:

   Table 2

x	calculation	y
1	$y = –3(1) + 3.9$	6.9
2	$y = –3(2) + 3.9$	9.9
0	$y = –3(0) + 3.9$	3.9
–1	$y = –3(–1) + 3.9$	0.9

   The equation in the problem is in $y = mx + b$ form (also known as **slope-intercept form**), where b is the y-value when $x = 0$, and m is the slope of the line.
2. $m = –3$
3. $b = 3.9$

Example 3

Given the equation $y = -2.8x - 9.1$:

1. Create an x–y table to describe the values of at least four points.
2. What is the slope of the line?
3. What is the y-intercept?

Solution

1. Pick a value for x, substitute the chosen value for x in the equation, and calculate y:

Table 3

x	**calculation**	***y***
1	$y = -2.8(1) - 9.1$	–11.9
2	$y = -2.8(2) - 9.1$	–14.7
0	$y = -2.8(0) - 9.1$	–9.1
–1	$y = -2.8(-1) - 9.1$	6.3

   The equation in the problem is in $y = mx + b$ form (also known as **slope-intercept form**), where b is the y-value when $x = 0$, and m is the slope of the line.
2. $m = -2.8$
3. $b = -9.1$

Intro Problem Revisited

Imagine walking through the electronics section of your local department store. On the wall are examples of dozens of television sets, from little 1900 units made to sit on a kitchen counter to 72″+ monsters meant to be the centerpiece of a home theatre. Looking at the prices, you note without surprise that the 72″ model is more expensive than the 19″, and a 42″ model is priced in between. It seems rather clear that as the TV gets larger, the price goes up. Does that mean increased screen size causes increased price?

No, it does not. This is an example of the difficulty associated with examining linear relationships. **Correlation does not imply causation.** Just because a pair of variables exhibit a relationship, linear or otherwise, does not mean that one variable causes changes in the other variable.

Vocabulary

Cartesian Graph: a “plus-shaped” graph, with the *explanatory variable* (the input value) plotted horizontally on the x-axis, and the *response variable* (the output value) plotted vertically on the y-axis.
Deterministic linear relationship: a relationship that plots a reliably straight and accurate single line.
Slope of a line: (commonly denoted m) describes the angle of a plotted line on a graph.
Scatter plot: a graph of individual points on an x–y graph.

Guided Practice

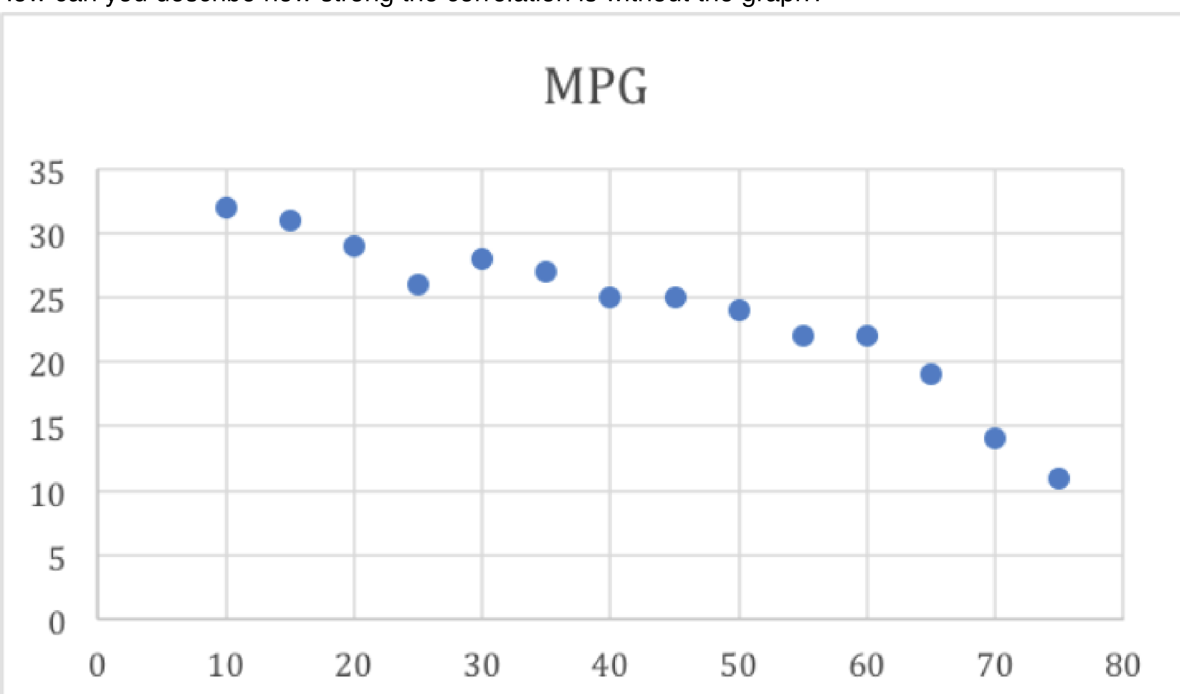
1. If a linear graph exhibits a positive slope, what can you predict will happen to the response variable as the explanatory variable increases?
2. If a linear graph has no slope, what does that mean?
3. Given the linear equation $2y = 5.2x + 7$:
    1. What is the slope?
    2. What is the y-interept?
    3. What happens to y as x increases?
4. Given the equation $y = 2x^2 + 4$:
    1. Is this a linear equation? Why or why not?
    2. Does this equation represent a relationship?

Solutions

1. A positive slope indicates that the variables increase and decrease together.
2. A line with no slope is a horizontal line, since the only defined variable is the output. No matter what value is given for the explanatory variable, the response is the same.
3. Answers:
    1. The slope, m, is 5.2.
    2. The y-intercept, b, is 7.
    3. Since this line has a positive slope, y increases as x increases.
4. Answers:
    1. No, the explanatory variable is squared, this graph would form a parabola.
    2. Yes! It is just not a **linear** relationship.

Linear Correlation Coefficient

Suppose you have noted that your car seems to use more gas when you drive fast than when you drive more slowly. You decide to see how strong the relationship is, so you do some research, collect the data, and plot the data on the graph below, where the explanatory variable x is mph, and the response variable y is mpg. How can you describe how strong the correlation is without the graph?



Watch This: The Correlation Coefficient and Coefficient of Determination

Watch this video online: https://youtu.be/dCgIavyFWIo

The linear correlation coefficient (sometimes called Pearson's Correlation Coefficient), commonly denoted *r*, is a measure of the strength of the linear relationship between two variables. The value of r has the following properties:

- *r* is always a value between −1 and +1
- The further an *r* value is from zero, the stronger the relationship between the two variables.
- The sign of *r* indicates the nature of the relationship: A positive *r* indicates a positive relationship, and a negative *r* indicates a negative relationship.

Generally speaking, you may think of the values of r in the following manner:

- If $|r|$ is between 0.85 and 1, there is a strong correlation.
- If $|r|$ is between 0.5 and 0.85, there is a moderate correlation.
- If $|r|$ is between 0.1 and 0.5, there is a weak correlation.
- If $|r|$ is less than 0.1, there is no apparent correlation.

Naturally, *r*-value can be calculated, but the formula is a bit beyond the scope of this course. Fortunately, there are many excellent and free online calculators for determining the r-value of a set of data. In this reading, I will be using the one at easycalculation.com, but a search for "correlation calculator online" will yield the most current options.

At the risk of overloading you with new terms, there is one more that I think it is worth learning in this reading, the **coefficient of determination**. The coefficient of determination is very simple to calculate if you know the correlation coefficient, since it is just r^2. The reason I mention it is that the coefficient of determination can be interpreted as the percentage of variation of the *y* variable that can be attributed to the relationship. In other words, a value of $r^2 = .63$ can be interpreted as "63% of the changes between one *y* value and another can be attributed to *y*'s relationship with *x*."

Example 4

Elaina is curious about the relationship between the weight of a dog and the amount of food it eats. Specifically, she wonders if heavier dogs eat more food, or if age and size factor in. She works at the Humane Society, and does some research. After some calculation, she determines that dog weight and food weight exhibit an *r*-value of 0.73.

What can Elaina say about the relationship, based on her research? What percentage of the increases in food intake can she attribute to weight, according to her research?

Solution

The calculated *r*-value of 0.73 tells us that Elaina's data demonstrates a moderate to strong correlation between the variables.

Since the coefficient of determination tells us the percentage of changes in the output variable that can be attributed to the input variable, we need to calculate r^2:

$r^2 = (0.73)^2 = .5329$

Approximately 53% of increases in food intake can be attributed to the linear relationship between food intake and the weight of the dog, suggesting that other factors, perhaps age and size, are also involved.

Example 5

Tuscany wonders if barrel racing times are related to the age of the horse. Specifically, she wonders if older horses take longer to complete a barrel racing run. As a member of the Pony Club, she does some research, and determines that horse age to barrel run time exhibits an *r*-value of 0.52.

What can Tuscany say about horse age vs barrel race time, according to her research?

Solution

Tuscany's research suggests that there is a moderate to weak correlation between horse age and barrel run time. In other words, the research suggests that $(0.52)^2 = .27 = 27\%$ of the differences between barrel run times could be attributable to the linear relationship between barrel run time and the age of the horse.

Example 6

Sayber has collected the following data regarding player score vs age in his favorite online game. He suspects that increased age is not a good indicator of gaming ability. What are the linear correlation coefficient and coefficient of determination values of his data, and how do they support or not support Sayber's hypothesis?

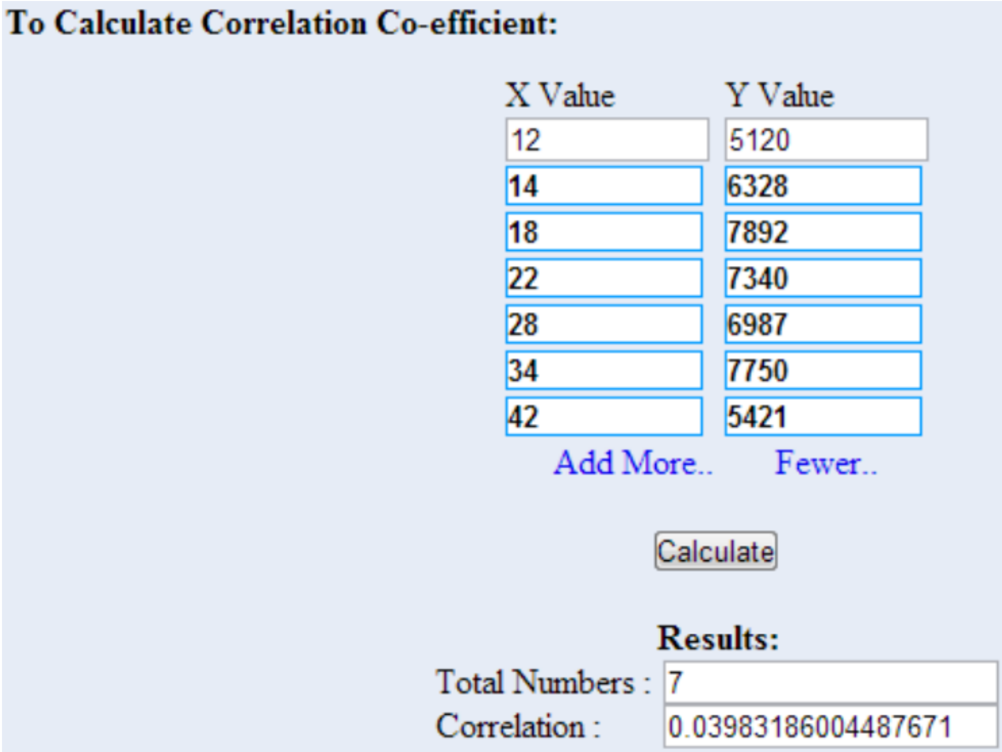
Table 4

Age	**Avg. Player Score**
12	5,120
14	6,328
18	7,892
22	7,340
28	6,987
34	7,750
42	5,421

Solution

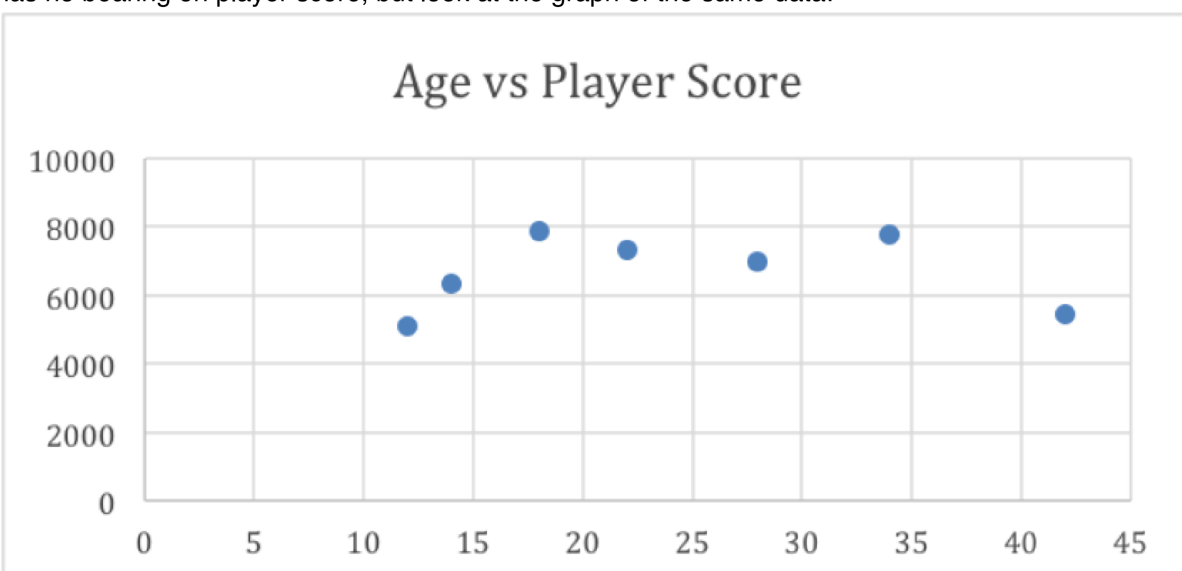
Let's use the online calculator at easycalculation.com for this one.

I entered the explanatory (Age) and response (Player Score) values into the calculator:



The linear correlation coefficient of approximately 0.04 suggests that there is no appreciable linear correlation. The coefficient of determination of 0.0016 suggests that perhaps 0.16% (practically none) of the variability of the player score is dependent on age.

Looking at the scores, however, something seems a miss with our findings. The scores suggest that age has no bearing on player score, but look at the graph of the same data:

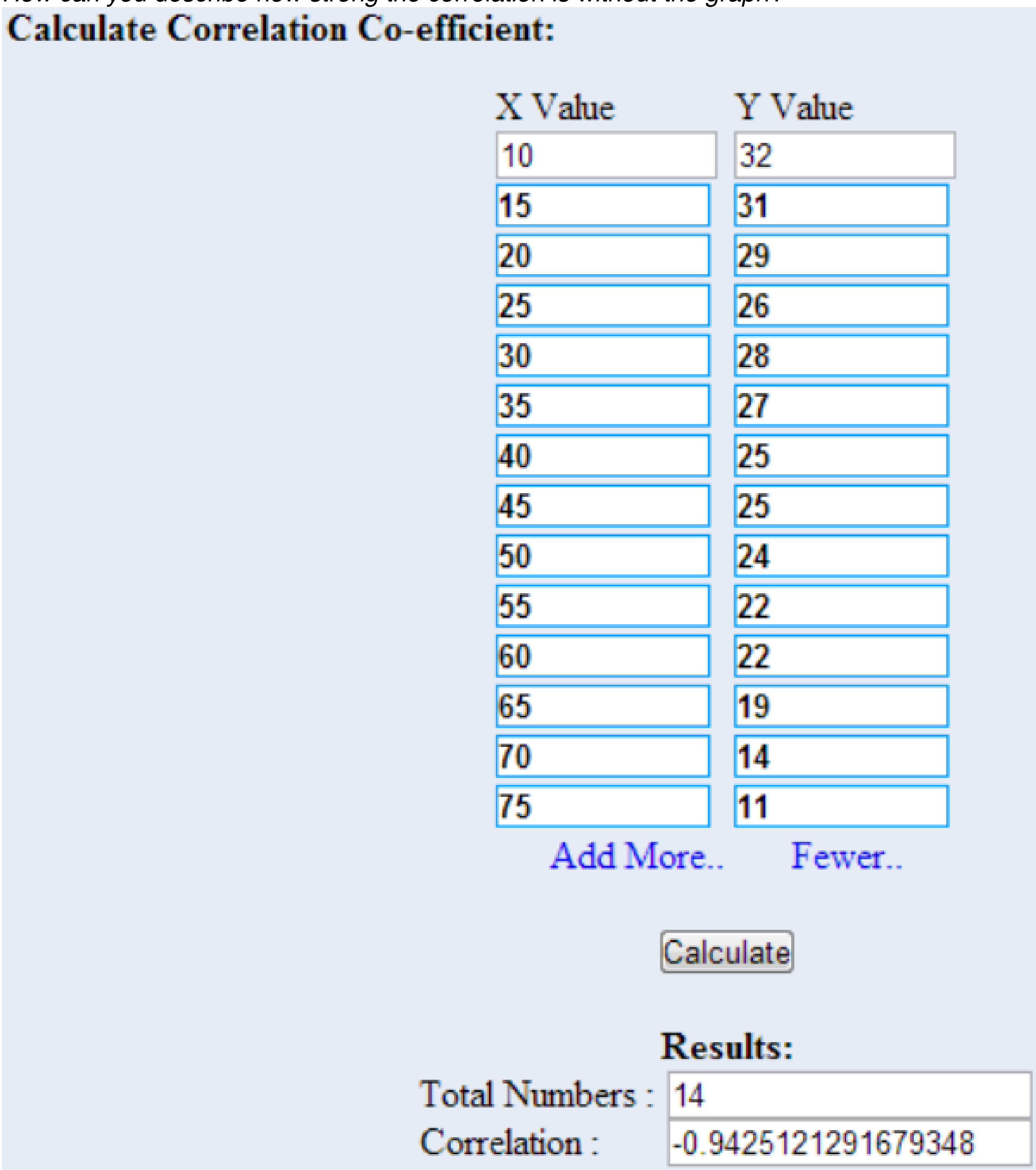


The graph suggests that the youngest and oldest polled players score less than players in late teens to mid-thirties, which seems reasonable.

This is an important example of the weakness of using just one indicator of the relationship between two variables. As I noted early in the reading, the r-value is only an indicator of linear correlation, it says nothing at all about other kinds of variable relationships. It is always a good idea to review your data in different ways to evaluate your initial conclusions.

Intro Problem Revisited

Suppose you have noted that your car seems to use more gas when you drive fast than when you drive more slowly. You decide to see how strong the relationship is, so you do some research, collect the data, and plot the data on the graph below, where the explanatory variable x is mph, and the response variable y is mpg. How can you describe how strong the correlation is without the graph?



After the reading above, we know that the r-value or r^2-value of the relationship between MPG and MPH would describe the strength of the linear relationship in a single value.

By taking the data points detailed on the graph (in practice, of course, I would have had them in table format already, since I would have needed them to build the graph in the first place), and entering them into a free linear coefficient calculator online, I get an *r*-value of –0.943, indicating a strong negative relationship. This also translates into an r^2-value of $(-0.943)^2 = 0.89$, indicating that the research suggests that approximately 89% of the decrease in MPG from left to right across the graph can be attributed to the increase in MPH.

Vocabulary

Linear correlation coefficient or ***r*-value of a relationship:** describes the strength of the linear relationship.

Coefficient of determination or ***r^2*-value of a relationship:** indicates the approximate percentage of variation in the response variable that can be attributed to the linear relationship between the response and explanatory variables, according to the data presented.

Guided Practice

1. What can you say about the strength of a linear relationship with an *r*-value of –0.87?
2. What can you say about the level of negative correlation of a relationship if you know the coefficient of determination is 0.82?
3. How much of the variability of *y* is attributable to x in a relationship with an *r*-value of 0.76?

Solutions

1. An $|r|$ of > 0.85 indicates a strong linear relationship. The fact that r is negative indicates that as x increases, y decreases.
2. Nothing! The coefficient of determination is r^2, and therefore always positive. We know that $|r|$= $\sqrt{.82} \approx .91$, so this is a strong linear correlation, but we have no idea if it is positive or negative.
3. The coefficient of determination describes the variation in y attributable to *x*, so we need to find r^2: (0.76)2 = .5776. Approximately 57.76% of the change in y-values can be attributed to the change in *x*.

Practice Questions

For the following questions, find the *x* and *y* intercepts of the given equations.

1. $-x + 4y = 8$
2. $3x + 5y = 15$
3. $-3x + 4y = 36$
4. $-8x + 5y = 40$
5. $5x - 6y = -30$
6. $-9x - 3y = -54$
7. $-x + 5y = -10$
8. $-3x + 8y = -72$

For the following questions, graph the line.

