Business Calculus

SBCTC & Lumen Learning

Cover Image: Map of scheduled airline traffic around the world, circa June 2009. Contains 54317 routes, rendered at 25% transparency. Base map is NASA Blue Marble (PD) plus airports from file:World-airport-map-2008.png, route data is fromAirline Route Mapper, rendering by OpenFlights (Open Database License). PHP source code for rendering available at the OpenFlights SVN.

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CONTENTS

- Why It Matters
- Reading: The Definite Integral
- Video: Area under the Curve Using Geometric Shapes
- Video: Area under the Curve Using Rectangles
- Video: Definition of the Definite Integral
- Video: Interpret the Meaning of Area under the Curve
- Reading: Antiderivative Formulas
- Video: The Antiderivative
- Reading: Other Methods of Integration
- Reading: The Definite Integral Applied to Area
- Video: Area between Two Curves
- Reading: Accumulation in Real Life
- Reading: Average Value
- Video: Average Value
- Reading: Continuous Income Stream
- Video: Present and Future Values
- Putting It Together

REFRESHER & REVIEW

READING: RULES OF EXPONENTS AND EXPONENTIAL EQUATIONS

Some of the rules for exponents that you will need:

1. $(b^m)(b^n) = b^{m+n}$
2. $\frac{b^m}{b^n} = b^{m-n}$
3. $(ab^m)^n = a^n b^{mn}$
4. $\frac{a}{b^n} = ab^{-n}$
5. $\sqrt[n]{b^m} = b^{m/n}$
6. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Text:

http://www.wallace.ccfaculty.org/book/5.1%20Exponents.pdf
http://www.wallace.ccfaculty.org/book/10.4%20Exponential%20Equations.pdf

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READING: LINEAR AND QUADRATIC EQUATIONS

Linear Equations

A linear equation is an equation in which the highest power (or exponent) is a 1 (one).

Examples:
$2x + 3 = 7$

$6(9x-4)+2=-3(5-2x)+1$

Quadratic Equations

A quadratic equation is an equation in which the highest power (or exponent) is a 2 (two).

Examples:
$4x^2+5x-3=0$
$x^2=4$
$-9(x+4)^2-6=0$

Text:

Linear Equations:

http://www.wallace.ccfaculty.org/book/1.1%20One-Step%20Equations.pdf
http://www.wallace.ccfaculty.org/book/1.3%20General%20Linear%20Equations.pdf
http://www.wallace.ccfaculty.org/book/1.4%20Fractions.pdf

Quadratic Equations:

http://www.wallace.ccfaculty.org/book/6.3%20Trinomials%20a=1.pdf
http://www.wallace.ccfaculty.org/book/6.4%20Trinomials%20a%20not%201.pdf
http://www.wallace.ccfaculty.org/book/6.6%20Factoring%20Strategy.pdf
http://www.wallace.ccfaculty.org/book/6.7%20Solve%20by%20Factoring.pdf
http://www.wallace.ccfaculty.org/book/9.4%20Quadratic%20Formula.pdf

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READING: GRAPHING

A **graph** is a visual way to represent functions and equations.

Text:

http://www.wallace.ccfaculty.org/book/2.1%20Points%20and%20Lines.pdf
http://www.wallace.ccfaculty.org/book/2.2%20Slope.pdf
http://www.wallace.ccfaculty.org/book/2.3%20Slope%20Intercept.pdf

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READING: FUNCTIONS

A **function** is a rule for a relationship between an input (or independent variable) and an output (or dependent) in which each input value uniquely determines one output value. We say "the output is a function of the input."

Text:

Introduction to Functions

http://www.opentextbookstore.com/appcalc/Chapter1-1.pdf
http://www.opentextbookstore.com/appcalc/Chapter1-2.pdf

Exponential Functions

http://www.opentextbookstore.com/appcalc/Chapter1-7.pdf
http://www.wallace.ccfaculty.org/book/10.4%20Exponential%20Equations.pdf

Logarithmic Functions

http://www.opentextbookstore.com/appcalc/Chapter1-8.pdf
http://www.wallace.ccfaculty.org/book/10.5%20Logarithm.pdf

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READING: BUSINESS TERMINOLOGY

Suppose you are producing and selling an item. The profit you make is the amount of money you take in minus what you have to pay to produce the items. Both of these quantities depend on how many you make and sell. Here is a list of definitions for some of the terminology that we'll use in this course:

- Your **Cost**, **C**, is the money you have to spend to produce your items.
- The **Fixed Cost** is the amount of money you have to spend regardless of how many items you produce. Fixed costs can include things like rent, purchase costs of machinery, and salaries for office staff. You have to pay the fixed costs even if you don't produce anything.
- The **Variable Cost** for q items is the amount of money you spend to actually produce them. Variable costs include things like the materials you use, the electricity to run the machinery, gasoline for your delivery vans, or the hourly wages of your production workers. These costs will vary according to how many items you produce.
- The **Total Cost** of q items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing q items. Usually when people discuss the *cost* of running a business, they mean the total cost.
- The **Average Cost** of q items is the total cost divided by q. You can also talk about the average fixed cost or the average variable cost.