1. $x + 3y = 2$
2. $m = -4$, $b = \frac{4}{3}$
3. *x*-intercept = –1, *y*-intercept = 2
4. $y = -4x + 2$
5. $m = -1$, $b = \frac{1}{2}$
6. $x + 2y = 5$
7. $-3x + 2y = -3$

For the following questions, describe the relationship based on the r-value.

1. $r = 0$
2. $r = 0.91$
3. $r = -0.49$
4. $r = 0.05$
5. $r = 1$

For the following questions, describe the relationship based on the coefficient of determination:

1. $r^2 = 0.82$
2. $r^2 = 0.15$
3. $r^2 = 0.47$
4. $r^2 = 1$
5. $r^2 = 0$

The following questions refer to the data in the following table:

Table 5	
x	***y***
5	70
7	69
13	58
22	47
36	36
38	25
45	14

1. What is the linear correlation coefficient of the data?
2. What does r tell you about the relationship?
3. What is the r^2 value of the data?
4. What does the coefficient of determination tell you about this relationship?
5. What would a graph of the data look like?

SUPPLEMENTAL VIDEOS

This YouTube playlist contains several videos that supplement the reading in this unit. You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

EXERCISES

Skills

1. A political scientist surveys 28 of the current 106 representatives in a state's congress. Of them, 14 said they were supporting a new education bill, 12 said there were not supporting the bill, and 2 were undecided.
    a. What is the population of this survey?
    b. What is the size of the population?
    c. What is the size of the sample?
    d. Give the sample statistic for the proportion of voters surveyed who said they were supporting the education bill.
    e. Based on this sample, we might expect how many of the representatives to support the education bill?
2. The city of Raleigh has 9500 registered voters. There are two candidates for city council in an upcoming election: Brown and Feliz. The day before the election, a telephone poll of 350 randomly selected registered voters was conducted. 112 said they'd vote for Brown, 207 said they'd vote for Feliz, and 31 were undecided.
    a. What is the population of this survey?
    b. What is the size of the population?
    c. What is the size of the sample?
    d. Give the sample statistic for the proportion of voters surveyed who said they'd vote for Brown.
    e. Based on this sample, we might expect how many of the 9500 voters to vote for Brown?
3. Identify the most relevant source of bias in this situation: A survey asks the following: Should the mall prohibit loud and annoying rock music in clothing stores catering to teenagers?
4. Identify the most relevant source of bias in this situation: To determine opinions on voter support for a downtown renovation project, a surveyor randomly questions people working in downtown businesses.
5. Identify the most relevant source of bias in this situation: A survey asks people to report their actual income and the income they reported on their IRS tax form.
6. Identify the most relevant source of bias in this situation: A survey randomly calls people from the phone book and asks them to answer a long series of questions.
7. Identify the most relevant source of bias in this situation: A survey asks the following: Should the death penalty be permitted if innocent people might die?
8. Identify the most relevant source of bias in this situation: A study seeks to investigate whether a new pain medication is safe to market to the public. They test by randomly selecting 300 men from a set of volunteers.
9. In a study, you ask the subjects their age in years. Is this data qualitative or quantitative?
10. In a study, you ask the subjects their gender. Is this data qualitative or quantitative?

11. Does this describe an observational study or an experiment: The temperature on randomly selected days throughout the year was measured.
12. Does this describe an observational study or an experiment? A group of students are told to listen to music while taking a test and their results are compared to a group not listening to music.
13. In a study, the sample is chosen by separating all cars by size, and selecting 10 of each size grouping. What is the sampling method?
14. In a study, the sample is chosen by writing everyone's name on a playing card, shuffling the deck, then choosing the top 20 cards. What is the sampling method?
15. A team of researchers is testing the effectiveness of a new HPV vaccine. They randomly divide the subjects into two groups. Group 1 receives new HPV vaccine, and Group 2 receives the existing HPV vaccine. The patients in the study do not know which group they are in.
    a. Which is the treatment group?
    b. Which is the control group (if there is one)?
    c. Is this study blind, double-blind, or neither?
    d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?
16. For the clinical trials of a weight loss drug containing *Garcinia cambogia t*he subjects were randomly divided into two groups. The first received an inert pill along with an exercise and diet plan, while the second received the test medicine along with the same exercise and diet plan. The patients do not know which group they are in, nor do the fitness and nutrition advisors.
    a. Which is the treatment group?
    b. Which is the control group (if there is one)?
    c. Is this study blind, double-blind, or neither?
    d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?

Concepts

17. A teacher wishes to know whether the males in his/her class have more conservative attitudes than the females. A questionnaire is distributed assessing attitudes.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?
    c. Are there any possible sources of bias in this study?
18. A study is conducted to determine whether people learn better with spaced or massed practice. Subjects volunteer from an introductory psychology class. At the beginning of the semester 12 subjects volunteer and are assigned to the massed-practice group. At the end of the semester 12 subjects volunteer and are assigned to the spaced-practice condition.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?
    c. This study involves two kinds of non-random sampling: (1) Subjects are not randomly sampled from some specified population and (2) Subjects are not randomly assigned to groups. Which problem is more serious? What affect on the results does each have?
19. A farmer believes that playing Barry Manilow songs to his peas will increase their yield. Describe a controlled experiment the farmer could use to test his theory.
20. A sports psychologist believes that people are more likely to be extroverted as adults if they played team sports as children. Describe two possible studies to test this theory. Design one as an observational study and the other as an experiment. Which is more practical?

Exploration

21. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 50 AIDS patients from the start of treatment until their deaths.
    a. What is the population of this study?
    b. List two reasons why the data may differ.
    c. Can you tell if one researcher is correct and the other one is incorrect? Why?
    d. Would you expect the data to be identical? Why or why not?
    e. If the first researcher collected her data by randomly selecting 40 states, then selecting 1 person from each of those states. What sampling method is that?

f. If the second researcher collected his data by choosing 40 patients he knew. What sampling method would that researcher have used? What concerns would you have about this data set, based upon the data collection method?

22. Find a newspaper or magazine article, or the online equivalent, describing the results of a recent study (the results of a poll are not sufficient). Give a summary of the study's findings, then analyze whether the article provided enough information to determine the validity of the conclusions. If not, produce a list of things that are missing from the article that would help you determine the validity of the study. Look for the things discussed in the text: population, sample, randomness, blind, control, placebos, etc.

PROBABILITY

INTRODUCTION

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with *n* equally likely possible outcomes and that *m* of those *n* outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

BASIC CONCEPTS

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes
The result of an experiment is called an **outcome**.
An **event** is any particular outcome or group of outcomes.
A **simple event** is an event that cannot be broken down further
The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: {1,2,3,4,5,6}

Some examples of simple events:

We roll a 1

We roll a 5

Some compound events:

We roll a number bigger than 4

We roll an even number

Basic Probability
Given that all outcomes are equally likely, we can compute the probability of an event *E* using this formula:
$P(E)=\frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally-likely outcomes}}$

Example 2

If we roll a 6-sided die, calculate

a) P(rolling a 1)

b) P(rolling a number bigger than 4)

Recall that the sample space is {1,2,3,4,5,6}

a) There is one outcome corresponding to "rolling a 1", so the probability is $\frac{1}{6}$

b) There are two outcomes bigger than a 4, so the probability is $\frac{2}{6}=\frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20}=\frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now 1

At some random moment, you look at your clock and note the minutes reading.

a. What is probability the minutes reading is 15?

b. What is the probability the minutes reading is 15 or less?

Cards
A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

There are 52 cards in the deck and 4 Aces so $P(Ace) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

Certain and Impossible events
An impossible event has a probability of 0.
A certain event has a probability of 1.
The probability of any event must be $0 \leq P(E) \leq 1$

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work*.

WORKING WITH EVENTS

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is *P*(six) =1/6. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is *P*(not a six) = $\frac{5}{6}$. Notice that

$P(\text{six}) + P(\text{notasix}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$

This is not a coincidence. Consider a generic situation with *n* possible outcomes and an event *E* that corresponds to *m* of these outcomes. Then the remaining *n* – *m* outcomes correspond to *E* not happening, thus

$P(\text{not}E) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$

Complement of an Event

The **complement** of an event is the event "*E* doesn't happen"
The notation $\bar{E}$ is used for the complement of event *E*.
We can compute the probability of the complement using $P\left(\bar{E}\right) = 1 - P(E)$
Notice also that $P(E) = 1 - P\left(\bar{E}\right)$

Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

There are 13 hearts in the deck, so $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$.

The probability of *not* drawing a heart is the complement: $P(\text{notheart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$

Probability of two independent events

Example 6

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

We could list all possible outcomes: {H1,H2,h2,H4,H5,H6,T1,T2,T3,T4,T5,T6}.

Notice there are 2 · 6 = 12 total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 7

Are these events independent?

a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

c) You draw a card from a deck, then draw a second card without replacing the first.

a) The probability that a head comes up on the second toss is 1/2 regardless of whether or not a head came up on the first toss, so these events are independent.

b) These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

c) The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

> *P*(*A* and *B*) for independent events
> If events *A* and *B* are independent, then the probability of both *A* and *B* occurring is
> $P(A \text{ and } B) = P(A) \cdot P(B)$
> where *P*(*A* and *B*) is the probability of events *A* and *B* both occurring, *P*(*A*) is the probability of event *A* occurring, and *P*(*B*) is the probability of event *B* occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 8

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

Try it Now 2

A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 9

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Here, there are still 12 possible outcomes: {H1,H2,h2,H4,H5,H6,T1,T2,T3,T4,T5,T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are H1, H2, h2, H4, H5, H6, T6), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2}+\frac{1}{6}=\frac{6}{12}+\frac{2}{12}=\frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12}-\frac{1}{12}=\frac{7}{12}$.

P(*A* or *B*)
The probability of either *A* or *B* occurring (or both) is
P(*A* or *B*) = *P*(*A*) + *P*(*B*) – *P*(*A* and *B*)

Example 10

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$P(\text{KingorQueen})=\frac{8}{52}$

Note that in this case, there are no cards that are both a Queen and a King, so . Using our probability rule, we could have said:

$P(\text{KingorQueen})=P(\text{King})+P(\text{Queen})-P(\text{KingandQueen})=\frac{4}{52}+\frac{4}{52}-0=\frac{8}{52}$

In the last example, the events were **mutually exclusive**, so *P*(*A* or *B*) = *P*(*A*) + *P*(*B*).

Example 11

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Half the cards are red, so $P(\text{red})=\frac{26}{52}$

There are four kings, so $P(\text{King})=\frac{4}{52}$

There are two red kings, so $P(\text{RedandKing})=\frac{2}{52}$

We can then calculate

$P(\text{RedorKing})=P(\text{Red})+P(\text{King})-P(\text{RedandKing})=\frac{26}{52}+\frac{4}{52}-\frac{2}{52}=\frac{28}{52}$

Try it Now 3

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Example 12

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a red car *and* got a speeding ticket

b) Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:

P(had a red car) + P(got a speeding ticket) – P(had a red car and got a speeding ticket)

= $\frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}$.

Conditional Probability

Often it is required to compute the probability of an event given that another event has occurred.

Example 13

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

P(ace on second draw I an ace on the first draw).

The vertical bar "I" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

> Conditional Probability
> The probability the event *B* occurs, given that event *A* has happened, is represented as
> *P*(*B* | *A*)
> This is read as "the probability of *B* given *A*"

Example 14

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 15

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a speeding ticket *given* they have a red car

b) Has a red car *given* they have a speeding ticket

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

P(ticket | red car) = $\frac{15}{150} = \frac{1}{10} = 0.1$

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

P(red car | ticket) = $\frac{15}{60} = \frac{1}{4} = 0.25$.

Notice from the last example that P(B | A) is **not** equal to P(A | B).

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

> Conditional Probability Formula
> If Events *A* and *B* are not independent, then
> *P*(*A* and *B*) = *P*(*A*) · *P*(*B* | *A*)

Example 16

If you pull 2 cards out of a deck, what is the probability that both are spades?

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

Example 17

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is *P*(A) + *P*(B) – *P*(A and B). In this problem, *P*(A and B) = 0 since the first card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is $\frac{1}{101} + \frac{1}{101} = \frac{2}{101}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{2}{101}$.

Try it Now 4

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Example 18

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a) *P*(not pregnant I positive test result)

b) *P*(positive test result I not pregnant)

	Positive test	Negative test	Total
Pregnant	70	4	74

Not Pregnant	5	14	19
Total	75	18	93

a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant. *P*(not pregnant I positive test result) = $\frac{5}{75} \approx 0.067$.

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. *P*(positive test result I not pregnant) = $\frac{5}{19} \approx 0.263$

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

BAYES THEOREM

In this section we concentrate on the more complex conditional probability problems we began looking at in the last section.

Example 19

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: 1/1000. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write *P*(disease)=0.001.)

A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say *P*(positive I disease)=1.)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that *P*(positive I no disease)=0.05.)

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute *P*(diseaseIpositive). We already know that

P(positive|disease)=1, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about 1/1000 of all people are afflicted with the disease, 1/1000 of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect (0.05)(999)=49.95 (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing P(disease|positive). There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

P(disease | positive) $\approx \frac{1}{51} \approx 0.0196$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease*.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and (0.05)(99,900)=4995 test positive but do not have the disease, so the exact probability of having the disease if you test positive is

P(disease | positive) $\approx \frac{100}{5095} \approx 0.0196$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

Bayes' Theorem

$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A)+P(\bar{A})P(B|\bar{A})}$

In our earlier example, this translates to

$$P(\text{disease}|\text{positive}) = \frac{P(\text{disease})P(\text{positive}|\text{disease})}{P(\text{disease})P(\text{positive}|\text{disease}) + P(\text{nodisease})P(\text{positive}|\text{nodisease})}$$

Plugging in the numbers gives

$$P(\text{disease}|\text{positive}) = \frac{(0.001)(1)}{(0.001)(1)+(0.999)(0.05)} \approx 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

Example 20

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 98 will test positive. So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{disease}|\text{positive}) = \frac{180}{278} \approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{disease}|\text{positive}) = \frac{(0.02)(0.90)}{(0.02)(0.90)+(0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

Try it Now 5

A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 21

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger   soup + sandwich   soup + quiche

soup + fajita   soup + pizza   salad + hamburger

salad + sandwich   salad + quiche   salad + fajita

salad + pizza   breadsticks + hamburger   breadsticks + sandwich

breadsticks + quiche   breadsticks + fajita   breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

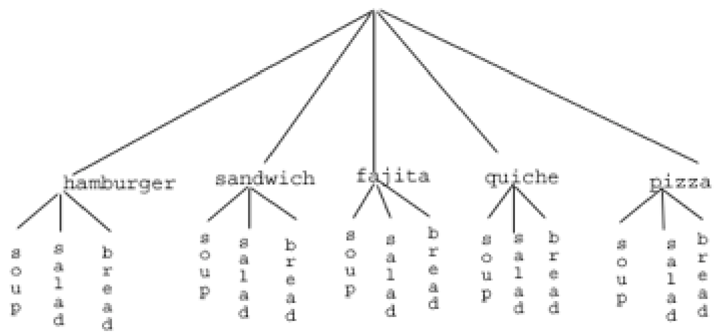
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

	hamburger	sandwich	quiche	fajita	pizza
soup	soup+burger				
salad	salad+burger				
bread	etc				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose form instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule
If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$.
This is sometimes called the multiplication rule for probabilities.

Example 22

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 23

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are

$3 \cdot 5 \cdot 2 = 30$ possibilities.

Example 24

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 23, which is much more compact.

Try it Now 6

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 25

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code worth with the letters MATH.

In this example, we needed to calculate $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$

> Factorial
> $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

Example 26

How many ways can five different door prizes be distributed among five people?

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 27

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of $100, the second is worth $25 and the third is worth $10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Using the Basic Counting Rule, there are 25 choices for the person who receives the $100 certificate, 24 remaining choices for the $25 certificate and 23 choices for the $10 certificate, so there are $25 \cdot 24 \cdot 23 =$ 13,800 ways in which the prizes can be awarded.

Example 28

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the $100 prize, or a different goldmedalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a $10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$

If you don't see why $(n - r + 1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get 13,800. In this case $n = 25$ and $r = 3$, so $n - r + 1 = 25 - 3 + 1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations
$nPr = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$
We say that there are nPr **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*.
It turns out that we can express this result more simply using factorials.
$_nP_r = \frac{n!}{(n-r)!}$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 29

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are $9P4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3{,}024$ permutations.

Example 30

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is $16P4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43{,}680$.

Try it Now 7

How many 5 character passwords can be made using the letters A through Z

a. if repeats are allowed

b. if no repeats are allowed

Combinations

In the previous section we considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 31

A charity benefit is attended by 25 people at which three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13{,}800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get $50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be 13,800/6 = 2300.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is **not** important*. If we are choosing *r* items out of *n* possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by $\frac{{}_nP_r}{{}_rP_r}$, and we could use this formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing *r* items out of *n* possibilities *without replacement* where the *order of selection is **not** important*.

Combinations
$_nC_r = \frac{{}_nP_r}{{}_rP_r}$
We say that there are *nCr* **combinations** of size *r* that may be selected from among *n* choices *without replacement* where *order doesn't matter*.
We can also write the combinations formula in terms of factorials:
$_nC_r = \frac{n!}{(n-r)!r!}$

Example 32

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection is **not** important* there are $_{35}C_4 = \frac{35\cdot34\cdot33\cdot32}{4\cdot3\cdot2\cdot1}$ = 52,360 combinations.

Try it Now 8

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 33

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are 15*C*10 = 3003 ways to choose the 10 Republicans and 14*C*9 = 2002 ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be 3003 · 2002 = 6,012,006 possible ways of selecting the members of the Defense Subcommittee.

Probability using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions

Example 34

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are 10 · 10 · 10 · 10 = 104 = 10000 total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute 10 · 9 · 8 · 7, or notice that this is the same as the permutation 10*P*4 = 5040.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$

Example 35

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is 48*C*6 = 12,271,512. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

Example 36

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

As above, the number of possible outcomes of the lottery drawing is 48*C*6 = 12,271,512. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by 6*C*5 = 6 and the number of ways to choose 1 out of the 42 losing numbers is given by 42*C*1 = 42. Thus the number of favorable outcomes is then given by the Basic Counting Rule: 6*C*5 · 42*C*1 = 6 · 42 = 252. So the probability of winning the second prize is.

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

Try it Now 9

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Example 37

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, $52C5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be $4C1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be $48C4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be $4C1 \cdot 48C4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{oneAce}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example 38

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{twoAces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 10

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Birthday Problem

Let’s take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let’s see if we should listen to our intuition. Let’s start with a simpler problem, however.

Example 39

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves “What is the alternative to having at least one shared birthday?” In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to “at least one” is having **none**. In other words, since this is a complementary event,

P(at least one) = 1 – P(none)

We will start, then, by computing the probability that there is no shared birthday. Let’s imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let’s ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is 364/365. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person’s, so the probability that the third person does not share a birthday with the first two is 363/365.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{nosharedbirthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

P(shared birthday) = 1 – P(no shared birthday) = 1 – 0.9918 = 0.0082.

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let’s make our group a bit bigger.

Example 40

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Continuing the pattern of the previous example, the answer should be

$$P(\text{sharedbirthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{sharedbirthday}) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 41

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Here we can calculate

$$P(\text{sharedbirthday}) = 1 - \frac{{}_{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: http://www-stat.stanford.edu/~susan/surprise/Birthday.html, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now 11

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

EXPECTED VALUE

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 42

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets $1 on a single number. If that number is spun on the wheel, then they receive $36 (their original $1 + $35). Otherwise, they lose their $1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Suppose you bet $1 on each of the 38 spaces on the wheel, for a total of $38 bet. When the winning number is spun, you are paid $36 on that number. While you won on that one number, overall you've lost $2. On a per-space basis, you have "won" -$2/$38 ≈ -$0.053. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose $1 and a very few people (about 1 person out of every 38) gain $35 (the $36 they win minus the $1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
-\$1	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get

0.9211 + (-0.9737) ≈ -0.053, which is the expected value we computed above.

Expected Value
Expected Value is the average gain or loss of an event if the procedure is repeated many times.
We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

Try it Now 12

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Example 43

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$ for all 6 numbers,

$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$ for 5 numbers.

Our probabilities and outcome values are:

Outcome	Probability of outcome
\$999,999	$\frac{1}{12271512}$
\$999	$\frac{252}{12271512}$
-\$1	$1 - \frac{253}{12271512} = \frac{12271259}{12271512}$

The expected value, then is:

$$(\$999,999) \cdot \frac{1}{12271512} + (\$999) \cdot \frac{252}{12271512} + (-\$1) \cdot \frac{12271259}{12271512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Try it Now 13

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any two dice match values, you get $2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 44

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year. ((Note: According to the estimator at http://www.numericalexample.com/index.php?view=article&id=91)) An insurance company charges $275 for a life-insurance policy that pays a $100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and outcomes are

Outcome	Probability of outcome
$100,000 – $275 = $99,725	0.00242
-$275	1 – 0.00242 = 0.99758

The expected value is ($99,725)(0.00242) + (-$275)(0.99758) = -$33.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

Try it Now Answers

1. There are 60 possible readings, from 00 to 59. a. $\frac{1}{60}$ b. $\frac{16}{60}$ (counting 00 through 15)
2. Since the second draw is made after replacing the first card, these events are independent. The probability of an ace on each draw is $\frac{4}{52} = \frac{1}{13}$, so the probability of an Ace on both draws is $\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$
3. P(white sock and white tee) = $\frac{6}{10} \cdot \frac{3}{7} = \frac{9}{35}$

P(white sock or white tee) = $\frac{6}{10}+\frac{3}{7}-\frac{9}{35}=\frac{27}{35}$

4. a. $\frac{6}{10}\cdot\frac{5}{9}=\frac{30}{90}=\frac{1}{3}$
5. Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

P(disease I positive) = $\frac{500}{500+2985}=\frac{500}{3485}$ ≈ 14.3%

6. 8 · 11 · 5 = 440 menu combinations
7. There are 26 characters. a. 265 = 11,881,376.   b. 26P5 = 26·25·24·23·22 = 7,893,600
8. Order does not matter. 29C19 = 20,030,010 possible subcommittees
9. There are 510 = 9,765,625 different ways the exam can be answered. There are 9 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 36 ways the test could be answered with one wrong answer.

P(9 answers correct) = $\frac{36}{5^{10}}$ ≈ 0.0000037 chance

10. $P(\text{threeAcesandtwoKings})=\frac{({}_4C_3)({}_4C_2)}{{}_{52}C_5}=\frac{24}{2598960}$ ≈ 0.0000092
11. $P(\text{sharedbirthday})=1-\frac{{}_{365}P_{10}}{365^{10}}$ ≈ 0.117
12. $(\$3,995)\cdot\frac{1}{2000}+(-\$5)\cdot\frac{1999}{2000}$ ≈ -$3.00
13. Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous two is $\frac{4}{6}$, so the probability that the three dice are different is $\frac{5}{6}\cdot\frac{4}{6}=\frac{20}{36}$. The probability that two dice will match is the complement, $1-\frac{20}{36}=\frac{16}{36}$.

The expected value is: $(\$2)\cdot\frac{16}{36}+(-\$1)\cdot\frac{20}{36}=\frac{12}{36}$≈ $0.33. Yes, it is in your advantage to play. On average, you'd win $0.33 per play.

PROBABILITY READING II

We can use permutations and combinations to help us answer more complex probability questions

Example 1

A 4 digit PIN is selected. What is the probability that there are no repeated digits?

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are 10 × 10 × 10 × 10 = 10^4 = 10000 total possible PINs.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute 10 × 9 × 8 × 7, or notice that this is the same as the permutation ${}_{10}P_4$ = 5040.

The probability of no repeated digits is the number of 4 digit PINs with no repeated digits divided by the total number of 4 digit PINs. This probability is

$$\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$$

Example 2

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6$ = 12,271,512. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_{6}C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx= 0.0000000815$$

Example 3

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6$ = 12,271,512. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_{6}C_5$ = 6 and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1$ = 42. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_{6}C_5 \times {}_{42}C_1 = 6 \times 42 = 252$. So the probability of winning the second prize is

$$\frac{({}_{6}C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

Try it Now

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Example 4

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \times {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$

Example 5

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$P(\text{two Ace}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

> Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 6

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event, *P*(at least one) = 1 – *P*(none)

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is 364/365. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is 363/365.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

P(shared birthday) = 1 – P(no shared birthday) = 1 – 0.9918 = 0.0082.

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 7

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 8

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Here we can calculate

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 9

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won"—\$2/\$38 ≈ –\$0.053. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
–\$1	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get \$35 × $\frac{1}{38}$ = 0.9211 and –\$1 × $\frac{37}{38}$ = –0.9737, and if we add these numbers we get 0.9211 + (–0.9737) ≈ –0.053, which is the expected value we computed above.

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Expected Value is the average gain or loss of an event if the procedure is repeated many times.
We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

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You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Example 10

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$ for all 6 numbers,

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Our probabilities and outcome values are:

Outcome	Probability of outcome
\$999,999	$\frac{1}{12271512}$

Outcome	Probability of outcome
\$999	$\frac{252}{12271512}$
–\$1	$1 - \frac{253}{12271512} = \frac{12271259}{12271512}$

The expected value, then is:

$$(\$999,999) \times \frac{1}{12271512} + (\$999) \times \frac{252}{12271512} + (-\$1) \times \frac{12271259}{12271512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose \$1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try it Now

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 11

According to the estimator at numericalexample.com, a 40-year-old man in the US has a 0.242% risk of dying during the next year. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and outcomes are

Outcome	Probability of outcome
\$100,000 – \$275 = \$99,725	0.00242
–\$275	1 – 0.00242 = 0.99758

The expected value is (\$99,725)(0.00242) + (–\$275)(0.99758) = –\$33.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

CALCULATING THE ODDS OF AN EVENT

Here you'll calculate odds by using outcomes or probability. Have you ever thought about the likelihood of an event happening? Take a look at this dilemma:

Telly and Carey were already hard at work when Ms. Kelley came into the bike shop on Thursday morning. It was three days before the big race and there was still a lot of work to be done.

"I can't believe it!" Ms. Kelley exclaimed as she came into the shop.

"What?" both girl asked alarmed.

"There is a 4 to 5 chance that it is going to rain on Saturday. I just heard the weather report," Ms. Kelley said sighing.

"Well, there is still a chance that it won't," Telly said trying to cheer her up.

When we think about chances and odds, we can calculate the likelihood that an event will or won't occur. In this case, there are odds that it will rain and odds that it won't. We can also express those odds as a fraction or a percentage. Learn about odds in this reading, and you can work on the odds of the rainstorm at the end.

Guidance

You've seen that the probability of an event is defined as a ratio that compares the favorable out comes to the total outcomes. We can write this ratio in fraction form.

$$P(\text{event}) = \frac{\text{favorable outcomes}}{\text{total outcomes}}$$

Sometimes people express the likelihood of events in terms of **odds** rather than probabilities. The odds of an event occurring are equal to the ratio of favorable outcomes to **unfavorable outcomes**.

Think about the odds for the arrow of the spinner above landing on red:

- favorable outcomes = 1(red)
- unfavorable outcomes = 2(blue, yellow)
- total outcomes = 3

So the *probability* of spinning red is:

$$P(\text{red}) = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{1}{3}$$

While the *odds* in favor of red are:

$$\text{Odds(in favor of red)} = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}} = \frac{1}{2}$$

Odds against an event occurring are defined as:

$$\text{Odds(against red)} = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{2}{1}$$

You can solve any probability problem in terms of odds rather than probabilities. Notice that the ratio represents what is being compared. Be sure that your numbers match the comparison.

We can use odds to calculate how likely an event is to happen. We can compare the odds in favor of an event with the probability that the event will actually occur. Let's look at an example.

Take a look at this situation.

You've seen that the odds in favor of an event (*E*) occurring are shown in this ratio.

$$\text{Odds(in favor of)}E = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}} = \frac{1}{2}$$

And the odds against the same event occurring are:

$$\text{Odds(against)}E = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{2}{1}$$

You can use these two facts to compute the ratio of things happening and not happening.

For example, suppose the weather forecast states:

Odds in favor of rain: 7 to 3

These odds tell you not only the odds of rain, but also the odds of not raining.

If the odds in favor or rain are 7 to 3, then the odds against rain are:

Odds against rain: 3 to 7

Another way of saying that is:

Odds that it will NOT rain: 3 to 7

You can use this idea in many different situations. If you know the odds that something will happen, then you also know the odds that it will not happen.

Use this spinner to calculate odds.

Example A

Odds in favor of spinning a blue.

Solution: $\frac{1}{2}$

Example B

Odds in favor of spinning a red or blue.

Solution: $\frac{2}{1}$

Example C

Odds against spinning a red or blue.

Solution: $\frac{1}{2}$

Intro Problem Revisited

Now let's go back to the dilemma from the beginning of the reading.

Answer all three questions.

What are the chances that it won't rain? We know that the odds of it raining is 4 to 5. Therefore it is a 1 out of 5 chance that it won't rain. Not very good odds.

What are the odds that it will as a percentage? 4 to 5 can be written as a percentage: 80% chance of rain.

What are the odds that it won't as a percentage? 1 to 5 can be written as a percentage: 20% chance that it won't rain.

Guided Practice

Here is one for you to try on your own.

What are the odds in favor of a number cube landing on 4?

Step 1

Find the favorable and unfavorable outcomes.

- favorable outcomes = 1(4)
- unfavorable outcomes = 5(1,2,3,5,6)

Step 2

Write the ratio of favorable to unfavorable outcomes.

$$\text{Odds}(4) = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}} = \frac{1}{5}$$

The odds in favor of rolling a 4 are 1 to 5.

Vocabulary

Disjoint events: events that don't have any outcomes in common.
Complementary events: probability that has a sum of 100%. Either/Or events are complementary events.

Watch This: Video Review

Watch this video online: https://youtu.be/76XEBRaelD0

Practice Questions

Solve the problems.

1. For rolling a number cube, what are the odds in favor of rolling a 2?
2. For rolling a number cube, what are the odds against rolling a 2?
3. For rolling a number cube, what are the odds in favor of rolling a number greater than 3?
4. For rolling a number cube, what are the odds in favor rolling a number less than 5?
5. For rolling a number cube, what are the odds against rolling a number less than 5?
6. For rolling a number cube, what are the odds in favor of rolling an even number?
7. For rolling a number cube, what are the odds against rolling an even number?

For a spinner numbered 1 –10, answer the following questions.

1. For spinning the spinner, what are the odds in favor of the arrow landing on 10?
2. For spinning the spinner, what are the odds in favor of the arrow landing on a 2 or 3?
3. For spinning the spinner, what are the odds in favor of the arrow landing on 7, 8 or 9?
4. For spinning the spinner, what are the odds in favor of NOT landing on an even number?
5. For spinning the spinner, what are the odds of the arrow NOT landing on 10?
6. For spinning the spinner, what are the odds in favor of the arrow landing on a number greater than 2?
7. For spinning the spinner, what are the odds in favor of the arrow NOT landing on a number greater than 2?
8. For spinning the spinner, what are the odds of the arrow not landing on a number greater than 3?

HOW TO CALCULATE THE ODDS OF WINNING THE LOTTERY

Visual explanation of how to calculate the odds of winning the lottery using probability and using combination theory.

Watch this video online: https://youtu.be/_aBKxnUtIOk

SUPPLEMENTAL VIDEOS

Watch these videos that supplement the reading you've just completed.

Basics of Probability—Events and Outcomes

Watch this video online: https://youtu.be/37P01dt0zsE

Basic Probabilities

Watch this video online: https://youtu.be/EBqj_R3dzd4

Probability—Complements

Watch this video online: https://youtu.be/RnljiW6ZM6A

Joint Probabilities of Independent Events: *P*(*A* and *B*)

Watch this video online: https://youtu.be/6F17WLp-EL8

Probability of Two Events: *P*(*A* or *B*)

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EXERCISES

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.
    a. A red ball is drawn
    b. A white ball is drawn
2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:
    a. A consonant
    b. A vowel
3. A group of people were asked if they had run a red light in the last year. 150 responded “yes”, and 185 responded “no”. Find the probability that if a person is chosen at random, they have run a red light in the last year.
4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dots, and 40 indicated they don’t enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.
5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.
6. Compute the probability of tossing a six-sided die and getting a 7.
7. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26

Total	18	22	25	65

8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

9. Compute the probability of tossing a six-sided die and getting an even number.
10. Compute the probability of tossing a six-sided die and getting a number less than 3.
11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?
12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?
13. Compute the probability of rolling a 12-sided die and getting a number other than 8.
14. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?
15. Referring to the grade table from question #7, what is the probability that a student chosen at random did NOT earn a C?
16. Referring to the credit card table from question #8, what is the probability that a person chosen at random has at least one credit card?
17. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?
18. A fair coin is flipped twice. What is the probability of showing heads on both flips?
19. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?
20. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?
21. Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random, find the probability that both are red.
22. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.
23. Bert and Ernie each have a well-shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:
    a. Bert and Ernie both draw an Ace.
    b. Bert draws an Ace but Ernie does not.
    c. neither Bert nor Ernie draws an Ace.
    d. Bert and Ernie both draw a heart.
    e. Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.
24. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that:
    a. Bert gets a Jack and Ernie rolls a five.
    b. Bert gets a heart and Ernie rolls a number less than six.
    c. Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
    d. Bert gets a red card and Ernie rolls a fifteen.
    e. Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.
25. Compute the probability of drawing a King from a deck of cards and then drawing a Queen.
26. Compute the probability of drawing two spades from a deck of cards.
27. A math class consists of 25 students, 14 female and 11 male. Two students are selected at random to participate in a probability experiment. Compute the probability that
    a. a male is selected, then a female.
    b. a female is selected, then a male.
    c. two males are selected.
    d. two females are selected.
    e. no males are selected.
28. A math class consists of 25 students, 14 female and 11 male. Three students are selected at random to participate in a probability experiment. Compute the probability that

a. a male is selected, then two females.
b. a female is selected, then two males.
c. two females are selected, then one male.
d. three males are selected.
e. three females are selected.

29. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female and earned an A.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

30. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

31. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.
32. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.
33. Referring to the table from #29, find the probability that a student chosen at random is female or earned a B.
34. Referring to the table from #30, find the probability that a person chosen at random is male or has no credit cards.
35. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.
36. Compute the probability of drawing a King or a heart from a deck of cards.
37. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Even-numbered given that the marble is red.
    b. Red given that the marble is even-numbered.
38. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Odd-numbered given that the marble is blue.
    b. Blue given that the marble is odd-numbered.
39. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.
40. Find the probability of rolling a “1” on a fair die, given that the last 3 rolls were all ones.
41. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.
42. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.
43. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event “the person is infected” and B be the event “the person tests positive”.
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find P(A I B).
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find P(not A I not B).

44. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find P(A I B).
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find P(not A I not B).
45. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%, compute the probability that a person who tests positive actually has the disease.
46. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.
47. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%. The false positive rate is 7%. If a the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?
48. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?
49. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?
50. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?
51. How many three-letter "words" can be made from 4 letters "FGHI" if
    a. repetition of letters is allowed
    b. repetition of letters is not allowed
52. How many four-letter "words" can be made from 6 letters "AEBWDP" if
    a. repetition of letters is allowed
    b. repetition of letters is not allowed
53. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles?
54. A computer password must be eight characters long. How many passwords are possible if only the 26 letters of the alphabet are allowed?
55. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?
56. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?
57. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?
58. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?
59. In western music, an octave is divided into 12 pitches. For the film *Close Encounters of the Third Kind*, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible if no note is repeated?
60. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the chromatic scale in any order, but with not pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?
61. In how many ways can 4 pizza toppings be chosen from 12 available toppings?
62. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?
63. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn't matter?
64. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn't matter?
65. A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?
66. The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?
67. You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?

68. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.
69. In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, they player wins second prize. What is the probability of winning this prize?
70. In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, they player wins third prize. What is the probability of winning this prize?
71. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.
72. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.
73. A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select on marble from the bag. If it is gold, you win $3. If it is silver, you win $2. If it is black, you lose $1. What is your expected value if you play this game?
74. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you $3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him $1. Compute the expected value for this game. Should you play this game?
75. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win 30,000 dollars. Otherwise, they lose $1. Find the expected value of this game.
76. A game is played by picking two cards from a deck. If they are the same value, then you win $5, otherwise you lose $1. What is the expected value of this game?
77. A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of $350. If they offer a 2 year extended warranty for $48, what is the company's expected value of each warranty sold?
78. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be $60,000. If the company offers earthquake insurance for $100, what is their expected value of the policy?

Exploration

Some of these questions were adapted from puzzles at mindyourdecisions.com.

79. A small college has been accused of gender bias in its admissions to graduate programs.
    a. Out of 500 men who applied, 255 were accepted. Out of 700 women who applied, 240 were accepted. Find the acceptance rate for each gender. Does this suggest bias?
    b. The college then looked at each of the two departments with graduate programs, and found the data below. Compute the acceptance rate within each department by gender. Does this suggest bias?

Department	Men		Women	
	Applied	Admitted	Applied	Admitted
Dept A	400	240	100	90
Dept B	100	15	600	150

c. Looking at our results from Parts *a* and *b*, what can you conclude? Is there gender bias in this college's admissions? If so, in which direction?

80. A bet on "black" in Roulette has a probability of 18/38 of winning. If you win, you double your money. You can bet anywhere from $1 to $100 on each spin.
    a. Suppose you have $10, and are going to play until you go broke or have $20. What is your best strategy for playing?
    b. Suppose you have $10, and are going to play until you go broke or have $30. What is your best strategy for playing?
81. Your friend proposes a game: You flip a coin. If it's heads, you win $1. If it's tails, you lose $1. However, you are worried the coin might not be fair coin. How could you change the game to make the game fair, without replacing the coin?
82. Fifty people are in a line. The first person in the line to have a birthday matching someone in front of them will win a prize. Of course, this means the first person in the line has no chance of winning. Which person has the highest likelihood of winning?
83. Three people put their names in a hat, then each draw a name, as part of a randomized gift exchange. What is the probability that no one draws their own name? What about with four people?

84. How many different "words" can be formed by using all the letters of each of the following words exactly once?
    a. "ALICE"
    b. "APPLE"
85. How many different "words" can be formed by using all the letters of each of the following words exactly once?
    a. "TRUMPS"
    b. "TEETER"
86. The *Monty Hall problem* is named for the host of the game show *Let's make a Deal*. In this game, there would be three doors, behind one of which there was a prize. The contestant was asked to choose one of the doors. Monty Hall would then open one of the other doors to show there was no prize there. The contestant was then asked if they wanted to stay with their original door, or switch to the other unopened door. Is it better to stay or switch, or does it matter?
87. Suppose you have two coins, where one is a fair coin, and the other coin comes up heads 70% of the time. What is the probability you have the fair coin given each of the following outcomes from a series of flips?
    a. 5 Heads and 0 Tails
    b. 8 Heads and 3 Tails
    c. 10 Heads and 10 Tails
    d. 3 Heads and 8 Tails
88. Suppose you have six coins, where five are fair coins, and one coin comes up heads 80% of the time. What is the probability you have a fair coin given each of the following outcomes from a series of flips?
    a. 5 Heads and 0 Tails
    b. 8 Heads and 3 Tails
    c. 10 Heads and 10 Tails
    d. 3 Heads and 8 Tails
89. In this problem, we will explore probabilities from a series of events.
    a. If you flip 20 coins, how many would you *expect* to come up "heads", on average? Would you expect *every* flip of 20 coins to come up with exactly that many heads?
    b. If you were to flip 20 coins, what would you consider a "usual" result? An "unusual" result?
    c. Flip 20 coins (or one coin 20 times) and record how many come up "heads". Repeat this experiment 9 more times. Collect the data from the entire class.
    d. When flipping 20 coins, what is the theoretic probability of flipping 20 heads?
    e. Based on the class's experimental data, what appears to be the probability of flipping 10 heads out of 20 coins?
    f. The formula ${}_nC_x p^x (1-p)^{n-x}$ will compute the probability of an event with probability *p* occurring *x* times out of *n*, such as flipping *x* heads out of *n* coins where the probability of heads is *p* = ½. Use this to compute the theoretic probability of flipping 10 heads out of 20 coins.
    g. If you were to flip 20 coins, based on the class's experimental data, what range of values would you consider a "usual" result? What is the combined probability of these results? What would you consider an "unusual" result? What is the combined probability of these results?
    h. We'll now consider a simplification of a case from the 1960s. In the area, about 26% of the jury eligible population was black. In the court case, there were 100 men on the juror panel, of which 8 were black. Does this provide evidence of racial bias in jury selection?

GROWTH MODELS

LINEAR (ALGEBRAIC) GROWTH

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$P_0 = 437$

$P_n = P_{n-1} + 32$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1}. Using this relationship, we could calculate:

$P_1 = P_0 + 32 = 437 + 32 = 469$

$P_2 = P_1 + 32 = 469 + 32 = 501$

$P_3 = P_2 + 32 = 501 + 32 = 533$

$P_4 = P_3 + 32 = 533 + 32 = 565$

$P_5 = P_4 + 32 = 565 + 32 = 597$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An **explicit equation** allows us to calculate P_n directly, without needing to know P_{n-1}. While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$P_1 = 437 + 32 \qquad = 437 + 1(32)$

$P_2 = P_1 + 32 = 437 + 32 + 32 \qquad = 437 + 2(32)$

$P_3 = P_2 + 32 = (437 + 2(32)) + 32 \qquad = 437 + 3(32)$

$P4 = P3 + 32 = (437 + 3(32)) + 32 \quad = 437 + 4(32)$

You can probably see the pattern now, and generalize that

$Pn = 437 + n(32) = 437 + 32n$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$P5 = 437 + 32(5) = 437 + 160 = 597$

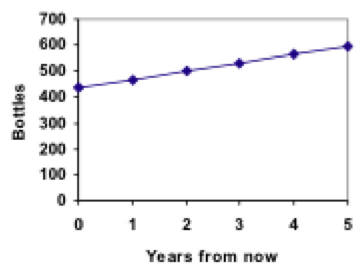
We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for Pn and solving for n

$1000 = 437 + 32n$

$563 = 32n$

$n = 563/32 = 17.59$

So Marco will reach 1000 bottles in 18 years.