- The **Marginal Cost** at q items is the cost of producing the *next* item. Really it's the cost of producing $q + 1$ items minus the cost of producing q items. In this course, we'll approximate this difference using calculus (we'll discuss how later).
- **Demand** is a relationship between the price p and the quantity q that can be sold (that is, demanded). Depending on your situation, you might think of p as a function of q, or of q as a function of p.
- Your **Revenue**, **R**, is the amount of money you actually take in from selling your products. Revenue is price × quantity.
- The **Average Revenue** of q items is the revenue divided by q.
- The **Marginal Revenue** at q items is the money gained from selling the next item. Again, we'll approximate this difference using calculus.
- The **Profit**, **P**, is the amount of money left over after all the bills are paid. P = R – C.
- The **Average Profit** for q items is the profit divided by q.
- The **Marginal Profit** at q items is the profit that results from making and selling the next item. As with the other "marginal" words, we'll use calculus to approximate this difference.

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LIMITS AND CONTINUITY

READING: LIMITS

Properties of Limits:

1. $\lim_{x \to a} (c) = c$ (where c is a constant)
2. $\lim_{x \to a} (x) = a$

Suppose that *n* is a constant, $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} g(x) = M$, then

1. $\lim_{x \to a} (n \cdot f(x)) = n \cdot L$
2. $\lim_{x \to a} (f(x) \pm g(x)) = L \pm M$
3. $\lim_{x \to a} (f(x) \cdot g(x)) = L \cdot M$
4. $\lim_{x \to a} (f(x)^n) = L^n$
5. $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ (as long as $M \neq 0$)

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READING: CONTINUITY

A function $f(x)$ is **continuous at the point** $x = a$ if the following three things are true:

1. $f(a)$ exists (We're allowed to plug the number into the function.)
2. $\lim_{x \to a} f(x)$ exists (We can get the limit as *x*→*a*.)
3. $\lim_{x \to a} f(x) = f(a)$ (The number we get from plugging in is the same as the limit.)

A function $f(x)$ is **continuous over the interval** (*s*, *t*) if it is continuous at every *x* in the interval (everywhere between *s* and *t*).

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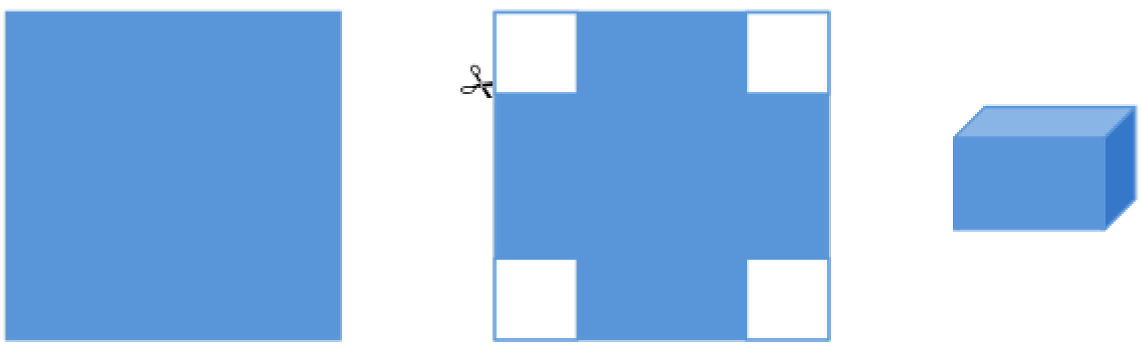
BASICS OF DERIVATIVES

WHY IT MATTERS

Introduction

In algebra, one of techniques we learned was how to take data and find a function that best models that data. Having a function is one thing, but knowing what that functions tells us is a whole other trick. Let's say we are given a sheet of cardboard and told to cut congruent squares out of each corner and then fold up the flaps to make a box.



We have a limited supply of cardboard but what we are shipping is flexible and can find in most boxes. We are tasked with finding the maximum volume box that we can create using this method. How would we go about doing that?

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MOTIVATION: VELOCITY

Suppose we drop a tomato from the top of a 100 foot building and time its fall (see figure 1).

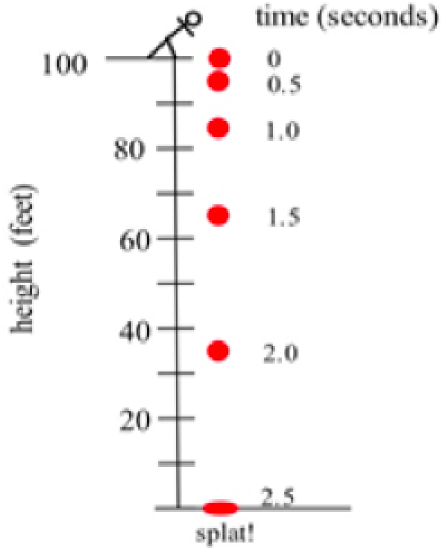
Some questions are easy to answer directly from the table given in figure 1:

1. How long did it take for the tomato *n* to drop 100 feet?
   2.5 seconds

2. How far did the tomato fall during the first second?
   100 – 84 = 16 feet
3. How far did the tomato fall during the last second?
   64 – 0 = 64 feet
4. How far did the tomato fall between *t* =.5 and *t* = 1?
   96 – 84 = 12 feet

Some other questions require a little calculation:

1. What was the average velocity of the tomato during its fall?
   Average velocity =



time (s)	height (ft)
0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

Figure 1

$$\frac{\text{Distance Fallen}}{\text{Total Time}} = \frac{\text{Change in Position}}{\text{Change in Time}} = \frac{-100ft}{2.5s} = -40ft/s$$

2. What was the average velocity between *t* = 1 and *t* = 2 seconds?
   Average velocity = $\frac{\text{Change in Position}}{\text{Change in Time}} = \frac{36ft-84ft}{2s-1s} = \frac{-48ft}{1s} = -48ft/s$

Some questions are more difficult.

1. How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, –40 ft/s . But the tomato fell slowly at the beginning and rapidly near the end so the “–40 ft/s” estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at t=1 by calculating the average velocities over a short time interval near *t* = 1 . The average velocity between *t* = 0.5 and *t* = 1 is $\frac{-12\text{feet}}{0.5s}$ = –24 ft/s, and the average velocity between *t* = 1 and *t* = 1.5 is $\frac{-20\text{feet}}{0.5s}$ = –40 ft/s so we can be reasonably sure that the instantaneous velocity is between –24 ft/s and –40 ft/s.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is $\frac{\text{Change in Position}}{\text{Change in Time}}$, which is the slope of the line **secant** through two points on the graph of height versus time (see figure 2). The instantaneous velocity at a particular time and height is the slope of the line **tangent** to the graph at the point given by that time and height.

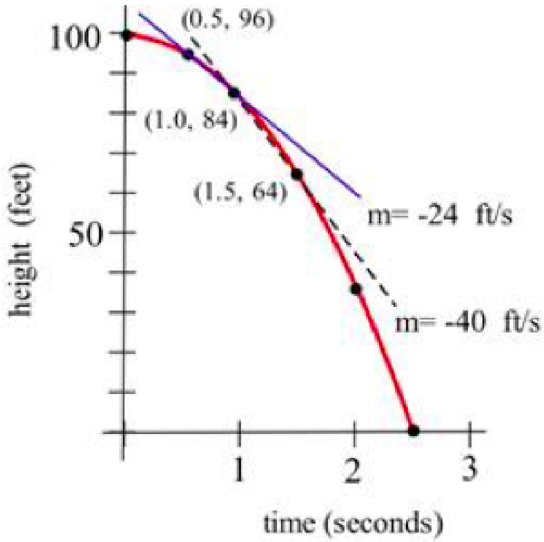


Figure 2

Average velocity = $\frac{\text{Change in Position}}{\text{Change in Time}}$ = slope of the line secant through 2 points.

Instantaneous velocity = slope of the line tangent to the graph.

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READING: CONCEPT OF DERIVATIVE

Slope and Rate of Change

The slope of a line measures how fast a line rises or falls as we move from left to right along the line. It measures the rate of change of the *y*-coordinate with respect to changes in the *x*-coordinate. If the line represents the distance traveled over time, for example, then its slope represents the velocity. In figure 1, you can remind yourself of how we calculate slope using two points on the line:

m = { slope from P to Q } = $\frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

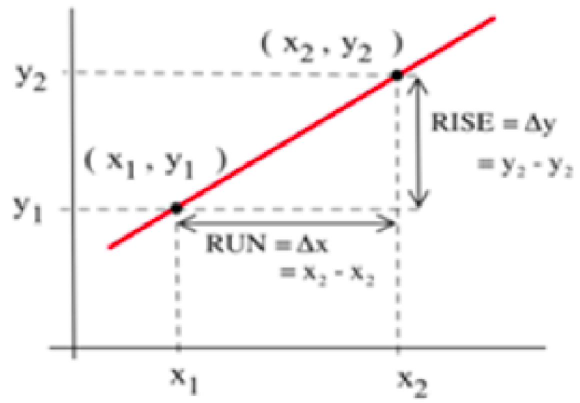

Figure 1

We would like to be able to get that same sort of information (how fast the curve rises or falls, velocity from distance) even if the graph is not a straight line. But what happens if we try to find the slope of a curve, as in figure 2? We need two points in order to determine the slope of a line. How can we find a slope of a curve, at just one point?

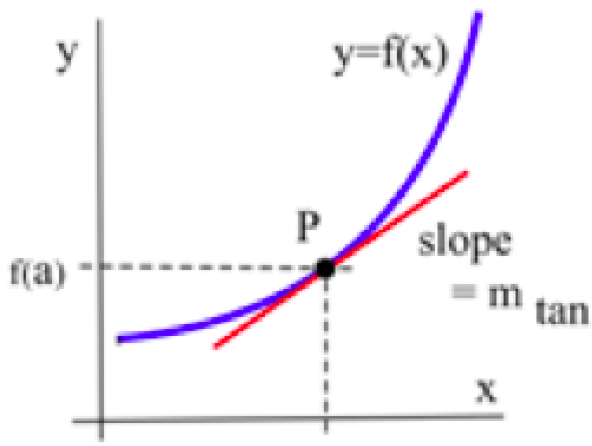

Figure 2

The answer, as suggested in Figure 2 is to find the slope of the line tangent to the curve at that point. Unfortunately, “tangent line” is hard to define precisely.