In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

Linear Growth
If a quantity starts at size $P0$ and grows by d every time period, then the quantity after n time periods can be determined using either of these relations:
Recursive form:
$Pn = Pn\text{-}1 + d$
Explicit form:
$Pn = P0 + d\, n$
In this equation, d represents the **common difference** – the amount that the population changes each time n increases by 1

Connection to prior learning – slope and intercept

You may recognize the common difference, d, in our linear equation as *slope*. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$.

In the standard algebraic equation $y = mx + b$, b was the y-intercept, or the y value when x was zero. In the form of the equation we're using, we are using $P0$ to represent that initial amount.

In the $y = mx + b$ equation, recall that m was the slope. You might remember this as “rise over run”, or the change in y divided by the change in x. Either way, it represents the same thing as the common difference, d, we are using – the amount the output P_n changes when the input n increases by 1.

The equations $y = mx + b$ and $P_n = P_0 + d\,n$ mean the same thing and can be used the same ways, we’re just writing it somewhat differently.

Example 1

The population of elk in a national forest was measured to be 12,000 in 2003, and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

To begin, we need to define how we’re going to measure n. Remember that P_0 is the population when $n = 0$, so we probably don’t want to literally use the year 0. Since we already know the population in 2003, let us define $n = 0$ to be the year 2003. Then

$P_0 = 12{,}000$.

Next we need to find d. Remember d is the growth per time period, in this case growth per year. Between the two measurements, the population grew by 15,000-12,000 = 3,000, but it took 2007-2003 = 4 years to grow that much. To find the growth per year, we can divide: 3000 elk / 4 years = 750 elk in 1 year.

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$$d = slope = \frac{\text{changeinoutput}}{\text{changeininput}} = \frac{15{,}000-12{,}000}{2007-2003} = \frac{3000}{4} = 750$$

We can now write our equation in whichever form is preferred.

Recursive form:

$P_0 = 12{,}000$

$P_n = P_{n-1} + 750$

Explicit form:

$P_n = 12{,}000 + 750n$

To answer the question, we need to first note that the year 2014 will be $n = 11$, since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

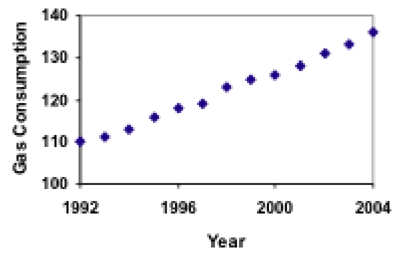
$P_{11} = 12{,}000 + 750(11) = 20{,}250$ elk

Example 2

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below ((Note: http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html)). Find a model for this data, and use it to predict consumption in 2016. If the trend continues, when will consumption reach 200 billion gallons?

Year	’92	’93	’94	’95	’96	’97	’98	’99	’00	’01	’02	’03	’04
Consumption (billion of gallons)	110	111	113	116	118	119	123	125	126	128	131	133	136

Plotting this data, it appears to have an approximately linear relationship:



While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

Letting $n = 0$ correspond with 1993 would give $P_0 = 111$ billion gallons.

To find d, we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of 133 – 111 = 22 billion gallons, over 10 years. This gives us an average change of 22 billion gallons / 10 year = 2.2 billion gallons per year.

Equivalently,

$d = slope = \frac{\text{changeinoutput}}{\text{changeininput}} = \frac{133-111}{10-0} = \frac{22}{10} = 2.2$billion gallons per year

We can now write our equation in whichever form is preferred.

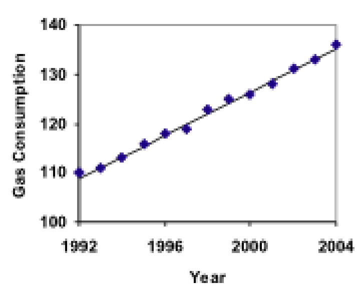
Recursive form:

$P_0 = 111$

$P_n = P_{n-1} + 2.2$

Explicit form:

$P_n = 111 + 2.2n$



Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2016:

$n = 23$ (2016 – 1993 = 23 years later)

$P_{23} = 111 + 2.2(23) = 161.6$

Our model predicts that the US will consume 161.6 billion gallons of gasoline in 2016 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set $Pn = 200$, and solve for n:

$Pn = 200$ Replace Pn with our model

$111 + 2.2n = 200$ Subtract 111 from both sides

$2.2n = 89$ Divide both sides by 2.2

$n = 40.4545$

This tells us that consumption will reach 200 billion about 40 years after 1993, which would be in the year 2033.

Example 3

The cost, in dollars, of a gym membership for n months can be described by the explicit equation $Pn = 70 + 30n$. What does this equation tell us?

The value for $P0$ in this equation is 70, so the initial starting cost is $70. This tells us that there must be an initiation or start-up fee of $70 to join the gym.

The value for d in the equation is 30, so the cost increases by $30 each month. This tells us that the monthly membership fee for the gym is $30 a month.

Try it Now 1

The number of stay-at-home fathers in Canada has been growing steadily ((Note: http://www.fira.ca/article.php?id=140)). While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers, then use it to predict the number if 2020.

When good models go bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

Example 4

Suppose a four year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year.

We can set up a growth model, with $n = 0$ corresponding to 4 years old.

Recursive form:

$P0 = 39$

$Pn = Pn\text{-}1 + 2.5$

Explicit form:

$Pn = 39 + 2.5n$

So at 6 years old, we would expect him to be

$P_2 = 39 + 2.5(2) = 44$ inches tall

Any mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$P_{46} = 39 + 2.5(46) = 154$ inches tall = 12.8 feet tall!

When using any mathematical model, we have to consider which inputs are reasonable to use. Whenever we **extrapolate**, or make predictions into the future, we are assuming the model will continue to be valid.

EXPONENTIAL (GEOMETRIC) GROWTH

Suppose that every year, only 10% of the fish in a lake have surviving offspring. If there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year.

Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1? The new population will be the old population, plus an additional 10%. Symbolically:

$P_1 = P_0 + 0.10P_0$

Notice this could be condensed to a shorter form by factoring:

$P_1 = P_0 + 0.10P_0 = 1P_0 + 0.10P_0 = (1+ 0.10)P_0 = 1.10P_0$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as "the original 100% plus an additional 10%"

For our fish population,

$P_1 = 1.10(1000) = 1100$

We could then calculate the population in later years:

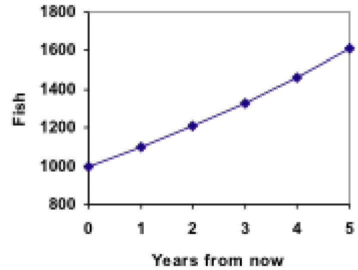
$P_2 = 1.10P_1 = 1.10(1100) = 1210$

$P_3 = 1.10P_2 = 1.10(1210) = 1331$

Notice that in the first year, the population grew by 100 fish, in the second year, the population grew by 110 fish, and in the third year the population grew by 121 fish.

While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

Graphing these values we see that this growth doesn't quite appear linear.



To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$P1 = 1.10P0 = 1.10(1000)$

$P2 = 1.10P1 = 1.10(1.10(1000)) = 1.102(1000)$

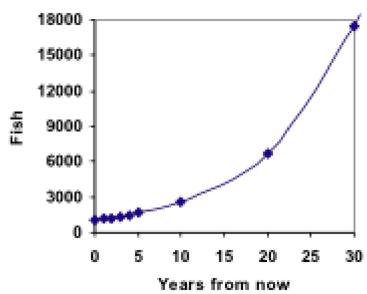
$P3 = 1.10P2 = 1.10(1.102(1000)) = 1.103(1000)$

$P4 = 1.10P3 = 1.10(1.103(1000)) = 1.104(1000)$

Observing a pattern, we can generalize the explicit form to be:

$Pn = 1.10n(1000)$, or equivalently, $Pn = 1000(1.10n)$

From this, we can quickly calculate the number of fish in 10, 20, or 30 years:



$P10 = 1.1010(1000) = 2594$

$P20 = 1.1020(1000) = 6727$

$P30 = 1.1030(1000) = 17449$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years compared to almost 18000 with this percent-based growth, called **exponential growth.**

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

Exponential Growth
If a quantity starts at size $P0$ and grows by $R\%$ (written as a decimal, r) every time period, then the quantity after n time periods can be determined using either of these relations:
Recursive form:

$P_n = (1+r)\, P_{n-1}$
Explicit form:
$P_n = (1+r)^n\, P_0$ or equivalently, $P_n = P_0\,(1+r)^n$
We call r the **growth rate**.
The term $(1+r)$ is called the **growth multiplier**, or common ratio.

Example 5

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245{,}000$. The year 2014 would then be $n = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$P_6 = (1+0.03)^6\,(245{,}000) = 1.19405(245{,}000) = 292{,}542.25$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Evaluating exponents on the calculator
To evaluate expressions like $(1.03)^6$, it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically either labeled like:
^ , y^x , or x^y .
To evaluate 1.03^6 we'd type 1.03 ^ 6, or 1.03 yx 6. Try it out – you should get an answer around 1.1940523.

Try it Now 2

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

Example 6

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

In the equation, $P_0 = 4600$, which is the starting value of the tuition when $n = 0$. This tells us that the tuition in 2010 was $4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was $4,600, and is expected to grow by 7.2% each year.

Example 7

In 1990, the residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons ((Note: http://www.eia.doe.gov/oiaf/1605/ggrpt/carbon.html)). If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Similar to before, we will correspond $n = 0$ with 1990, as that is the year for the first piece of data we have. That will make $P_0 = 962$ (million metric tons of CO_2). In this problem, we are not given the growth rate, but instead are given that $P_{10} = 1182$.

When $n = 10$, the explicit equation looks like:

$P_{10} = (1+r)^{10} P_0$

We know the value for P_0, so we can put that into the equation:

$P_{10} = (1+r)^{10}\, 962$

We also know that $P_{10} = 1182$, so substituting that in, we get

$1182 = (1+r)^{10}\, 962$

We can now solve this equation for the growth rate, r. Start by dividing by 962.

$\frac{1182}{962} = (1+r)^{10}$ Take the 10th root of both sides

$\sqrt[10]{\frac{1182}{962}} = 1 + r$ Subtract 1 from both sides

$r = \sqrt[10]{\frac{1182}{962}} - 1 = 0.0208 = 2.08\%$

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding P_{60}

$P_{60} = (1+0.0208)^{60}\, 962 = 3308.4$ million metric tons of CO_2 in 2050

Rounding

As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

If you need to round, **keep at least three significant digits** – numbers after any leading zeros. So 0.4162 could be reasonably rounded to 0.416. A growth rate of 0.001027 could be reasonably rounded to 0.00103.

Evaluating roots on the calculator

In the previous example, we had to calculate the 10th root of a number. This is different than taking the basic square root, √. Many scientific calculators have a button for general roots. It is typically labeled like:

$\sqrt[n]{\ }$, $\sqrt[x]{\ }$, or $\sqrt[y]{x}$

To evaluate the 3rd root of 8, for example, we'd either type 3 $\sqrt[x]{\ }$ 8, or 8 $\sqrt[x]{\ }$ 3, depending on the calculator.

Try it on yours to see which to use – you should get an answer of 2.

If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that . So, to compute the 3rd root of 8, you could use your calculator's exponent key to evaluate . To do this, type:

8 y^x (1 ÷ 3)

The parentheses tell the calculator to divide 1/3 before doing the exponent.

Try it Now 3

The number of users on a social networking site was 45 thousand in February when they officially went public, and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should then expect two years after they went public?

Example 8

Looking back at the last example, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Again we will get $n = 0$ correspond with 1990, giving $P_0 = 962$. To find d, we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When $n = 10$, the explicit linear equation looks like:

$P_{10} = P_0 + 10d$

We know the value for P_0, so we can put that into the equation:

$P_{10} = 962 + 10d$

Since we know that $P_{10} = 1182$, substituting that in we get

$1182 = 962 + 10d$

We can now solve this equation for the common difference, d.

$1182 - 962 = 10d$

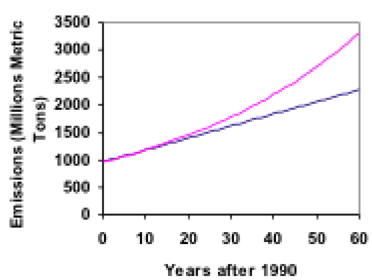
$220 = 10d$

$d = 22$

This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$P_{60} = 962 + 22(60) = 2282$ million metric tons.

You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.



So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

- 1) Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?
- 2) Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

SOLVE EXPONENTIALS FOR TIME: LOGARITHMS

Earlier, we found that since Olympia, WA had a population of 245 thousand in 2008 and had been growing at 3% per year, the population could be modeled by the equation

Pn = (1+0.03)*n* (245,000), or equivalently, *Pn* = 245,000(1.03)*n*.

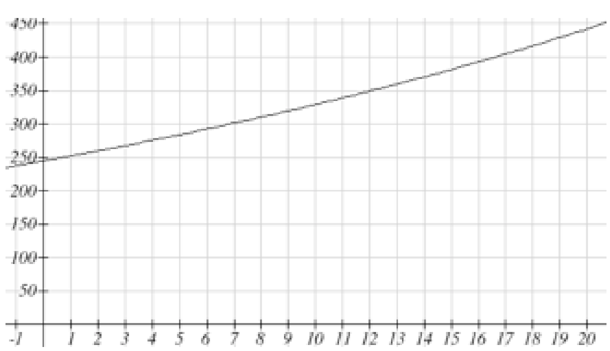
Using this equation, we were able to predict the population in the future.

Suppose we wanted to know when the population of Olympia would reach 400 thousand. Since we are looking for the year *n* when the population will be 400 thousand, we would need to solve the equation

400,000 = 245,000(1.03)*n* dividing both sides by 245,000 gives

1.6327 = 1.03*n*

One approach to this problem would be to create a table of values, or to use technology to draw a graph to estimate the solution.



From the graph, we can estimate that the solution will be around 16 to 17 years after 2008 (2024 to 2025). This is pretty good, but we'd really like to have an algebraic tool to answer this question. To do that, we need to introduce a new function that will undo exponentials, similar to how a square root undoes a square. For exponentials, the function we need is called a logarithm. It is the inverse of the exponential, meaning it undoes the exponential. While there is a whole family of logarithms with different bases, we will focus on the common log, which is based on the exponential 10*x*.

> Common Logarithm
> The common logarithm, written log(*x*), undoes the exponential 10*x*
> This means that log(10*x*) = *x*, and likewise 10log(*x*) = *x*
> This also means the statement 10*a* = *b* is equivalent to the statement log(*b*) = *a*
> log(*x*) is read as "log of *x*", and means "the logarithm of the value *x*". It is important to note that this is *not* multiplication – the log doesn't mean anything by itself, just like √ doesn't mean anything by itself; it has to be applied to a number.

Example 9

Evaluate each of the following

a) log(100) b) log(1000) c) log(10000) d) log(1/100) e) log(1)

a) log(100) can be written as log(102). Since the log undoes the exponential, log(102) = 2

b) log(1000) = log(103) = 3

c) log(10000) = log(104) = 4

d) Recall that $x^{-n} = \frac{1}{x^n}$. $\log\left(\frac{1}{100}\right) = \log\left(10^{-2}\right) = -2$

e) Recall that *x*0 = 1. log(1) = log(100) = 0

It is helpful to note that from the first three parts of the previous example that the number we're taking the log of has to get *10 times bigger* for the log to increase in value by 1.

Of course, most numbers cannot be written as a nice simple power of 10. For those numbers, we can evaluate the log using a scientific calculator with a log button.

Example 10

Evaluate log(300)

Using a calculator, log(300) is approximately 2.477121

With an equation, just like we can add a number to both sides, multiply both sides by a number, or square both sides, we can also take the logarithm of both sides of the equation and end up with an equivalent equation. This will allow us to solve some simple equations.

Example 11

a) Solve $10x = 1000$ b) Solve $10x = 3$ c) Solve $2(10x) = 8$

a) Taking the log of both sides gives $\log(10x) = \log(1000)$

Since the log undoes the exponential, $\log(10x) = x$. Similarly $\log(1000) = \log(103) = 3$.

The equation simplifies then to $x = 3$.

b) Taking the log of both sides gives $\log(10x) = \log(3)$.

On the left side, $\log(10x) = x$, so $x = \log(3)$. We can approximate this value with a calculator. $x \approx 0.477$

c) Here we would first want to isolate the exponential by dividing both sides of the equation by 2, giving

$10x = 4$.

Now we can take the log of both sides, giving $\log(10x) = \log(4)$, which simplifies to

$x = \log(4) \approx 0.602$

This approach allows us to solve exponential equations with powers of 10, but what about problems like $2 = 1.03n$ from earlier, which have a base of 1.03? For that, we need the exponent property for logs.

Properties of Logs: Exponent Property
$\log(A^r) = r\log(A)$

To show why this is true, we offer a proof.

Since the logarithm and exponential undo each other, $10^{\log A} = A$.

So $A^r = \left(10^{\log A}\right)^r$

Utilizing the exponential rule that states $\left(x^a\right)^b = x^{ab}$,

$A^r = \left(10^{\log A}\right)^r = 10^{r \log A}$

So then $\log(A^r) = \log\left(10^{r \log A}\right)$

Again utilizing the property that the log undoes the exponential on the right side yields the result

$\log(A^r) = r \log A$

Example 12

Rewrite log(25) using the exponent property for logs

log(25) = log(52) = 2log(5)

This property will finally allow us to answer our original question.

Solving exponential equations with logarithms
1. Isolate the exponential. In other words, get it by itself on one side of the equation. This usually involves dividing by a number multiplying it.
2. Take the log of both sides of the equation.
3. Use the exponent property of logs to rewrite the exponential with the variable exponent multiplying the logarithm.
4. Divide as needed to solve for the variable.

Example 13

If Olympia is growing according to the equation, *Pn* = 245(1.03)*n*, where *n* is years after 2008, and the population is measured in thousands. Find when the population will be 400 thousand.

We need to solve the equation

400 = 245(1.03)*n* Begin by dividing both sides by 245 to isolate the exponential

$1.633 = 1.03n$ Now take the log of both sides

$\log(1.633) = \log(1.03n)$ Use the exponent property of logs on the right side

$\log(1.633) = n \log(1.03)$ Now we can divide by log(1.03)

$\frac{\log(1.633)}{\log(1.03)} = n$ We can approximate this value on a calculator

$n \approx 16.591$

Alternatively, after applying the exponent property of logs on the right side, we could have evaluated the logarithms to decimal approximations and completed our calculations using those approximations, as you'll see in the next example. While the final answer may come out slightly differently, as long as we keep enough significant values during calculation, our answer will be close enough for most purposes.

Example 14

Polluted water is passed through a series of filters. Each filter removes 90% of the remaining impurities from the water. If you have 10 million particles of pollutant per gallon originally, how many filters would the water need to be passed through to reduce the pollutant to 500 particles per gallon?

In this problem, our "population" is the number of particles of pollutant per gallon. The initial pollutant is 10 million particles per gallon, so $P0$ = 10,000,000. Instead of changing with time, the pollutant changes with the number of filters, so n will represent the number of filters the water passes through.

Also, since the amount of pollutant is *decreasing* with each filter instead of increasing, our "growth" rate will be negative, indicating that the population is decreasing instead of increasing, so $r = -0.90$.

We can then write the explicit equation for the pollutant:

$Pn = 10{,}000{,}000(1 - 0.90)n = 10{,}000{,}000(0.10)n$

To solve the question of how many filters are needed to lower the pollutant to 500 particles per gallon, we can set Pn equal to 500, and solve for n.

$500 = 10{,}000{,}000(0.10)n$ Divide both sides by 10,000,000

$0.00005 = 0.10n$ Take the log of both sides

$\log(0.00005) = \log(0.10n)$ Use the exponent property of logs on the right side

$\log(0.00005) = n \log(0.10)$ Evaluate the logarithms to a decimal approximation

-4.301 = *n* (-1) Divide by -1, the value multiplying *n*

4.301 = *n*

It would take about 4.301 filters. Of course, since we probably can't install 0.3 filters, we would need to use 5 filters to bring the pollutant below the desired level.

Try it Now 4

India had a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, when will India's population reach 1.2 billion?

LOGISTIC GROWTH

In our basic exponential growth scenario, we had a recursive equation of the form

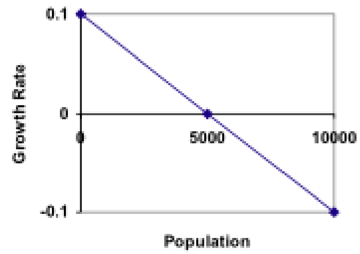
Pn = Pn-1 + r Pn-1

In a confined environment, however, the growth rate may not remain constant. In a lake, for example, there is some *maximum sustainable population* of fish, also called a **carrying capacity**.

Carrying Capacity
The **carrying capacity**, or **maximum sustainable population**, is the largest population that an environment can support.