Definition: A **secant line** is a line between two points on a curve.

Not-Quite Definition: A **tangent line** is a line at one point on a curve . . . that does its best to be the curve at that point.

It turns out that the easiest way to define the tangent line is to define its slope.

Definition of the Derivative

The tangent line problem and the instantaneous velocity problem are the same problem. In each problem we want to know how rapidly something is **changing at an instant in time**, and the answer turns out to be finding the **slope of a tangent line,** which we approximated with the **slope of a secant line**. This idea is the key to defining the slope of a curve.

The Derivative

- The **derivative** of a function *f* at a point (*x*, *f*(*x*)) is the instantaneous rate of change.
- The **derivative** is the slope of the line tangent to the graph of f at the point (*x*, *f*(*x*)).
- The **derivative** is the slope of the curve *f*(*x*) at the point (*x*, *f*(*x*)).
- A function is called **differentiable** at (*x*, *f*(*x*)) if its derivative exists at (*x*, *f*(*x*)).

Notation for the Derivative

The **derivative of *f*(*x*) with respect to *x*** is written as $f'(x)$ (read aloud as "*f* prime of *x*"), or $\frac{df}{dx}$ (read aloud as "dee eff dee ex").

The notation that resembles a fraction is called **Leibniz notation**. It displays not only the name of the function, but also the name of the variable (in this case, *x*). It looks like a fraction because the derivative is a slope.

Using the Concept in English

We **find the derivative** of a function, **take the derivative** of a function, or **differentiate** a function. We use an adaptation of the notation to mean "find the derivative of *f*(*x*):"$\frac{d}{dx}(f(x)) = \frac{df}{dx}$

Practical Definition

The derivative can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval. The tinier the interval, the closer this is to the true instantaneous rate of change, slope of the tangent line, or slope of the curve.

Formal Algebraic Definition

The official way to find a function's derivative, called the **limit definition of the derivative**, is using

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The Derivative as a Function

We know how to find (or at least approximate) the derivative of a function for any *x*-value; this means we can think of the derivative as a function, too. The inputs are the same *x*s; the output is the value of the derivative at the *x* value.

Text:

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http://www.opentextbookstore.com/appcalc/Chapter2-2.pdf

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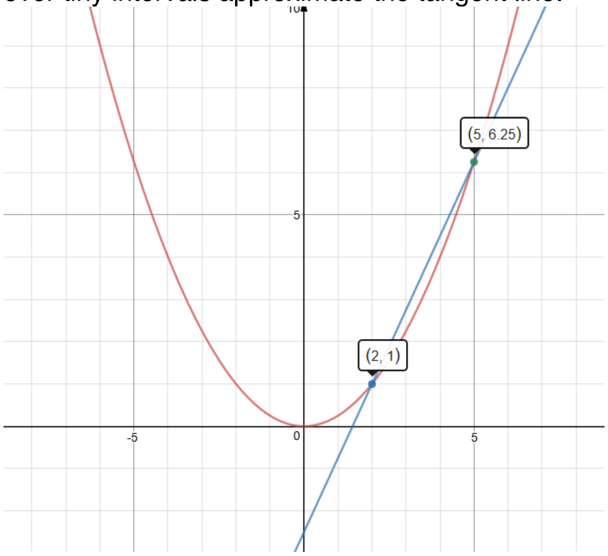
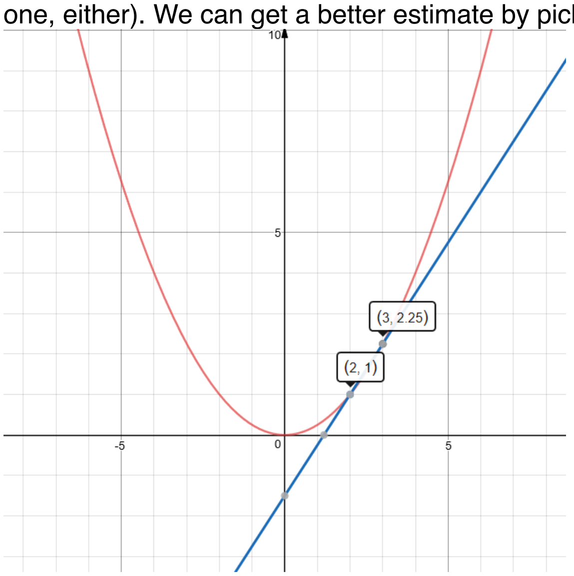
READING: CALCULATING TANGENT LINES

Let's look at the problem of finding the slope of the line L which is tangent to $y = 0.25x^2$ at the point (2, 1).

We could estimate the slope of L from the graph, but we won't. Instead, we will use the idea that secant lines over tiny intervals approximate the tangent line.



The line through (2, 1) and (5, 6.25) on the graph of *f* is a secant line, and we can calculate that slope exactly: $m = \frac{(6.25-1)}{(5-2)} = \frac{5.25}{3} = 1.75$. But this is only an estimate of the slope of the tangent line (and not a good one, either). We can get a better estimate by picking a second point on the graph of closer to (2, 1):



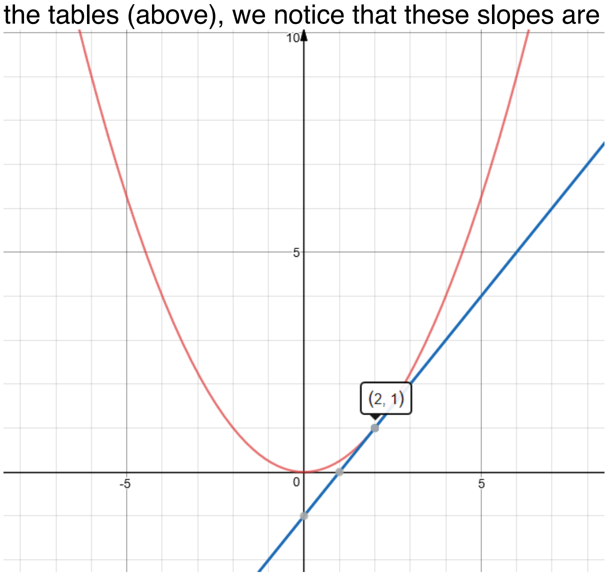
In the above, we see that the slope of the line through the points (2, 1) and (3, 2.25) is a better approximation of the slope of the tangent line at (2, 1): $m = \frac{(2.25-1)}{(3-2)} = \frac{1.25}{1} = 1.25$, a better estimate, but still an approximation. We can continue picking points closer and closer to (2, 1) on the graph of f, and then calculating the slopes of the lines through each of these points and the point (2, 1):

Points to the left of (2, 1)		
x	$y = 0.25x^2$	slope of line through (x, y) and (2, 1)
1.5	0.5625	0.875

Points to the left of (2, 1)		
1.9	0.9025	0.975
1.99	0.990025	0.9975

Points to the right of (2, 1)		
x	$y = 0.25x^2$	slope of line through (x, y) and (2, 1)
2.5	1.5625	1.125
2.1	1.1025	1.025
2.01	1.010025	1.0025

As the points we pick get closer and closer to the point (2, 1) on the graph of $y = 0.25x^2$, the slopes of the lines through the points and (2, 1) are better and better approximations of the slope of the tangent line. If we look at the tables (above), we notice that these slopes are getting closer and closer to $m = 1$.



We can bypass much of the calculating by not picking specific points: let's look at a general point near (2, 1). Let's call h the distance from 2 to some nearby point. Then $2 + h$ is the x-value of the nearby point and $f(2 + h)$ is the y-value of the nearby point. If h is small, then $x = 2 + h$ is close to 2 and the point $(x, y) = (2 + h, f(2 + h)) = (2 + h, 0.25(2 + h)^2)$ is close to (2, 1). The slope m of the line through the points (2, 1) and $(2 + h, 0.25(2 + h)^2)$ is the slope of the secant line:

$$m = \frac{0.25(2+h)^2-1}{(2+h)-2} = \frac{0.25(4+4h+h^2)-1}{h} = \frac{1+h+0.25h^2-1}{h} = \frac{h+0.25h^2}{h} = \frac{h(1+0.25h)}{h} = 1 + 0.25h$$

If h is very small, then $m = 1 + 0.25h$ is a very good approximation to the slope of the tangent line, and $m = 1 + 0.25h$ is very close to the value 1.

As the interval between two points gets smaller and smaller, the secant line becomes more like the tangent line and its slope gets closer to the slope of the tangent line. That's good news—we've figured out how to find the slope of a secant line.

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INTERACTIVE: SLOPE AND THE DERIVATIVE OF A FUNCTION

You see here the function $f(x) = \frac{x^2}{6} - 2x + 8$ and its tangent line *t* together with a slope triangle. The slope *s* of the tangent line is drawn again starting at the x-axis.

Visit this page in your course online to use this activity.

1. Drag point *T* with the mouse. This produces a trace of the slope creating the graph of the slope function. Which kind of function is this slope function? Try to find the slope function's equation, too. Write down all your results.
2. Calculate the first derivative of the function *f* on paper.
3. Double click on the drawing pad, type the first derivative of f into the input field and press **Enter**. Choose the "move mode" and drag point *T* with your mouse again. What is happening? Write down your observations.

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INTERACTIVE: TANGENT OF F(X)

Turn off the blue line f"(x) by clicking on the blue circle below and to the left. Then grab the blue dot A on the graph and trace it along its curve ($y = x^3 - 5x^2 + 3x - 7$). The red dot B draws $y = 3x^2 - 10x + 3$ which is the slope of the line tangent to the curve you're tracing.

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READING: RULES FOR DERIVATIVES

In this section, we'll get the derivative rules that will let us find formulas for derivatives when our function comes to us as a formula. This is a very algebraic section, and you should get lots of practice. When you tell someone you have studied calculus, this is the one skill they will expect you to have. There's not a lot of deep meaning here—these are strictly algebraic rules.

Derivative Rules: Building Blocks

In what follows, *f* and *g* are differentiable functions of *x* and *k* and *n* are constants.