For our fish, the carrying capacity is the largest population that the resources in the lake can sustain. If the population in the lake is far below the carrying capacity, then we would expect the population to grow essentially exponentially. However, as the population approaches the carrying capacity, there will be a scarcity of food and space available, and the growth rate will decrease. If the population exceeds the carrying capacity, there won't be enough resources to sustain all the fish and there will be a negative growth rate, causing the population to decrease back to the carrying capacity.

If the carrying capacity was 5000, the growth rate might vary something like that in the graph shown. Note that this is a linear equation with intercept at 0.1 and slope $-\frac{0.1}{5000}$, so we could write an equation for this adjusted growth rate as:



radjusted = $0.1 - \frac{0.1}{5000}P = 0.1\left(1 - \frac{P}{5000}\right)$

Substituting this in to our original exponential growth model for *r* gives

$$P_n = P_{n-1} + 0.1\left(1 - \frac{P_{n-1}}{5000}\right)P_{n-1}$$

> Logistic Growth
> If a population is growing in a constrained environment with carrying capacity *K*, and absent constraint would grow exponentially with growth rate *r*, then the population behavior can be described by the logistic growth model:
> $$P_n = P_{n-1} + r\left(1 - \frac{P_{n-1}}{K}\right)P_{n-1}$$

Unlike linear and exponential growth, logistic growth behaves differently if the populations grow steadily throughout the year or if they have one breeding time per year. The recursive formula provided above models generational growth, where there is one breeding time per year (or, at least a finite number); there is no explicit formula for this type of logistic growth.

Example 15

A forest is currently home to a population of 200 rabbits. The forest is estimated to be able to sustain a population of 2000 rabbits. Absent any restrictions, the rabbits would grow by 50% per year. Predict the future population using the logistic growth model.

Modeling this with a logistic growth model, *r* = 0.50, *K* = 2000, and *P0* = 200. Calculating the next year:

$$P_1 = P_0 + 0.50\left(1 - \frac{P_0}{2000}\right)P_0 = 200 + 0.50\left(1 - \frac{200}{2000}\right)200 = 290$$

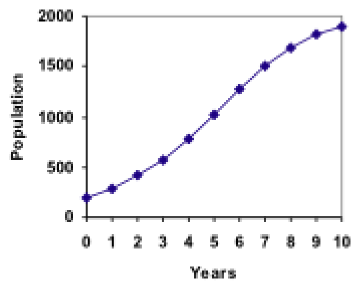
We can use this to calculate the following year:

$$P_2 = P_1 + 0.50\left(1 - \frac{P_1}{2000}\right)P_1 = 290 + 0.50\left(1 - \frac{290}{2000}\right)290 \approx 414$$

A calculator was used to compute several more values:

n	0	1	2	3	4	5	6	7	8	9	10
P_n	200	290	414	578	784	1022	1272	1503	1690	1821	1902

Plotting these values, we can see that the population starts to increase faster and the graph curves upwards during the first few years, like exponential growth, but then the growth slows down as the population approaches the carrying capacity.



Example 16

On an island that can support a population of 1000 lizards, there is currently a population of 600. These lizards have a lot of offspring and not a lot of natural predators, so have very high growth rate, around 150%. Calculating out the next couple generations:

$$P_1 = P_0 + 1.50\left(1 - \frac{P_0}{1000}\right)P_0 = 600 + 1.50\left(1 - \frac{600}{1000}\right)600 = 960$$

$$P_2 = P_1 + 1.50\left(1 - \frac{P_1}{1000}\right)P_1 = 960 + 1.50\left(1 - \frac{960}{1000}\right)960 = 1018$$

Interestingly, even though the factor that limits the growth rate slowed the growth a lot, the population still overshot the carrying capacity. We would expect the population to decline the next year.

$$P_3 = P_2 + 1.50\left(1 - \frac{P_3}{1000}\right)P_3 = 1018 + 1.50\left(1 - \frac{1018}{1000}\right)1018 = 991$$

Calculating out a few more years and plotting the results, we see the population wavers above and below the carrying capacity, but eventually settles down, leaving a steady population near the carrying capacity.

Try it Now 5

A field currently contains 20 mint plants. Absent constraints, the number of plants would increase by 70% each year, but the field can only support a maximum population of 300 plants. Use the logistic model to predict the population in the next three years.

Example 17

On a neighboring island to the one from the previous example, there is another population of lizards, but the growth rate is even higher – about 205%.

Calculating out several generations and plotting the results, we get a surprise: the population seems to be oscillating between two values, a pattern called a 2-cycle.

While it would be tempting to treat this only as a strange side effect of mathematics, this has actually been observed in nature. Researchers from the University of California observed a stable 2-cycle in a lizard population in California. ((Note: http://users.rcn.com/jkimball.ma.ultranet/BiologyPages/P/Populations2.html))

Taking this even further, we get more and more extreme behaviors as the growth rate increases higher. It is possible to get stable 4-cycles, 8-cycles, and higher. Quickly, though, the behavior approaches chaos (remember the movie Jurassic Park?).

Try it Now Answers

1. Letting $n = 0$ correspond with 1976, then $P_0 = 20{,}610$.

From 1976 to 2010 the number of stay-at-home fathers increased by

53,555 – 20,610 = 32,945

This happened over 34 years, giving a common different d of 32,945 / 34 = 969.

$P_n = 20{,}610 + 969n$

Predicting for 2020, we use $n = 44$

$P_{44} = 20{,}610 + 969(44) = 63{,}246$ stay-at-home fathers in 2020.

2. Using $n = 0$ corresponding with 2008,

$P_{12} = (1+0.0134)^{12} (1.14)$ = about 1.337 billion people in 2020

3. Here we will measure n in months rather than years, with $n = 0$ corresponding to the February when they went public. This gives $P_0 = 45$ thousand. October is 8 months later, so $P_8 = 60$.

$P_8 = (1+r)^8 P_0$

$60 = (1+r)^8 45$

$\frac{60}{45} = (1+r)^8$

$\sqrt[8]{\frac{60}{45}} = 1 + r$

$r = \sqrt[8]{\frac{60}{45}} - 1 = 0.0366$, or 3.66%

The general explicit equation is $P_n = (1.0366)^n 45$

Predicting 24 months (2 years) after they went public:

$P_{24} = (1.0366)^{24} 45 = 106.63$ thousand users.

4. $1.14(1.0134)^n = 1.2$. $n = 3.853$, which is during 2011

5. $P_1 = P_0 + 0.70\left(1 - \frac{P_0}{300}\right) P_0 = 20 + 0.70\left(1 - \frac{20}{300}\right) 20 = 33$

$P_2 = 54$

$P_3 = 85$

EXERCISES

Skills

1. Marko currently has 20 tulips in his yard. Each year he plants 5 more.
    a. Write a recursive formula for the number of tulips Marko has
    b. Write an explicit formula for the number of tulips Marko has
2. Pam is a Disc Jockey. Every week she buys 3 new albums to keep her collection current. She currently owns 450 albums.
    a. Write a recursive formula for the number of albums Pam has
    b. Write an explicit formula for the number of albums Pam has
3. A store's sales (in thousands of dollars) grow according to the recursive rule $P_n=P_{n-1} + 15$, with initial population $P_0=40$.
    a. Calculate P_1 and P_2
    b. Find an explicit formula for P_n
    c. Use your formula to predict the store's sales in 10 years
    d. When will the store's sales exceed $100,000?
4. The number of houses in a town has been growing according to the recursive rule $P_n=P_{n-1} + 30$, with initial population $P_0=200$.
    a. Calculate P_1 and P_2
    b. Find an explicit formula for P_n
    c. Use your formula to predict the number of houses in 10 years
    d. When will the number of houses reach 400 houses?
5. A population of beetles is growing according to a linear growth model. The initial population (week 0) was $P_0=3$, and the population after 8 weeks is $P_8=67$.
    a. Find an explicit formula for the beetle population in week n
    b. After how many weeks will the beetle population reach 187?
6. The number of streetlights in a town is growing linearly. Four months ago ($n = 0$) there were 130 lights. Now ($n = 4$) there are 146 lights. If this trend continues,
    a. Find an explicit formula for the number of lights in month n
    b. How many months will it take to reach 200 lights?
7. Tacoma's population in 2000 was about 200 thousand, and had been growing by about 9% each year.
    a. Write a recursive formula for the population of Tacoma
    b. Write an explicit formula for the population of Tacoma
    c. If this trend continues, what will Tacoma's population be in 2016?
    d. When does this model predict Tacoma's population to exceed 400 thousand?
8. Portland's population in 2007 was about 568 thousand, and had been growing by about 1.1% each year.
    a. Write a recursive formula for the population of Portland
    b. Write an explicit formula for the population of Portland
    c. If this trend continues, what will Portland's population be in 2016?
    d. If this trend continues, when will Portland's population reach 700 thousand?
9. Diseases tend to spread according to the exponential growth model. In the early days of AIDS, the growth rate was around 190%. In 1983, about 1700 people in the U.S. died of AIDS. If the trend had continued unchecked, how many people would have died from AIDS in 2005?
10. The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2015.
11. A bacteria culture is started with 300 bacteria. After 4 hours, the population has grown to 500 bacteria. If the population grows exponentially,
    a. Write a recursive formula for the number of bacteria
    b. Write an explicit formula for the number of bacteria
    c. If this trend continues, how many bacteria will there be in 1 day?
    d. How long does it take for the culture to triple in size?
12. A native wolf species has been reintroduced into a national forest. Originally 200 wolves were transplanted. After 3 years, the population had grown to 270 wolves. If the population grows exponentially,
    a. Write a recursive formula for the number of wolves
    b. Write an explicit formula for the number of wolves
    c. If this trend continues, how many wolves will there be in 10 years?
    d. If this trend continues, how long will it take the population to grow to 1000 wolves?
13. One hundred trout are seeded into a lake. Absent constraint, their population will grow by 70% a year. The lake can sustain a maximum of 2000 trout. Using the logistic growth model,
    a. Write a recursive formula for the number of trout
    b. Calculate the number of trout after 1 year and after 2 years.
14. Ten blackberry plants started growing in my yard. Absent constraint, blackberries will spread by 200% a month. My yard can only sustain about 50 plants. Using the logistic growth model,
    a. Write a recursive formula for the number of blackberry plants in my yard
    b. Calculate the number of plants after 1, 2, and 3 months

15. In 1968, the U.S. minimum wage was \$1.60 per hour. In 1976, the minimum wage was \$2.30 per hour. Assume the minimum wage grows according to an exponential model where *n* represents the time in years after 1960.
    a. Find an explicit formula for the minimum wage.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was \$5.15 in 1996, is this above, below or equal to what the model predicts?

Concepts

16. The population of a small town can be described by the equation $Pn = 4000 + 70n$, where *n* is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.
17. The population of a small town can be described by the equation $Pn = 4000(1.04)n$, where *n* is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.

Exploration

Most of the examples in the text examined growing quantities, but linear and exponential equations can also describe decreasing quantities, as the next few problems will explore.

18. A new truck costs \$32,000. The car's value will depreciate over time, which means it will lose value. For tax purposes, depreciation is usually calculated linearly. If the truck is worth \$24,500 after three years, write an explicit formula for the value of the car after *n* years.
19. Inflation causes things to cost more, and for our money to buy less (hence your grandparents saying, "In my day, you could buy a cup of coffee for a nickel"). Suppose inflation decreases the value of money by 5% each year. In other words, if you have \$1 this year, next year it will only buy you \$0.95 worth of stuff. How much will \$100 buy you in 20 years?
20. Suppose that you have a bowl of 500 M&M candies, and each day you eat ¼ of the candies you have. Is the number of candies left changing linearly or exponentially? Write an equation to model the number of candies left after *n* days.
21. A warm object in a cooler room will decrease in temperature exponentially, approaching the room temperature according to the formula where *Tn* is the temperature after *n* minutes, *r* is the rate at which temperature is changing, *a* is a constant, and *Tr* is the temperature of the room. Forensic investigators can use this to predict the time of death of a homicide victim. Suppose that when a body was discovered ($n = 0$) it was 85 degrees. After 20 minutes, the temperature was measured again to be 80 degrees. The body was in a 70 degree room.
    a. Use the given information with the formula provided to find a formula for the temperature of the body.
    b. When did the victim die, if the body started at 98.6 degrees?
22. Recursive equations can be very handy for modeling complicated situations for which explicit equations would be hard to interpret. As an example, consider a lake in which 2000 fish currently reside. The fish population grows by 10% each year, but every year 100 fish are harvested from the lake by people fishing.
    a. Write a recursive equation for the number of fish in the lake after *n* years.
    b. Calculate the population after 1 and 2 years. Does the population appear to be increasing or decreasing?
    c. What is the maximum number of fish that could be harvested each year without causing the fish population to decrease in the long run?
23. The number of Starbucks stores grew after first opened. The number of stores from 1990-2007, as reported on their corporate website ((Note: http://www.starbucks.com/aboutus/Company_Timeline.pdf retrieved May 2009)), is shown below.
    a. Carefully plot the data. Does is appear to be changing linearly or exponentially?
    b. Try finding an equation to model the data by picking two points to work from. How well does the equation model the data?
    c. Try using an equation of the form , where *k* is a constant, to model the data. This type of model is called a Power model. Compare your results to the results from part b. *Note: to use this model, you will need to have 1990 correspond with n = 1 rather than n = 0.*

Year	Number of Starbucks stores		Year	Number of Starbucks stores

1990	84		1999	2498
1991	116		2000	3501
1992	165		2001	4709
1993	272		2002	5886
1994	425		2003	7225
1995	677		2004	8569
1996	1015		2005	10241
1997	1412		2006	12440
1998	1886		2007	15756

24. Thomas Malthus was an economist who put forth the principle that population grows based on an exponential growth model, while food and resources grow based on a linear growth model. Based on this, Malthus predicted that eventually demand for food and resources would out outgrow supply, with doom-and-gloom consequences. Do some research about Malthus to answer these questions.
    a. What societal changes did Malthus propose to avoid the doom-and-gloom outcome he was predicting?
    b. Why do you think his predictions did not occur?
    c. What are the similarities and differences between Malthus's theory and the logistic growth model?

FINANCE

SIMPLE INTEREST

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = $5. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-time Interest

$I = P_0 r$

$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$

I is the interest

A is the end amount: principal plus interest

P0 is the principal (starting amount)

r is the interest rate (in decimal form. Example: 5% = 0.05)

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

P0 = $300 the principal

r = 0.03 3% rate

I = $300(0.03) = $9. You will earn $9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Each year, you would earn 5% interest: $1000(0.05) = $50 in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$I = P_0 rt$

$A = P_0 + I = P_0 + P_0 rt = P_0(1 + rt)$

I is the interest
A is the end amount: principal plus interest
P0 is the principal (starting amount)
r is the interest rate in decimal form
t is time
The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate
Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.
For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

P0 = $1000 the principal

r = 0.02 2% rate per half-year

t = 8 4 years = 8 half-years

I = $1000(0.02)(8) = $160. You will earn $160 interest total over the four years.

Try it Now 1

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

COMPOUND INTEREST

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3}{12}$ = 0.25% per month.

In the first month,

P0 = $1000

r = 0.0025 (0.25%)

I = $1000 (0.0025) = $2.50

A = $1000 + $2.50 = $1002.50

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

P0 = $1002.50

I = $1002.50 (0.0025) = $2.51 (rounded)

A = $1002.50 + $2.51 = $1005.01

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting balance	Interest earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85

12	1027.85	2.57	1030.42

To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

P_0 = \$1000

$P_m = (1+0.0025)P_{m-1}$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

P_0 = \$1000

$P_1 = 1.0025P_0 = 1.0025\ (1000)$

$P_2 = 1.0025P_1 = 1.0025\ (1.0025\ (1000)) = 1.0025^2(1000)$

$P_3 = 1.0025P_2 = 1.0025\ (1.0025^2(1000)) = 1.0025^3(1000)$

$P_4 = 1.0025P_3 = 1.0025\ (1.0025^3(1000)) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m$(\$1000)

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$$P_m = P_0\left(1+\frac{r}{k}\right)^m$$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual interest rate

k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If *N* is the number of years, then $m = N\,k$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$P_N = P_0\left(1+\frac{r}{k}\right)^{Nk}$$

P_N is the balance in the account after *N* years.

P_0 is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

In this example,

P0 = $3000 the initial deposit

$r = 0.06$ 6% annual rate

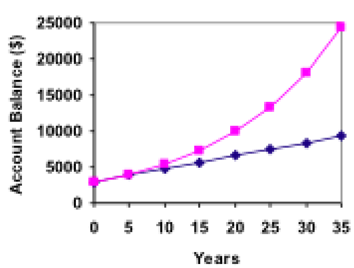
$k = 12$ 12 months in 1 year

$N = 20$ since we're looking for how much we'll have after 20 years

So $P_{20} = 3000\left(1 + \frac{0.06}{12}\right)^{20\times12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating exponents on the calculator
When we need to calculate something like 53 it is easy enough to just multiply 5·5·5=125. But when we need to calculate something like 1.005240 , it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators. Most scientific calculators have a button for exponents. It is typically either labeled like:
^ , yx , or xy .
To evaluate 1.005240 we'd type 1.005 ^ 240, or 1.005 yx 240. Try it out – you should get something around 3.3102044758.

Example 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

In this example,

We're looking for *P0*.

r = 0.04 4%

k = 4 4 quarters in 1 year

N = 18 Since we know the balance in 18 years

P18 = $40,000 The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for *P0*.

$$40000 = P_0\left(1+\frac{0.04}{4}\right)^{18\times 4}$$
$$40000 = P_0(2.0471)$$
$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding
It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 6

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

P0 = $1000 the initial deposit

r = 0.05 5%

$k = 12$ 12 months in 1 year

$N = 30$ since we're looking for the amount after 30 years

If we first compute *r/k*, we find 0.05/12 = 0.00416666666667

Here is the effect of rounding this to different values:

***r/k* rounded to:**	**Gives *P30* to be:**	**Error**
0.004	$4208.59	$259.15
0.0042	$4521.45	$53.71
0.00417	$4473.09	$5.35
0.004167	$4468.28	$0.54
0.0041667	$4467.80	$0.06
no rounding	$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough – $5 off of $4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate $P_{30} = 1000\left(1 + \frac{0.05}{12}\right)^{12\times30}$

We can quickly calculate 12×30 = 360, giving $P_{30} = 1000\left(1 + \frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type this	**Calculator shows**
0.05 ÷ 12 = .	0.00416666666667
+ 1 = .	1.00416666666667
yx 360 = .	4.46774431400613
× 1000 = .	4467.74431400613

Using your calculator continued

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 × (1 + 0.05 ÷ 12) y^x 360 = .

ANNUITIES

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, *d*, to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$r = 0.06$ (6%)

$k = 12$ (12 compounds/deposits per year)

$d = \$100$ (our deposit per month)

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)\, P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

Continuing this pattern, after *m* deposits, we'd have saved:

In other words, after *m* months, the first deposit will have earned compound interest for *m*-1 months. The second deposit will have earned interest for *m*-2 months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005\left(100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100\right)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \cdots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{array}{lclllll} 1.005P_m & = & 100(1.005)^m + & 100(1.005)^{m-1} & + \cdots + & 100(1.005) & \\ P_m & = & & 100(1.005)^{m-1} & + \cdots + & 100(1.005) & +100 \end{array}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for *Pm*

$$0.005P_m = 100\left((1.005)^m - 1\right)$$
$$P_m = \frac{100\left((1.005)^m - 1\right)}{0.005}$$

Replacing *m* months with 12*N*, where *N* is measured in years, gives

$$P_N = \frac{100\left((1.005)^{12N} - 1\right)}{0.005}$$

Recall 0.005 was *r/k* and 100 was the deposit *d.* 12 was *k*, the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$P_N = \frac{d\left(\left(1+\frac{r}{k}\right)^{Nk} - 1\right)}{\left(\frac{r}{k}\right)}$$

PN is the balance in the account after *N* years.
d is the regular deposit (the amount you deposit each year, each month, etc.)
r is the annual interest rate in decimal form.
k is the number of compounding periods in one year.
If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, *k* = 12.

If you make your deposits every year, use yearly compounding, *k* = 1.

If you make your deposits every quarter, use quarterly compounding, *k* = 4.

Etc.

When do you use this
Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.
Compound interest assumes that you put money in the account once and let it sit there earning interest.
Compound interest: One deposit
Annuity: Many deposits.

Example 7

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

In this example,

d = $100 the monthly deposit

r = 0.06 6% annual rate

k = 12 since we're doing monthly deposits, we'll compound monthly

N = 20 we want the amount after 20 years

Putting this into the equation:

$$P_{20} = \frac{100\left(\left(1+\frac{0.06}{12}\right)^{20(12)}-1\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_{20} = \frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$

$$P_{20} = \frac{100\,(3.310-1)}{(0.005)}$$

$$P_{20} = \frac{100\,(2.310)}{(0.005)} = \$46200$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $46,200 – $24,000 = $22,200.

Example 8

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

In this example,

We're looking for *d*.

r = 0.08 8% annual rate

k = 12 since we're depositing monthly

N = 30 30 years

P30 = $200,000 The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for *d*.

$$200,000 = \frac{d\left(\left(1+\frac{0.08}{12}\right)^{30(12)}-1\right)}{\left(\frac{0.08}{12}\right)}$$
$$200,000 = \frac{d\left((1.00667)^{360}-1\right)}{(0.00667)}$$
$$200,000 = d(1491.57)$$
$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit $134.09 each month to have $200,000 in 30 years if your account earns 8% interest

Try it Now 2

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

PAYOUT ANNUITIES

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d\left(1-\left(1+\frac{r}{k}\right)^{-Nk}\right)}{\left(\frac{r}{k}\right)}$$

P0 is the balance in the account at the beginning (starting amount, or principal).
d is the regular withdrawal (the amount you take out each year, each month, etc.)
r is the annual interest rate (in decimal form. Example: 5% = 0.05)
k is the number of compounding periods in one year.
N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.
Compound interest: One deposit
Annuity: Many deposits.
Payout Annuity: Many withdrawals

Example 9

After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

In this example,

d = $1000 the monthly withdrawal

r = 0.06 6% annual rate

k = 12 since we're doing monthly withdrawals, we'll compound monthly

N = 20 since were taking withdrawals for 20 years

We're looking for *P0*; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-20(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_0 = \frac{1000 \times \left(1-(1.005)^{-240}\right)}{(0.005)}$$

$$P_0 = \frac{1000 \times (1-0.302)}{(0.005)} = \$139,600$$

You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $240,000 – $139,600 = $100,400 in interest.

Evaluating negative exponents on your calculator
With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this (-) , some with +/- . The button is often near the = key or the decimal point.
If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005-240 you'd type something like: 1.005 ^ (-) 240
If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: 1.005 yx 240 (-) =
Give it a try – you should get 1.005-240 = 0.302096

Example 10

You know you will have $500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

In this example,

We're looking for *d*.

$r = 0.08$ 8% annual rate

$k = 12$ since we're withdrawing monthly

$N = 30$ 30 years

$P0 = \$500,000$ we are beginning with \$500,000

In this case, we're going to have to set up the equation, and solve for *d*.

$$500,000 = \frac{d\left(1-\left(1+\frac{0.08}{12}\right)^{-30(12)}\right)}{\left(\frac{0.08}{12}\right)}$$

$$500,000 = \frac{d\left(1-(1.00667)^{-360}\right)}{(0.00667)}$$

$$500,000 = d(136.232)$$

$$d = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw \$3,670.21 each month for 30 years.

Try it Now 3

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

LOANS

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d\left(1-\left(1+\frac{r}{k}\right)^{-Nk}\right)}{\left(\frac{r}{k}\right)}$$

P0 is the balance in the account at the beginning (the principal, or amount of the loan).
d is your loan payment (your monthly payment, annual payment, etc)
r is the annual interest rate in decimal form.
k is the number of compounding periods in one year.
N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this
The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.
Compound interest: One deposit
Annuity: Many deposits.
Payout Annuity: Many withdrawals
Loans: Many payments

Example 11

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

In this example,

d = $200 the monthly loan payment

r = 0.03 3% annual rate

k = 12 since we're doing monthly payments, we'll compound monthly

N = 5 since we're making monthly payments for 5 years

We're looking for *P0*, the starting amount of the loan.

$$P_0 = \frac{200\left(1-\left(1+\frac{0.03}{12}\right)^{-5(12)}\right)}{\left(\frac{0.03}{12}\right)}$$

$$P_0 = \frac{200\left(1-(1.0025)^{-60}\right)}{(0.0025)}$$

$$P_0 = \frac{200\,(1-0.861)}{(0.0025)} = \$11,120$$

You can afford a $11,120 loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $12,000-$11,120 = $880 interest total.

Example 12

You want to take out a $140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

In this example,

We're looking for *d*.

r = 0.06 6% annual rate

k = 12 since we're paying monthly

N = 30 30 years

$P0$ = $140,000 the starting loan amount

In this case, we're going to have to set up the equation, and solve for *d*.

$$140,000 = \frac{d\left(1-\left(1+\frac{0.06}{12}\right)^{-30(12)}\right)}{\left(\frac{0.06}{12}\right)}$$
$$140,000 = \frac{d\left(1-(1.005)^{-360}\right)}{(0.005)}$$
$$140,000 = d(166.792)$$
$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of $839.37 per month for 30 years.

You're paying a total of $302,173.20 to the loan company: $839.37 per month for 360 months. You are paying a total of $302,173.20 – $140,000 = $162,173.20 in interest over the life of the loan.

Try it Now 4

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

REMAINING LOAN BALANCE

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were $1,000 a month, after a year you will *not* have paid off $12,000 of the loan balance.

To determine the remaining loan balance, we can think “how much loan will these loan payments be able to pay off in the remaining time on the loan?”

Example 13

If a mortgage at a 6% interest rate has payments of $1,000 a month, how much will the loan balance be 10 years from the end the loan?

To determine this, we are looking for the amount of the loan that can be paid off by $1,000 a month payments in 10 years. In other words, we’re looking for P0 when

d = $1,000 the monthly loan payment

r = 0.06 6% annual rate

k = 12 since we’re doing monthly payments, we’ll compound monthly

N = 10 since we’re making monthly payments for 10 more years

$$P_0 = \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-10(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_0 = \frac{1000\left(1-(1.005)^{-120}\right)}{(0.005)}$$

$$P_0 = \frac{1000\,(1-0.5496)}{(0.005)} = \$90,073.45$$

The loan balance with 10 years remaining on the loan will be $90,073.45

Often times answering remaining balance questions requires two steps:

1) Calculating the monthly payments on the loan

2) Calculating the remaining loan balance based on the *remaining time* on the loan

Example 14

A couple purchases a home with a $180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

First we will calculate their monthly payments.

We’re looking for *d*.

r = 0.04 4% annual rate

k = 12 since they’re paying monthly

N = 30 30 years

P0 = $180,000 the starting loan amount

We set up the equation and solve for *d*.

$$180,000=\frac{d\left(1-\left(1+\frac{0.04}{12}\right)^{-30(12)}\right)}{\left(\frac{0.04}{12}\right)}$$
$$180,000=\frac{d\left(1-(1.00333)^{-360}\right)}{(0.00333)}$$
$$180,000=d(209.562)$$
$$d=\frac{180,000}{209.562}=\$858.93$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

d = \$858.93 the monthly loan payment we calculated above

r = 0.04 4% annual rate

k = 12 since they're doing monthly payments

N = 25 since they'd be making monthly payments for 25 more years

$$P_0=\frac{858.93\left(1-\left(1+\frac{0.04}{12}\right)^{-25(12)}\right)}{\left(\frac{0.04}{12}\right)}$$
$$P_0=\frac{858.93\left(1-(1.00333)^{-300}\right)}{(0.00333)}$$
$$P_0=\frac{858.93\,(1-0.369)}{(0.00333)}=\$155,793.91$$

The loan balance after 5 years, with 25 years remaining on the loan, will be \$155,793.91

Over that 5 years, the couple has paid off \$180,000 – \$155,793.91 = \$24,206.09 of the loan balance. They have paid a total of \$858.93 a month for 5 years (60 months), for a total of \$51,535.80, so \$51,535.80 – \$24,206.09 = \$27,329.71 of what they have paid so far has been interest.

WHICH EQUATION TO USE?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try it Now 5

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

SOLVING FOR TIME

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Example 15

If you invest $2000 at 6% compounded monthly, how long will it take the account to double in value?

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$P0$ = $2000 the initial deposit

r = 0.06 6% annual rate

k = 12 12 months in 1 year

So our general equation is $P_N = 2000\left(1+\frac{0.06}{12}\right)^{N\times 12}$. We also know that we want our ending amount to be double of $2000, which is $4000, so we're looking for *N* so that *PN* = 4000. To solve this, we set our equation for *PN* equal to 4000.

$4000 = 2000\left(1+\frac{0.06}{12}\right)^{N\times 12}$ Divide both sides by 2000

$2 = (1.005)^{12N}$ To solve for the exponent, take the log of both sides

$\log(2) = \log\left((1.005)^{12N}\right)$ Use the exponent property of logs on the right side

$\log(2) = 12N\log(1.005)$ Now we can divide both sides by 12log(1.005)

$\frac{\log(2)}{12\log(1.005)} = N$ Approximating this to a decimal

N = 11.581

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded log(2) to 0.301 and log(1.005) to 0.00217, then your final answer would have been about 11.577 years.

Example 16

If you invest $100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to $10,000?

This is a savings annuity problem since we are making regular deposits into the account.

d = $100 the monthly deposit

r = 0.03 3% annual rate

k = 12 since we're doing monthly deposits, we'll compound monthly

We don't know *N*, but we want *PN* to be $10,000.

Putting this into the equation:

$10,000 = \frac{100\left(\left(1+\frac{0.03}{12}\right)^{N(12)}-1\right)}{\left(\frac{0.03}{12}\right)}$ Simplifying the fractions a bit

$10,000 = \frac{100\left((1.0025)^{12N}-1\right)}{0.0025}$

We want to isolate the exponential term, 1.002512*N*, so multiply both sides by 0.0025

$25 = 100\left((1.0025)^{12N}-1\right)$ Divide both sides by 100

$0.25 = (1.0025)^{12N}-1$ Add 1 to both sides

$1.25 = (1.0025)^{12N}$ Now take the log of both sides

$\log(1.25) = \log\left((1.0025)^{12N}\right)$ Use the exponent property of logs

$\log(1.25) = 12N\log(1.0025)$ Divide by 12log(1.0025)

$\frac{\log(1.25)}{12\log(1.0025)} = N$ Approximating to a decimal

N = 7.447 years

It will take about 7.447 years to grow the account to $10,000.

Try it Now 6

Joel is considering putting a $1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of $30 a month?

Try it Now Answers

1.

I = $30 of interest

$P0$ = $500 principal

r = unknown

t = 1 month

Using $I = P0rt$, we get 30 = 500·r·1. Solving, we get r = 0.06, or 6%. Since the time was monthly, this is the monthly interest. The annual rate would be 12 times this: 72% interest.

2.

d = $5 the daily deposit

r = 0.03 3% annual rate

k = 365 since we're doing daily deposits, we'll compound daily

N = 10 we want the amount after 10 years

$P_{10} = \frac{5\left(\left(1+\frac{0.03}{365}\right)^{365\times10}-1\right)}{\frac{0.03}{365}}$ =$21,282.07

We would have deposited a total of $5·365·10 = $18,250, so $3,032.07 is from interest

3.

d = unknown

r = 0.04 4% annual rate

k = 1 since we're doing annual scholarships

N = 20 20 years

$P0$ = 100,000 we're starting with $100,000

$$100,000 = \frac{d\left(1-\left(1+\frac{0.04}{1}\right)^{-20\times1}\right)}{\frac{0.04}{1}}$$

Solving for *d* gives \$7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, \$100,000(0.04) = \$4,000 a year would have been available.

Try it Now Answers continued

4.

d = unknown

r = 0.16 16% annual rate

k = 12 since we're making monthly payments

N = 2 2 years to repay

P0 = 3,000 we're starting with a \$3,000 loan

$$3,000 = \frac{d\left(1-\left(1+\frac{0.16}{12}\right)^{-2\times 12}\right)}{\frac{0.16}{12}}$$

Solving for *d* gives \$146.89 as monthly payments.

In total, she will pay \$3,525.36 to the store, meaning she will pay \$525.36 in interest over the two years.

5.

6.

d = \$30 The monthly payments

r = 0.12 12% annual rate

k = 12 since we're making monthly payments

P0 = 1,000 we're starting with a \$1,000 loan

We are solving for *N*, the time to pay off the loan

$$1,000 = \frac{30\left(1-\left(1+\frac{0.12}{12}\right)^{-N(12)}\right)}{\frac{0.12}{12}}$$

Solving for *N* gives 3.396. It will take about 3.4 years to pay off the purchase.

EXERCISES

Skills

1. A friend lends you $200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
2. Suppose you obtain a $3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of $10,000 for $9,800. This means that in 13 weeks, the government will give you the face value, earning you $200. What annual interest rate have you earned?
4. Suppose you are looking to buy a $5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?
5. You deposit $300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
6. How much will $1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
7. You deposit $2000 in an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 20 years?
    b. How much interest will you earn?
8. You deposit $10,000 in an account earning 4% interest compounded monthly.
    a. How much will you have in the account in 25 years?
    b. How much interest will you earn?
9. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. How much would you need to deposit in an account now in order to have $20,000 in the account in 4 years? Assume the account earns 5% interest.
11. You deposit $200 each month into an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 30 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
12. You deposit $1000 each year into an account earning 8% compounded annually.
    a. How much will you have in the account in 10 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
13. Jose has determined he needs to have $800,000 for retirement in 30 years. His account earns 6% interest.
    a. How much would you need to deposit in the account each month?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
14. You wish to have $3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
15. You want to be able to withdraw $30,000 each year for 25 years. Your account earns 8% interest.
    a. How much do you need in your account at the beginning
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?
16. How much money will I need to have at retirement so I can withdraw $60,000 a year for 20 years from an account earning 8% compounded annually?
    a. How much do you need in your account at the beginning
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?
17. You have $500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?
18. Loren already knows that he will have $500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?
19. You can afford a $700 per month mortgage payment. You've found a 30 year loan at 5% interest.
    a. How big of a loan can you afford?
    b. How much total money will you pay the loan company?
    c. How much of that money is interest?
20. Marie can afford a $250 per month car payment. She's found a 5 year loan at 7% interest.
    a. How expensive of a car can she afford?
    b. How much total money will she pay the loan company?
    c. How much of that money is interest?
21. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
22. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?

23. You want to buy a $200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
    a. How much is the loan amount going to be?
    b. What will your monthly payments be if the interest rate is 5%?
    c. What will your monthly payments be if the interest rate is 6%?
24. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
    a. Find her monthly payments.
    b. How much interest will she pay over the life of the loan?
25. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate. How much does he still owe on the car?
26. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years. How much does she still owe on the mortgage?
27. Pat deposits $6,000 into an account earning 4% compounded monthly. How long will it take the account to grow to $10,000?
28. Kay is saving $200 a month into an account earning 5% interest. How long will it take her to save $20,000?
29. James has $3,000 in credit card debt, which charges 14% interest. How long will it take to pay off the card if he makes the minimum payment of $60 a month?
30. Chris has saved $200,000 for retirement, and it is in an account earning 6% interest. If she withdraws $3,000 a month, how long will the money last?

Concepts

31. Suppose you invest $50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?
32. Suppose you put off making investments for the first 5 years, and instead made deposits of $50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?
33. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing $10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?
34. Kendra wants to be able to make withdrawals of $60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?
35. You have $2,000 to invest, and want it to grow to $3,000 in two years. What interest rate would you need to find to make this possible?
36. You have $5,000 to invest, and want it to grow to $20,000 in ten years. What interest rate would you need to find to make this possible?
37. You plan to save $600 a month for the next 30 years for retirement. What interest rate would you need to have $1,000,000 at retirement?
38. You really want to buy a used car for $11,000, but can only afford $200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?

Exploration

39. Pay day loans are short term loans that you take out against future paychecks: The company advances you money against a future paycheck. Either visit a pay day loan company, or look one up online. Be forewarned that many companies do not make their fees obvious, so you might need to do some digging or look at several companies.
    a. Explain the general method by which the loan works.
    b. We will assume that we need to borrow $500 and that we will pay back the loan in 14 days. Determine the total amount that you would need to pay back and the effective loan rate. The effective loan rate is the percentage of the original loan amount that you pay back. It is not the same as the APR (annual rate) that is probably published.
    c. If you cannot pay back the loan after 14 days, you will need to get an extension for another 14 days. Determine the fees for an extension, determine the total amount you will be paying for the now 28 day loan, and compute the effective loan rate.
40. Suppose that 10 years ago you bought a home for $110,000, paying 10% as a down payment, and financing the rest at 9% interest for 30 years.
    a. Let's consider your existing mortgage:
        i. How much money did you pay as your down payment?
        ii. How much money was your mortgage (loan) for?

iii. What is your current monthly payment?
iv. How much total interest will you pay over the life of the loan?

b. This year, you check your loan balance. Only part of your payments have been going to pay down the loan; the rest has been going towards interest. You see that you still have $88,536 left to pay on your loan. Your house is now valued at $150,000.
   i. How much of the loan have you paid off? (i.e., how much have you reduced the loan balance by? Keep in mind that interest is charged each month – it's not part of the loan balance.)
   ii. How much money have you paid to the loan company so far?
   iii. How much interest have you paid so far?
   iv. How much equity do you have in your home (equity is value minus remaining debt)

c. Since interest rates have dropped, you consider refinancing your mortgage at a lower 6% rate.
   i. If you took out a new 30 year mortgage at 6% for your remaining loan balance, what would your new monthly payments be?
   ii. How much interest will you pay over the life of the new loan?

d. Notice that if you refinance, you are going to be making payments on your home for another 30 years. In addition to the 10 years you've already been paying, that's 40 years total.
   i. How much will you save each month because of the lower monthly payment?
   ii. How much total interest will you be paying (you need to consider the amount from 2c and 3b)
   iii. Does it make sense to refinance? (there isn't a correct answer to this question. Just give your opinion and your reason)

DESCRIBING DATA

PRESENTING CATEGORICAL DATA GRAPHICALLY

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table.**

Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Color	**Frequency**
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

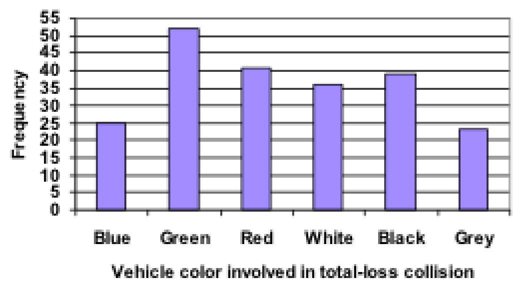
Bar graph

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

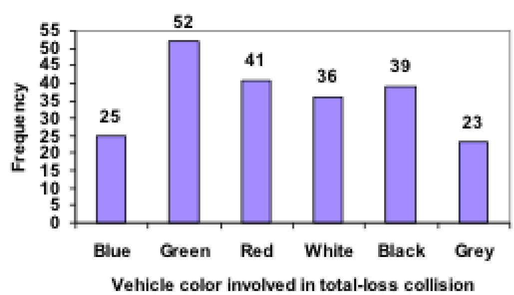
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. The construction of a bar chart is most easily described by use of an example.

Example 2

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



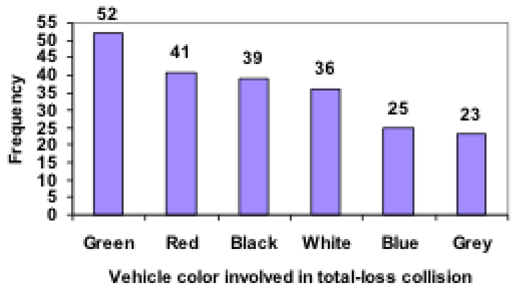
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

Pareto chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency

Example 3

Transforming our bar graph from earlier into a Pareto chart, we get:

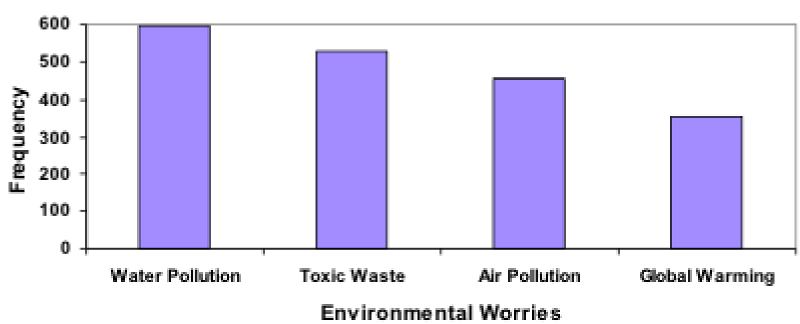


Example 4

In a survey ((Note: Gallup Poll. March 5-8, 2009. http://www.pollingreport.com/enviro.htm)), adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried "a great deal" about some selected concerns are summarized below.

Environmental Issue	**Frequency**
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

This data could be shown graphically in a bar graph:



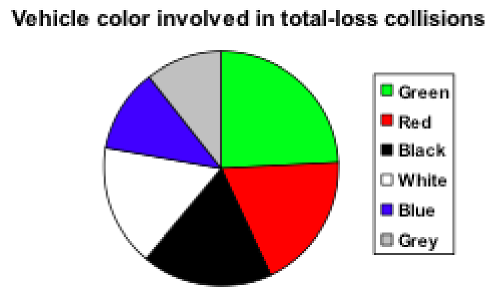
To show relative sizes, it is common to use a pie chart.

Pie Chart

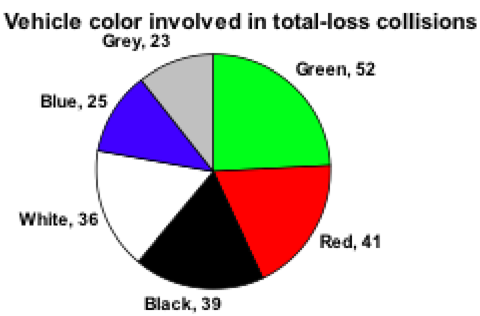
A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example 5

For our vehicle color data, a pie chart might look like this:



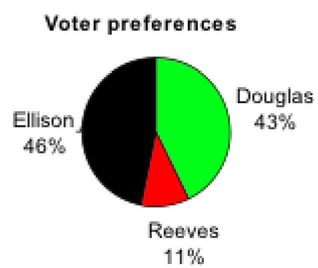
Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.