1. **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$ except when $n = 0$.
   Special case when $n = 0$: $\frac{d}{dx}(kx^0) = \frac{d}{dx}(k) = 0$ (because $k = kx^0$)
2. **Constant Multiple Rule:** $\frac{d}{dx}(kf) = kf'$
3. **Sum (or Difference) Rule:** $\frac{d}{dx}(f \pm g) = f' \pm g'$
4. **Exponential Functions:** $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(a^x) = a^x \cdot \ln(a)$
5. **Natural Logarithm:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

The sum, difference, and constant multiple rule combined with the power rule allow us to easily find the derivative of any polynomial.

This video provides an example of finding the derivative of a function containing radicals:

Watch this video online: https://youtu.be/OqsVPz8I-_Y

Product and Quotient Rules

The basic rules will let us tackle simple functions. But what happens if we need the derivative of a combination of these functions?

The rules for finding derivatives of products and quotients are a little complicated, but they save us the much more complicated algebra we might face if we were to try to multiply things out. They also let us deal with products where the factors are not polynomials. We can use these rules, together with the basic rules, to find derivatives of many complicated looking functions.

In what follows, *a* and *b* are differentiable functions of *x*.

1. **Product Rule:** If $f(x) = a(x)b(x)$ then $f'(x) = \frac{d}{dx}(a(x)b(x)) = a'b + b'a$
2. **Quotient Rule:** If $f(x) = \frac{a(x)}{b(x)}$ then $f'(x) = \frac{d}{dx}\left(\frac{a(x)}{b(x)}\right) = \frac{a'b - b'a}{b^2}$

Chain Rule

There is one more type of complicated function that we will want to know how to differentiate: composition. The Chain Rule will let us find the derivative of a composition.

In what follows, *a* and *b* are differentiable functions of *x.*

Chain Rule: If $f(x) = a(b(x))$ then $f'(x) = a'(b(x)) \cdot b'(x)$

It may help to remember the saying, "The derivative of a composition is the derivative of the outside TIMES the derivative of the inside."

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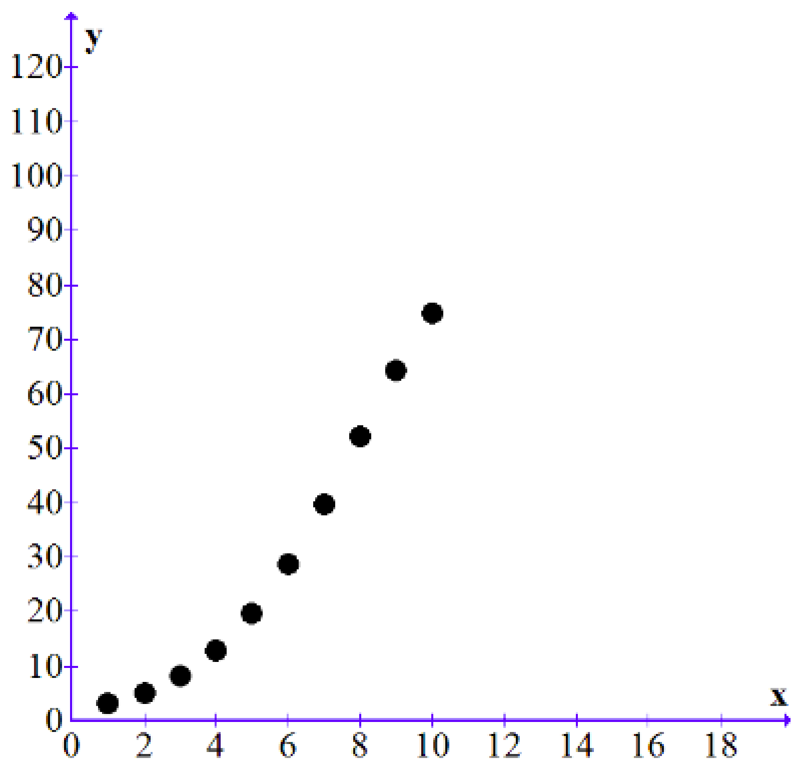
USING THE DERIVATIVE

Introduction

Let’s say you have an advertising campaign going on a non-disposable item you are selling. You are collecting data and notice that your sales (in millions) are based on the total amount you have invested in advertising (in thousands). The graph below reflects this.



Question 1: If I invest $2000 in advertising what would be the rate of return on sales (i.e. how much would sales increase or decrease based on my investment in advertising)?

Question 2: Is there a time when I should decrease the amount of money I am spending on advertising?

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READING: EXAMPLES OF INSTANTANEOUS RATES OF CHANGE

Uses of the Derivative

So far we have emphasized the derivative as the slope of the line tangent to a graph. That interpretation is very visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are used in a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few interpretations of the derivative that are commonly used:

Rate of Change

We call $f'(x)$ the **rate of change** of the function at x. If the units for x are years and the units for $f(x)$ are people, then the units for $\frac{df}{dx}$ are $\frac{\text{people}}{\text{year}}$, a rate of change in population.

Slope

We have already see the derivative used as a slope: $f'(x)$ is the **slope of the line tangent to the graph of *f* at the point (*x*, *f*(*x*)).**

Velocity

If $f(x)$ is the location of an object at time x, then $f'(x)$ is the **velocity** of the object at time x. If the units for x are hours and $f(x)$ is distance measured in miles, then the units for $f'(x)$ are $\frac{\text{miles}}{\text{hour}}$ (miles per hour), which is a measure of velocity. Note that velocity is another word for **speed**, except that we can have negative velocity (which means the object is moving in reverse).

Acceleration

If $f(x)$ is the velocity of an object at time x, then $f'(x)$ is the **acceleration** of the object at time If the units are for x are hours and $f(x)$ has the units $\frac{\text{miles}}{\text{hour}}$, then the units for the acceleration $f'(x)$ are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}^2}$ (miles per hour per hour). Since velocity itself is a derivative, then acceleration is the derivative of a derivative. Acceleration is the *second derivative* of an object's location.

Marginal Cost

If $f(x)$ is the total cost of making x objects, then $f'(x)$ is the **marginal cost**, at a production level of x. This marginal cost is approximately the **additional** cost of making one more object once we have already made x objects. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x)$ are $\frac{\text{dollars}}{\text{bicycle}}$, the cost per bicycle.

Marginal Revenue

If $f(x)$ is the total revenue from selling x objects, then $f'(x)$ is the **marginal revenue**, at a production level of x. The marginal revenue is an approximation of the revenue gained from selling **one more** object after having already sold x objects. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x)$ are $\frac{\text{dollars}}{\text{bicycle}}$, the revenue per bicycle.

Marginal Profit

If $f(x)$ is the total profit from producing and selling x objects, then $f'(x)$ is the **marginal profit**, the profit to be made from producing and selling one more object. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units $f'(x)$ are $\frac{\text{dollars}}{\text{bicycle}}$, dollars per bicycle, which is the profit per bicycle.

In business contexts, the word "marginal" usually means the derivative or rate of change of some quantity.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and notations of calculus are still useful.

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READING: OPTIMIZATION

In theory and applications, we often want to maximize or minimize some quantity. A store owner may want to maximize the number of "good" parking spots or minimize the time that a shopper spends waiting to check out. A manufacturer may want to maximize profits and market share or minimize costs and waste products.

The best way we have to find an optimal solution without calculus is to examine the graph of the function, but we are limited in a number of ways related to accuracy (Did we zoom in/out enough to see it all? Is the scale good enough to get a desired precision?).

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums. At the same time, it ensures that we haven't missed anything important.

Maxima and Minima

Before we examine how calculus can help us find maximums and minimums, we need to define the concepts we will use:

Definitions

- f has a **local (or relative) maximum** at c if $f(c) \geq f(x)$ for all x near c
- f has a **local (or relative) minimum** at c if $f(c) \leq f(x)$ for all x near c
- f has a **local (or relative) extreme** at c if $f(c)$ is a **local maximum or minimum.**
- f has a **global (or absolute) maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f.
- f has a **global (or absolute) minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f.
- f has a **global extreme** at c if $f(c)$ is a **global maximum or minimum**.

Note that the plural of maximum is maxima, and the plural of minimum is minima, but we often simply say "max" or "min" for singular or plural; it saves a lot of syllables.

Finding Local Maxima and Minima of a Function

Definitions

- A **critical number** for a function *f* is a value $x = c$ in the domain of *f* where either $f'(c) = 0$ or $f'(c)$ is undefined.
- A **critical point** for a function *f* is a point $(c, f(c))$ where *c* is a critical number of *f*.

A local max or min of *f* can only occur at a critical point.

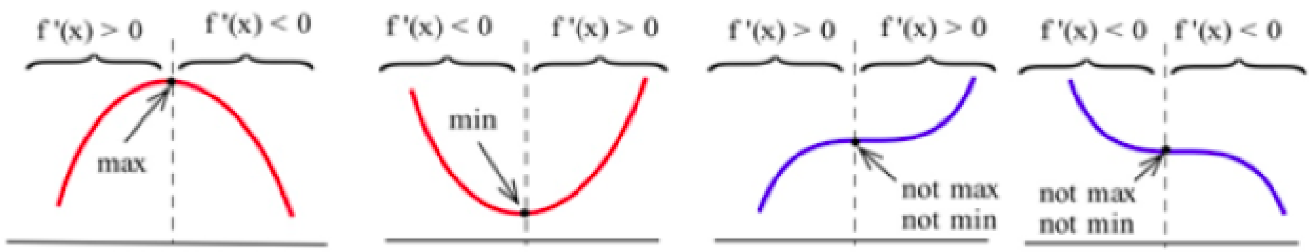
It is not enough to find the critical points—we can only say that a function ***might* have** a local extreme at the critical points.

Once we have found the critical points of *f*, we still have the problem of determining whether these points are maxima, minima, or neither.

Critical numbers only give the **possible** locations of extremes, and some critical numbers are not the locations of extremes. Critical numbers are the **candidates** for the locations of maxima and minima.

f' and Extreme Values of f

Four possible shapes of graphs are shown below. In each graph, the point marked by an arrow is a critical point, where $f'(x) = 0$.



At a local max, such as in the graph on the left, the function increases on the left of the local max, then decreases on the right. The derivative is first positive, then negative at a local max. At a local min, the function decreases to the left and increases to the right, so the derivative is first negative, then positive. When there isn't a local extreme, the function continues to increase (or decrease) right past the critical point—the derivative doesn't change sign.