Example 6

The pie chart to the right shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.



Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs. ((Note: For example: http://nces.ed.gov/nceskids/createAgraph/ or http://docs.google.com))

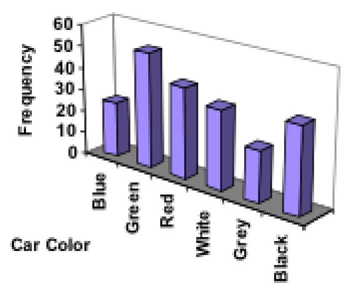
Try it Now 1

Create a bar graph and a pie chart to illustrate the grades on a history exam below.

A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.

Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.



Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example 7

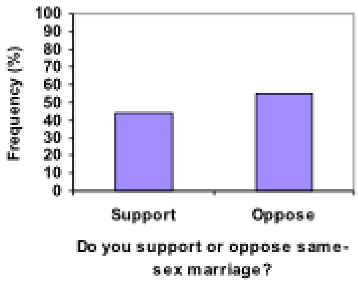
A labor union might produce the graph to the right to show the difference between the average manager salary and the average worker salary.

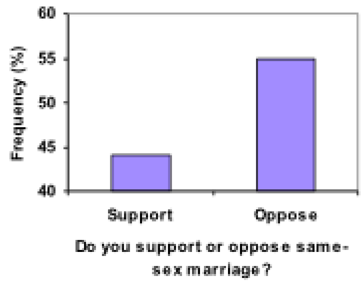
Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

Example 8

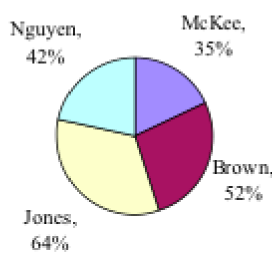
Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008 ((Note: CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from http://www.pollingreport.com/civil.htm)). The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.





Try it Now 2

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.



PRESENTING QUANTITATIVE DATA GRAPHICALLY

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 9

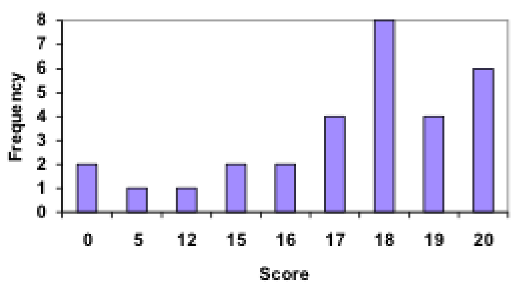
A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by grouping like values:

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary, like we did for categorical data:



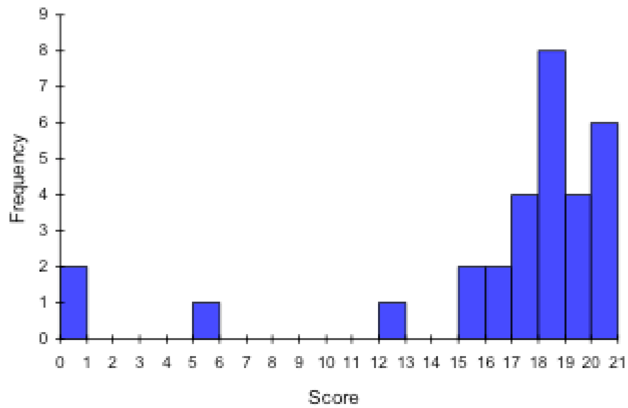
However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Histogram

A histogram is like a bar graph, but where the horizontal axis is a number line

Example 10

For the values above, a histogram would look like:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at ½ values to avoid this ambiguity.

Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

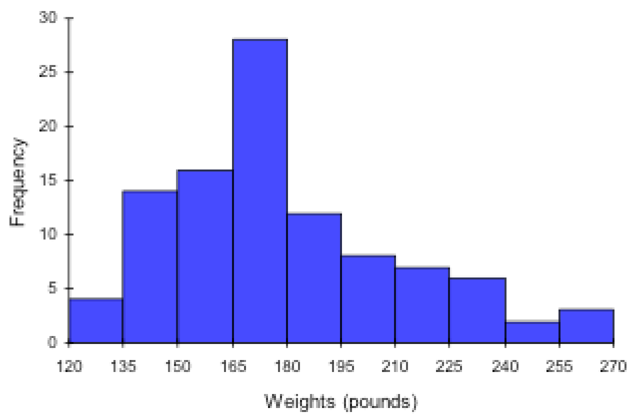
Example 11

Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

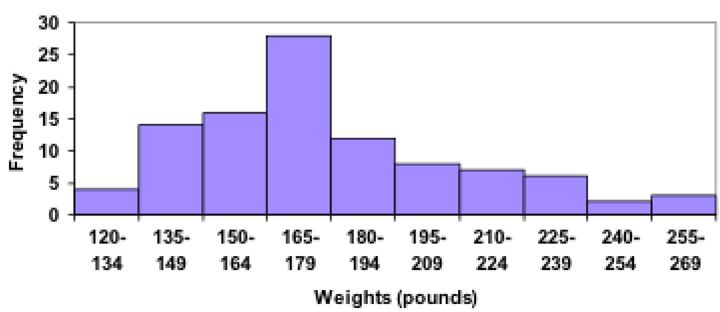
Interval	Frequency
120 – 134	4
135 – 149	14
150 – 164	16
165 – 179	28

180 – 194	12
195 – 209	8
210 – 224	7
225 – 239	6
240 – 254	2
255 – 269	3

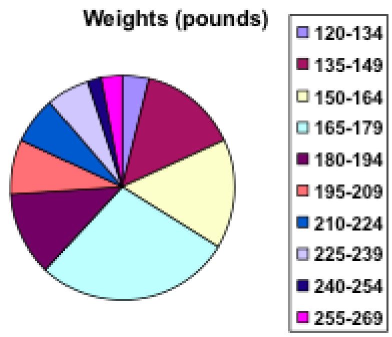
A histogram of this data would look like:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



Try it Now 3

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140	$160	$160	$165	$180	$220	$235	$240	$250	$260	$280	$285
$285	$285	$290	$300	$300	$305	$310	$310	$315	$315	$320	$320
$330	$340	$345	$350	$355	$360	$360	$380	$395	$420	$460	$460

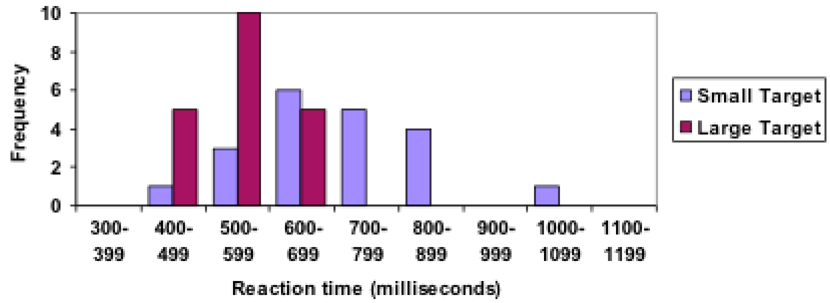
When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

Example 12

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Interval (milliseconds)	**Frequency small target**	**Frequency large target**
300-399	0	0
400-499	1	5
500-599	3	10
600-699	6	5
700-799	5	0
800-899	4	0
900-999	0	0
1000-1099	1	0
1100-1199	0	0

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.

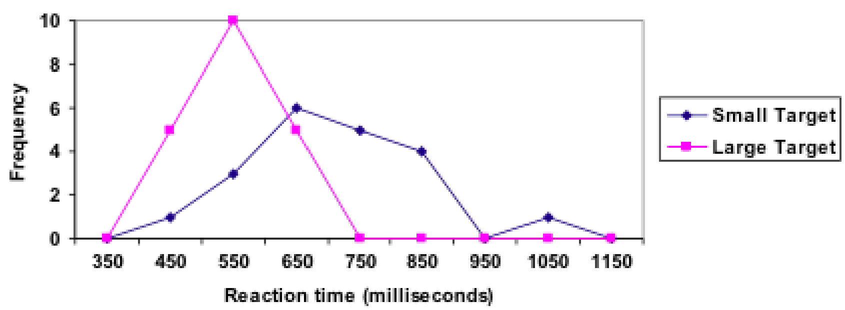


Frequency polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 13

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



MEASURES OF CENTRAL TENDENCY

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second section describes measures of variability.

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 14

Marci's exam scores for her last math class were: 79, 86, 82, 94. The mean of these values would be:

$\frac{79+86+82+94}{4} = 85.25$. Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 15

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get 634/31 = 20.4516. It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean."

Try it Now 4

The price of a jar of peanut butter at 5 stores was: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

Example 16

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below.

Income (thousands of dollars)	**Frequency**
15	6
20	8
25	11
30	17

35	19
40	20
45	12
50	7

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{\overbrace{15+\cdots+15}^{6\text{terms}}+\overbrace{20+\cdots+20}^{8\text{terms}}+\overbrace{25+\cdots+25}^{11\text{terms}}+\cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as = 90. Using this simplification, we get

$$\frac{15\cdot6+20\cdot8+25\cdot11+30\cdot17+35\cdot19+40\cdot20+45\cdot12+50\cdot7}{100}=\frac{3390}{100}=33.9$$

The mean household income of our sample is 33.9 thousand dollars ($33,900).

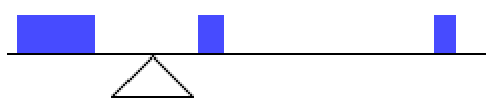
Example 17

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5000 thousand). Adding this to our sample, our mean is now:

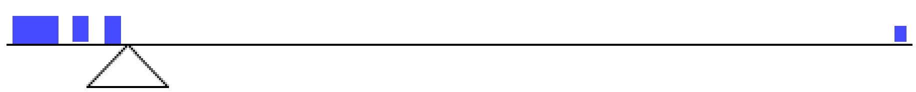
$$\frac{15\cdot6+20\cdot8+25\cdot11+30\cdot17+35\cdot19+40\cdot20+45\cdot12+50\cdot7+5000\cdot1}{101}=\frac{8390}{101}=83.069$$

While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

Median

The **median** of a set of data is the value in the middle when the data is in order
To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.
If the number of data values, *N*, is odd, then the median is the middle data value. This value can be found by rounding *N*/2 up to the next whole number.
If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values *N*/2 and *N*/2 + 1)

Example 18

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value (31/2 = 15.5, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 19

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get (6+7)/2 = 6.5.

The median quiz score was 6.5.

Try it Now 5

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Example 20

Let us return now to our original household income data

Income (thousands of dollars)	**Frequency**
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values – the 50th and 51st data values. To find these, we start counting up from the bottom:

There are 6 data values of $15, so	Values 1 to 6 are $15 thousand
The next 8 data values are $20, so	Values 7 to (6+8)=14 are $20 thousand
The next 11 data values are $25, so	Values 15 to (14+11)=25 are $25 thousand
The next 17 data values are $30, so	Values 26 to (25+17)=42 are $30 thousand
The next 19 data values are $35, so	Values 43 to (42+19)=61 are $35 thousand

From this we can tell that values 50 and 51 will be $35 thousand, and the mean of these two values is $35 thousand. The median income in this neighborhood is $35 thousand.

Example 21

If we add in the new neighbor with a $5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is $35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the “typical” value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 22

In our vehicle color survey, we collected the data

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try it Now 6

Reviewers were asked to rate a product on a scale of 1 to 5. Find

a. The mean rating

b. The median rating

c. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

MEASURES OF VARIATION

Consider these three sets of quiz scores:

Section A: 5 5 5 5 5 5 5 5 5 5

Section B: 0 0 0 0 0 10 10 10 10 10

Section C: 4 4 4 5 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range**.

Range

The range is the difference between the maximum value and the minimum value of the data set.

Example 23

Using the quiz scores from above,

For section A, the range is 0 since both maximum and minimum are 5 and 5 – 5 = 0

For section B, the range is 10 since 10 – 0 = 10

For section C, the range is 2 since 6 – 4 = 2

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Standard deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	deviation: data value – mean
0	0-5 = -5
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
10	10-5 = 5

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

data value	deviation: data value – mean	deviation squared
0	0-5 = -5	(-5)2 = 25
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0

5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
10	10-5 = 5	(5)2 = 25

We then add the squared deviations up to get 25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50. Ordinarily we would then divide by the number of scores, *n*, (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by *n* – 1 (in this case, 10 – 1 = 9). ((Note: The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.))

So in our example, we would have 50/10 = 5 if section D represents a population and 50/9 = about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

$$\text{populationstandarddeviation} = \sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$$

$$\text{or}$$

$$\text{samplestandarddeviation} = \sqrt{\frac{50}{9}} \approx 2.4$$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To compute standard deviation:

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by *n*, the number of data values, if the data represents a whole population; divide by *n* – 1 if the data is from a sample.
5. Compute the square root of the result.

Example 24

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	deviation: data value – mean	deviation squared

0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}} = \sqrt{\frac{250}{10}} = 5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Try it Now 7

The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, **quartiles** are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.
The first quartile (Q1) is the value so that 25% of the data values are below it; the third quartile (Q3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.
This divides the data into quarters; 25% of the data is between the minimum and Q1, 25% is between Q1 and the median, 25% is between the median and Q3, and 25% is between Q3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five number summary

The five number summary takes this form:
Minimum, Q1, Median, Q3, Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If n is the number of data values, we compute a locator by finding 25% of n. If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

To find the first quartile, Q1

Begin by ordering the data from smallest to largest
Compute the locator: $L = 0.25n$
If L is a decimal value:
Round up to $L+$
Use the data value in the $L+$th position
If L is a whole number:
Find the mean of the data values in the Lth and $L+1$th positions.

To find the third quartile, Q3

Use the same procedure as for Q1, but with locator: $L = 0.75n$
Examples should help make this clearer.

Example 25

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is 0.75(9) = 6.75. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Example 26

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value *is* a whole number, we will find the mean of the 2nd and 3rd data values: (60+62)/2 = 61, so the first quartile is 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: (67+69)/2 = 68, so Q3 is 68.

Note that the median could be computed the same way, using 50%.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 27

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary would be: 59, 61, 65, 68, 70.

Example 28

Returning to our quiz score data. In each case, the first quartile locator is 0.25(10) = 2.5, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Section and data	5-number summary
Section A: 5 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 5 6 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly, since the summary contains almost as many values as the original data.

Try it Now 8

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

$140	$160	$160	$165	$180	$220	$235	$240	$250	$260	$280	$285
$285	$285	$290	$300	$300	$305	$310	$310	$315	$315	$320	$320
$330	$340	$345	$350	$355	$360	$360	$380	$395	$420	$460	$460

Example 29

Returning to the household income data from earlier, create the five-number summary.

Income (thousands of dollars)	**Frequency**
15	6

20	8
25	11
30	17
35	19
40	20
45	12
50	7

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was $35 thousand. We can see in the table that the minimum income is $15 thousand, and the maximum is $50 thousand.

To find Q1, we calculate the locator: L = 0.25(100) = 25. This is a whole number, so Q1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of $15, so Values 1 to 6 are $15 thousand

The next 8 data values are $20, so Values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so Values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so Values 26 to (25+17)=42 are $30 thousand

The 25th data value is $25 thousand, and the 26th data value is $30 thousand, so Q1 will be the mean of these: (25 + 30)/2 = $27.5 thousand.

To find Q3, we calculate the locator: L = 0.75(100) = 75. This is a whole number, so Q3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are $35, so Values 43 to (42+19)=61 are $35 thousand

The next 20 data values are $40, so Values 61 to (61+20)=81 are $40 thousand

Both the 75th and 76th data values lie in this group, so Q3 will be $40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between $40 thousand and $50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

Box plot

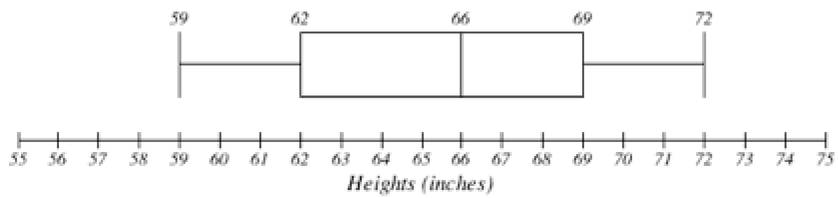
A **box plot** is a graphical representation of a five-number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

Example 30

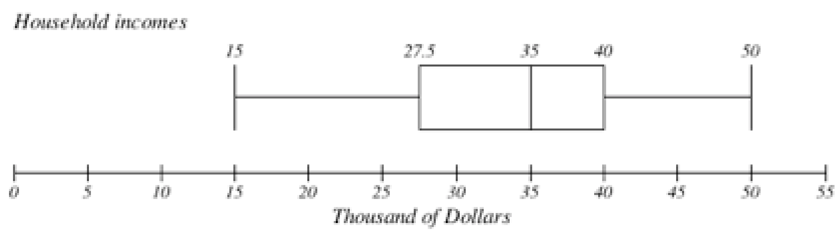
The box plot below is based on the 9 female height data with 5 number summary:

59, 62, 66, 69, 72.



Example 31

The box plot below is based on the household income data with 5 number summary:



15, 27.5, 35, 40, 50

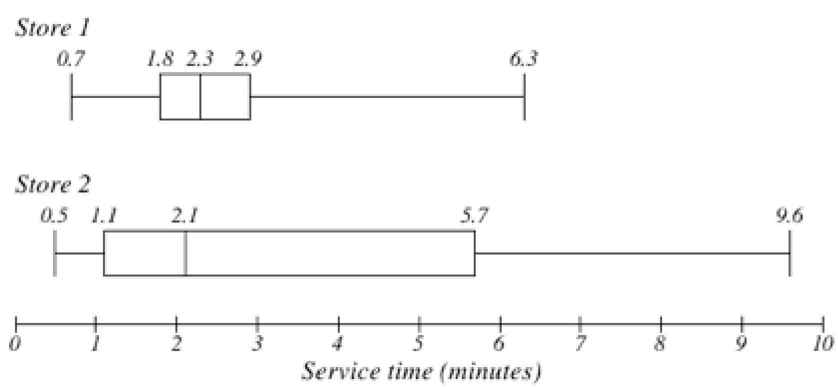
Try it Now 9

Create a boxplot based on the textbook price data from the last Try it Now.

Box plots are particularly useful for comparing data from two populations.

Example 32

The box plot of service times for two fast-food restaurants is shown below.



While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 33

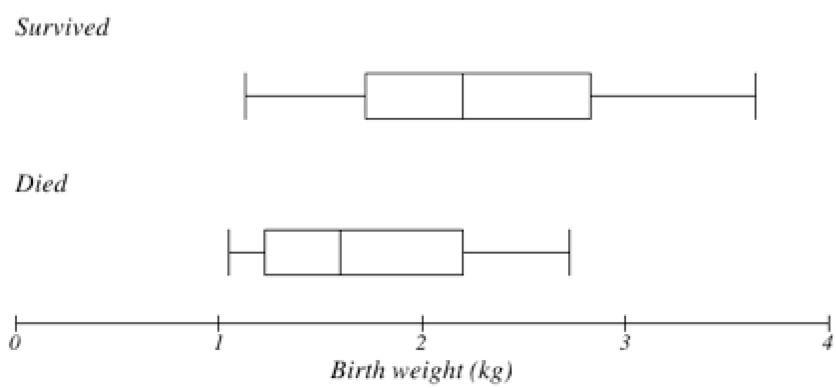
The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS) ((Note: van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. Arch. Disease in Childhood, 48, 249–255. As quoted on http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398296&section=1.1.3)). The boxplot is separated to show the birth weights of infants who survived and those that did not.

Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

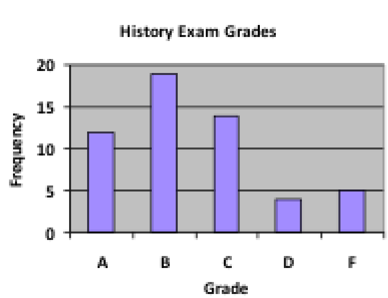
Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

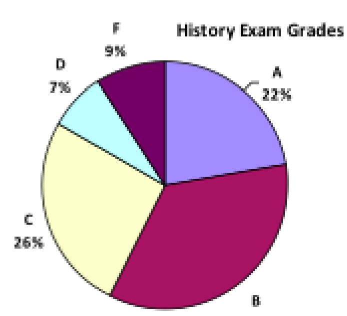
Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.



Try it Now Answers

1. 



2. While the pie chart accurately depicts the relative size of the people agreeing with each candidate, the chart is confusing, since usually percents on a pie chart represent the percentage of the pie the slice represents.
3. Using a class intervals of size 55, we can group our data into six intervals:

Cost interval	Frequency
$140-194	5
$195-249	3
$250-304	9
$305-359	12
$360-414	4
$415-469	3

We can use the frequency distribution to generate the histogram

4. Adding the prices and dividing by 5 we get the mean price: $3.682
5. First we put the data in order: $3.29, $3.59, $3.75, $3.79, $3.99. Since there are an odd number of data, the median will be the middle value, $3.75.
6. There are 23 ratings.
   a. The mean is $\frac{1\cdot4+2\cdot8+3\cdot7+4\cdot3+5\cdot1}{23} \approx 2.5$
   b. There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median is 2.
   c. The mode is the most frequent rating. The mode rating is 2.
7. Earlier we found the mean of the data was $3.682.

data value	deviation: data value – mean	deviation squared
3.29	3.29 – 3.682 = -0.391	0.153664
3.59	3.59 – 3.682 = -0.092	0.008464
3.79	3.79 – 3.682 = 0.108	0.011664
3.75	3.75 – 3.682 = 0.068	0.004624
3.99	3.99 – 3.682 = 0.308	0.094864

   This data is from a sample, so we will add the squared deviations, divide by 4, the number of data values minus 1, and compute the square root: $\sqrt{\frac{0.153664+0.008464+0.011664+0.004624+0.094864}{4}} \approx \$0.261$
8. The data is already in order, so we don't need to sort it first.The minimum value is $140 and the maximum is $460.There are 36 data values so *n* = 36. *n*/2 = 18, which is a whole number, so the median is the mean of the 18th and 19th data values, $305 and $310. The median is $307.50.

   To find the first quartile, we calculate the locator, L = 0.25(36) = 9. Since this is a whole number, we know Q1 is the mean of the 9th and 10th data values, $250 and $260. Q1 = $255.

   To find the third quartile, we calculate the locator, L = 0.75(36) = 27. Since this is a whole number, we know Q3 is the mean of the 27th and 28th data values, $345 and $350. Q3 = $347.50.

   The 5 number summary of this data is: $140, $255, $307.50, $347.50, $460
9. Boxplot of textbook costs

EXERCISES

Skills

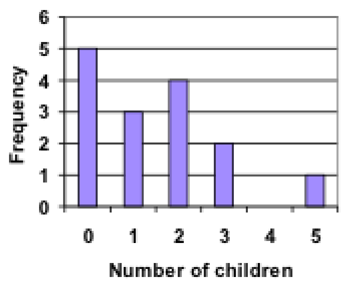
1. The table below shows scores on a Math test.
    a. Complete the frequency table for the Math test scores
    b. Construct a histogram of the data
    c. Construct a pie chart of the data

80	50	50	90	70	70	100	60	70	80	70	50
90	100	80	70	30	80	80	70	100	60	60	50

2. A group of adults where asked how many cars they had in their household
    a. Complete the frequency table for the car number data
    b. Construct a histogram of the data
    c. Construct a pie chart of the data

1	4	2	2	1	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

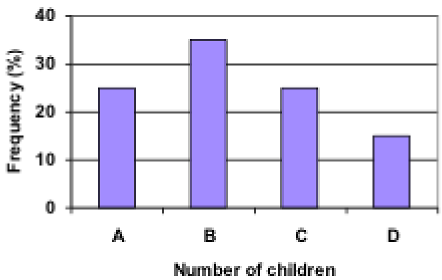
3. A group of adults were asked how many children they have in their families. The bar graph to the right shows the number of adults who indicated each number of children.
    a. How many adults where questioned?
    b. What percentage of the adults questioned had 0 children?



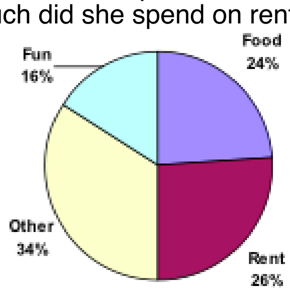
4. Jasmine was interested in how many days it would take an order from Netflix to arrive at her door. The graph below shows the data she collected.
    a. How many movies did she order?

b. What percentage of the movies arrived in one day?

5. The bar graph below shows the *percentage* of students who received each letter grade on their last English paper. The class contains 20 students. What number of students earned an A on their paper?



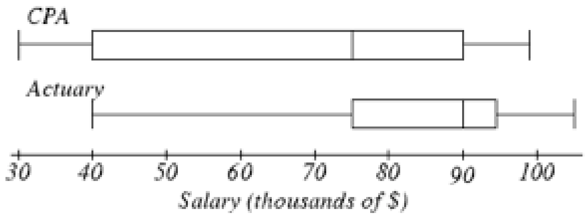
6. Kori categorized her spending for this month into four categories: Rent, Food, Fun, and Other. The percents she spent in each category are pictured here. If she spent a total of $2600 this month, how much did she spend on rent?



7. A group of diners were asked how much they would pay for a meal. Their responses were: $7.50, $8.25, $9.00, $8.00, $7.25, $7.50, $8.00, $7.00.
    1. Find the mean
    2. Find the median
    3. Write the 5-number summary for this data
8. You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
    1. Find the mean
    2. Find the median
    3. Write the 5-number summary for this data
9. Refer back to the histogram from question #3.
    1. Compute the mean number of children for the group surveyed
    2. Compute the median number of children for the group surveyed
    3. Write the 5-number summary for this data
    4. Create box plot
10. Refer back to the histogram from question #4.
    1. Computer the mean number of shipping days
    2. Compute the median number of shipping days
    3. Write the 5-number summary for this data
    4. Create box plot

Concepts

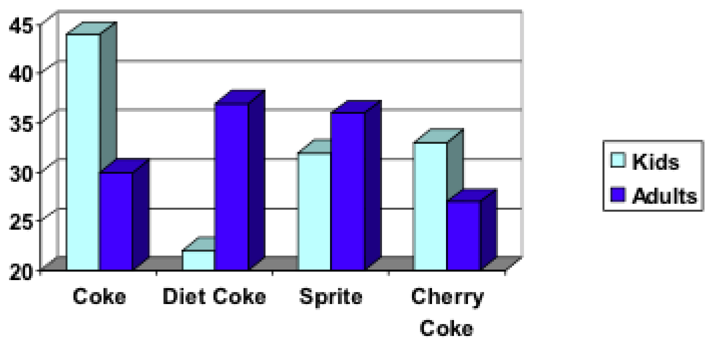
11. The box plot below shows salaries for Actuaries and CPAs. Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?



12. Referring to the boxplot above, what percentage of actuaries makes more than the median salary of a CPA?

Exploration

13. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected.Researcher 1: 3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34Researcher 2: 3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29
    a. Create comparative histograms of the data
    b. Create comparative boxplots of the data
14. A graph appears below showing the number of adults and children who prefer each type of soda. There were 130 adults and kids surveyed. Discuss some ways in which the graph below could be improved



15. Make up three data sets with 5 numbers each that have:
    a. the same mean but different standard deviations.
    b. the same mean but different medians.
    c. the same median but different means.
16. A sample of 30 distance scores measured in yards has a mean of 7, a variance of 16, and a standard deviation of 4.
    a. You want to convert all your distances from yards to feet, so you multiply each score in the sample by 3. What are the new mean, median, variance, and standard deviation?
    b. You then decide that you only want to look at the distance past a certain point. Thus, after multiplying the original scores by 3, you decide to subtract 4 feet from each of the scores. Now what are the new mean, median, variance, and standard deviation?
17. In your class, design a poll on a topic of interest to you and give it to the class.
    a. Summarize the data, computing the mean and five-number summary.
    b. Create a graphical representation of the data.
    c. Write several sentences about the topic, using your computed statistics as evidence in your writing.

SETS

BASICS

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

a) The set of all even numbers

b) The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

a) {1, 3, 9, 12}

b) {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.
The symbol $\in$ means “is an element of”.
A set that contains no elements, { }, is called the **empty set** and is notated $\emptyset$

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set *A* is another set that contains only elements from the set *A*, but may not contain all the elements of *A*.
If *B* is a subset of *A*, we write $B \subseteq A$
A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.
If *B* is a proper subset of *A*, we write $B \subset A$

Example 3

Consider these three sets

A = the set of all even numbers $\quad B = \{2, 4, 6\} \quad C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of *B* is also an even number, so is an element of *A*.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of *A*, since C contains an element, 3, that is not contained in *A*

Example 4

Suppose a set contains the plays *Much Ado About Nothing*, *MacBeth*, and *A Midsummer's Night Dream*. What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

UNION, INTERSECTION, AND COMPLEMENT

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).
The union is notated $A \cup B$.
More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)
The **intersection** of two sets contains only the elements that are in both sets.
The intersection is notated $A \cap B$.
More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$
The **complement** of a set *A* contains everything that is *not* in the set *A*.
The complement is notated *A'*, or *Ac*, or sometimes ~*A*.

Example 5

Consider the sets: *A* = {red, green, blue} *B* = {red, yellow, orange}

C = {red, orange, yellow, green, blue, purple}

a) Find $A \cup B$

The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange}

Notice we only list red once.

b) Find $A \cap B$

The intersection contains all the elements in both sets: $A \cap B$ = {red}

c) Find $Ac \cap C$

Here we're looking for all the elements that are *not* in set *A* and are also in *C*.

$Ac \cap C$ = {orange, yellow, purple}

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $Bc \cap A$

Notice that in the example above, it would be hard to just ask for *Ac*, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so *Ac* contains all the elements in the universal set that are not in *A*.

Example 6

a) If we were discussing searching for books, the universal set might be all the books in the library.

b) If we were grouping your Facebook friends, the universal set would be all your Facebook friends.

c) If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is U = all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then

$Ac = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $Ac \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8

Suppose H = {cat, dog, rabbit, mouse}, F = {dog, cow, duck, pig, rabbit}

W = {duck, rabbit, deer, frog, mouse}

a) Find $(H \cap F) \cup W$

We start with the intersection: $H \cap F$ = {dog, rabbit}

Now we union that result with W: $(H \cap F) \cup W$ = {dog, duck, rabbit, deer, frog, mouse}

b) Find $H \cap (F \cup W)$

We start with the union: $F \cup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}

Now we intersect that result with H: $H \cap (F \cup W)$ = {dog, rabbit, mouse}

c) Find $(H \cap F)c \cap W$

We start with the intersection: $H \cap F$ = {dog, rabbit}

Now we want to find the elements of W that are *not* in $H \cap F$

$(H \cap F)c \cap W$ = {duck, deer, frog, mouse}

VENN DIAGRAMS

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

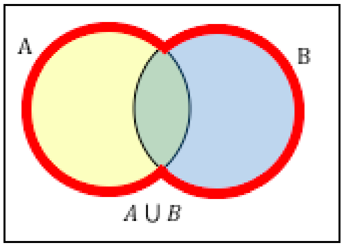
Venn Diagram

A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.
Basic Venn diagrams can illustrate the interaction of two or three sets.

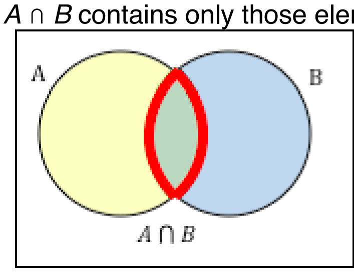
Example 9

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $Ac \cap B$

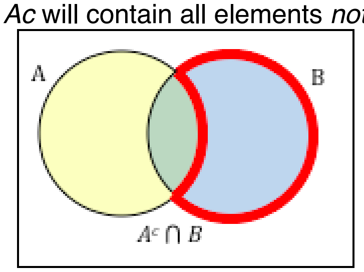
$A \cup B$ contains all elements in *either* set.



$A \cap B$ contains only those elements in both sets – in the overlap of the circles.



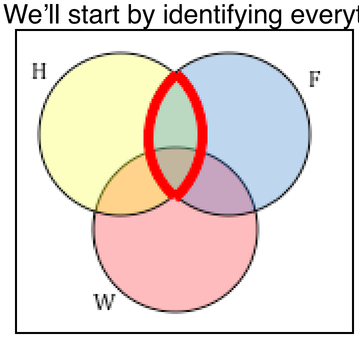
Ac will contain all elements *not* in the set A. $A^c \cap B$ will contain the elements in set B that are not in set A.



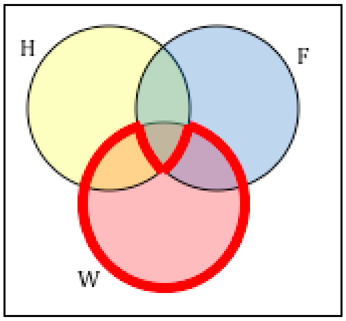
Example 10

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

We'll start by identifying everything in the set $H \cap F$



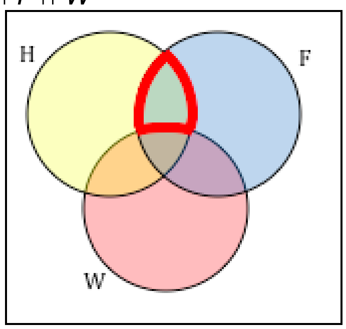
Now, $(H \cap F)c \cap W$ will contain everything *not* in the set identified above that is also in set W.



Example 11

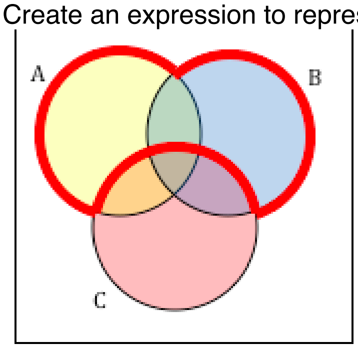
Create an expression to represent the outlined part of the Venn diagram shown.

The elements in the outlined set *are* in sets *H* and *F*, but are not in set *W*. So we could represent this set as $H \cap F \cap W^c$



Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown



CARDINALITY

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.
The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

What is the cardinality of B? $A \cup B$, $A \cap B$?

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of P = the set of English names for the months of the year?

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Example 14

A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

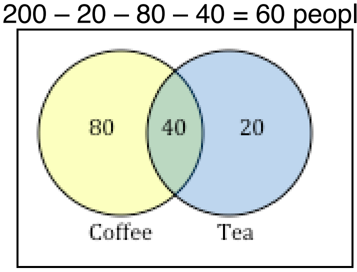
- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

200 – 20 – 80 – 40 = 60 people who drink neither.



Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice. In symbols,

$n(F \cup T) = n(F) + n(T) - n(F \cap T)$

$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)c$. Since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)c$ must be the other 10%.

The previous example illustrated two important properties

Cardinality properties

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$n(Ac) = n(U) - n(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$n(A \cap B) = n(A) + n(B) - n(A \cup B)$

Example 16

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

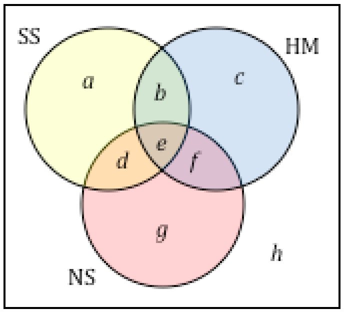
21 were taking a SS course 26 were taking a HM course

19 were taking a NS course 9 were taking SS and HM

7 were taking SS and NS 10 were taking HM and NS

3 were taking all three 7 were taking none

How many students are only taking a SS course?



It might help to look at a Venn diagram.

From the given data, we know that there are

3 students in region *e* and

7 students in region *h*.

Since 7 students were taking a SS and NS course, we know that n(*d*) + n(*e*) = 7. Since we know there are 3 students in region 3, there must be

7 – 3 = 4 students in region *d*.

Similarly, since there are 10 students taking HM and NS, which includes regions *e* and *f*, there must be

10 – 3 = 7 students in region *f*.

Since 9 students were taking SS and HM, there must be 9 – 3 = 6 students in region *b*.

Now, we know that 21 students were taking a SS course. This includes students from regions *a, b, d,* and *e*. Since we know the number of students in all but region *a*, we can determine that 21 – 6 – 4 – 3 = 8 students are in region *a*.

8 students are taking only a SS course.

Try it Now 4

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs

44 believed in ghosts

25 believed in Bigfoot

10 believed in UFOs and ghosts

8 believed in ghosts and Bigfoot

5 believed in UFOs and Bigfoot

2 believed in all three

How many people surveyed believed in at least one of these things?

Try it Now Answers

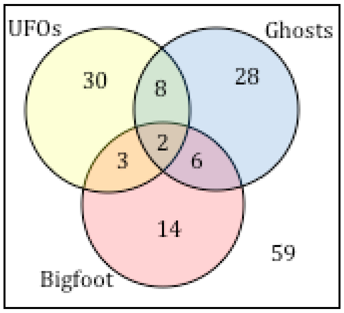
1. There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

2. $A \cup C$ = {red, orange, yellow, green, blue purple}

$Bc \cap A$ = {green, blue}

3. $A \cup B \cap Cc$

4. Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves 150 – 91 = 59 who believe in none.



EXERCISES

1. List out the elements of the set "The letters of the word Mississippi"
2. List out the elements of the set "Months of the year"
3. Write a verbal description of the set {3, 6, 9}
4. Write a verbal description of the set {a, i, e, o, u}
5. Is {1, 3, 5} a subset of the set of odd integers?
6. Is {A, B, C} a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.

7. A = {1, 2, 3, 4, 5}   B = {1, 3, 5}   C = {4, 6}   U = {numbers from 0 to 10}
8. 3 $\in$ *B* 8. 5 $\in$ *C*
9. $B \subset A$
10. $C \subset A$
11. $C \subset B$
12. $C \subset D$

Using the sets from above, and treating *U* as the Universal set, find each of the following:

13. $A \cup B$
14. $A \cup C$
15. $A \cap C$
16. $B \cap C$
17. *Ac*
18. *Bc*

Let D = {b, a, c, k},   E = {t, a, s, k},   F = {b, a, t, h}. Using these sets, find the following:

19. $Dc \cap E$
20. $Fc \cap D$
21. $(D \cap E) \cup F$
22. $D \cap (E \cup F)$
23. $(F \cap E)c \cap D$
24. $(D \cup E)c \cap F$

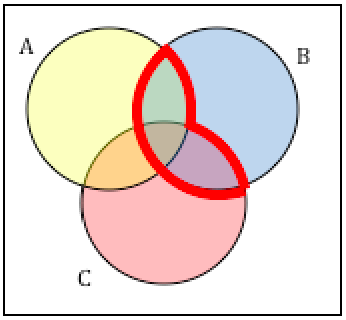
Create a Venn diagram to illustrate each of the following:

25. $(F \cap E) \cup D$

26. $(D \cup E)^c \cap F$
27. $(F^c \cap E^c) \cap D$
28. $(D \cup E) \cup F$

Write an expression for the shaded region.

29.



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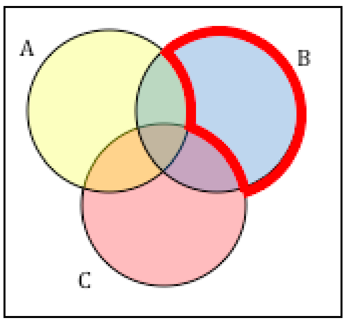
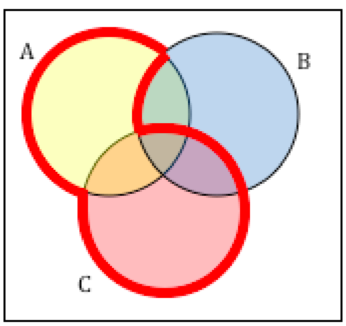
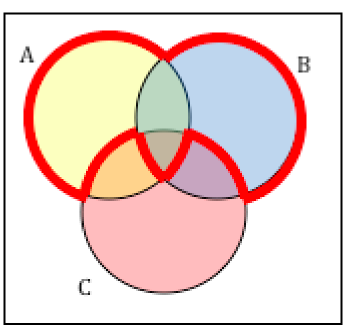


31.



32.



Let A = {1, 2, 3, 4, 5}   B = {1, 3, 5}   C = {4, 6}. Find the cardinality of the given set.

33. $n(A)$
34. $n(B)$
35. $n(A \cup C)$
36. $n(A \cap C)$

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.

37. n($A \cap C$)
38. n($B \cup C$)
39. n($A \cap B \cap Cc$)
40. n($A \cap Bc \cap C$)
41. If n(G) = 20, n(H) = 30, n($G \cap H$) = 5, find n($G \cup H$)
42. If n(G) = 5, n(H) = 8, n($G \cap H$) = 4, find n($G \cup H$)
43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.
    52 only use Netflix	62 only use Redbox
    24 only use a video store	16 use only a video store and Redbox
    48 use only Netflix and Redbox	30 use only a video store and Netflix
    10 use all three	25 use none of these
44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?
    5 only said color	8 only said size
    16 only said brand	20 said only color and size
    42 said only color and brand	53 said only size and brand
    102 said all three	20 said none of these
45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.
    18 had seen *The Matrix* (*M*)	24 had seen *Star Wars (SW)*
    20 had seen *Lord of the Rings (LotR)*	10 had seen *M* and *SW*
    14 had seen *LotR* and *SW*	12 had seen *M* and *LotR*
    6 had seen all three
46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.
    30% use the bus	20% ride a bicycle
    25% walk	5% use the bus and ride a bicycle
    10% ride a bicycle and walk	12% use the bus and walk
    2% use all three