The First Derivative Test for Extremes

Find the critical points of *f*. For each critical number *c*, examine the sign of f' to the left and to the right of *c*. What happens to the sign as you move from left to right?

1. If $f'(x)$ changes from **positive to negative** at $x = c$, then *f* has a local **maximum** at $(c, f(c))$.
2. If $f'(x)$ changes from **negative to positive** at $x = c$, then *f* has a local **minimum** at $(c, f(c))$.
3. If $f'(x)$ **does not change sign** at $x = c$, then $(c, f(c))$ is **neither** a local max nor a local min.

The Second Derivative Test for Extremes

Find all critical points of *f*. For those critical points where $f'(c) = 0$, find $f''(c)$.

1. If $f''(c) < 0$ then *f* is concave down and has a **local maximum** at $x = c$.
2. If $f''(c) > 0$ then *f* is concave up and has a **local minimum** at $x = c$.

3. If $f''(c) = 0$ then f may have a local maximum, a minimum or neither at $x = c$.

Please note: The Second Derivative Test doesn't always give an answer. The First Derivative Test will always give you an answer. Use whichever test you want to. But remember—you have to do *something* to be sure that your critical point actually is a local max or min.

Finding Global Maxima and Minima of a Function

In some problems, we may want to find the global extreme—the absolutely highest or lowest point on the graph or in a region. How do we do this?

To Find Global Extremes

The only places where a function can have a global extreme are critical points or endpoints.

1. If the function has only one critical point, and it's a local extreme, then it is also the global extreme.
2. If there are endpoints, find the global extremes by comparing y-values at all the critical points and at the endpoints.
3. When in doubt, graph the function to be sure.

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READING: SECOND DERIVATIVE AND CONCAVITY

Graphically, a function is **concave up** if its graph is curved with the opening upward (Figure 1a). Similarly, a function is **concave down** if its graph opens downward (Figure 1b).

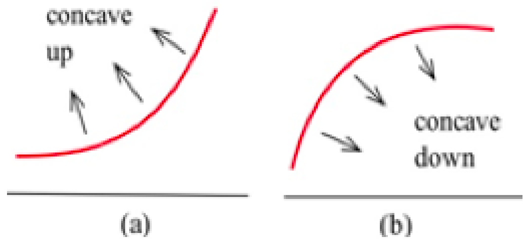

Figure 1

The derivative of a function *f* is a function that gives information about the slope of *f*. **The first derivative tells us if the original function is increasing or decreasing.**

Because *f′* is a function, we can take its derivative. This second derivative also gives us information about our original function *f*. The second derivative gives us a mathematical way to tell how the graph of a function is curved. **The second derivative tells us if the original function is concave up or down.**

Second Derivative

Let $f(x)$ be a function. The **second derivative of f** is the derivative of $f'(x)$. Using prime notation, this is $f''(x)$. You can read this aloud as "f double prime of x." Using Leibniz notation, the second derivative is written $\frac{d^2 f}{dx^2}$ or $\frac{d^2}{dx^2}(f(x))$. This is read aloud as "the second derivative of *f*.

If $f''(x)$ is positive on an interval, the graph of $f(x)$ is **concave up** on that interval**.**

If $f''(x)$ is negative on an interval, the graph of $f(x)$ is **concave down** on that interval**.**

Inflection Points

An **inflection point** is a point on the graph of a function where the concavity of the function changes from concave up to down or from concave down to up.

Alternately, an **inflection point** is a point on the graph where the second derivative changes sign.

The second derivative will change signs only when $f''(x) = 0$ or when $f''(x)$ is undefined (just like using the first derivative to find critical points).

Note that just having the second derivative be zero or undefined is not enough to decide that we have an inflection point. We still need to check that the sign of $f''(x)$ changes sign on either side.

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PUTTING IT TOGETHER – BOX

Summary

Back to our box problem at the beginning. If we let x = side of a square that is to be cut out, then the volume of the box we will wind up with is given by

$$V = 144x - 24x^2 + x^3$$

Since we are wanting to find the maximum volume, we know to take the derivative of this to calculate our minimums and maximums. By performing our first derivative and setting it equal to zero:

$$V^1 = 144 - 48x + 3x^2$$

We find we have critical points at x = 4 and x = 12

Using our second derivative test, we discover that 4″ squares will give us the maximum volume of the box that we are seeking.

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APPLICATION OF DERIVATIVES

WHY IT MATTERS

Introduction

You are in a boardroom and someone shows the following on a slide:

Demand function for Product X: $q = 400 - 2p$

Current Price: $p = \$100$

Should we raise or lower the price to increase our profits?

This is a great question, because all businesses want to optimize their profits. The issue is what will happen if we raise or lower the price? If it goes up will people stop buying it? If it goes down will more people be inclined to buy it? How can math help us determine this?

READING: APPLIED OPTIMIZATION

We have used derivatives to find the maxima and minima of some functions, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More likely, someone will describe a problem and ask you to maximize or minimize something. For example, he or she will say something like the following questions:

- "What is the largest volume package which the post office will take?"
- "What is the quickest way to get from here to there?"
- "What is the least expensive way to accomplish some task?"

In this section, we'll discuss how to find these using calculus.

Max/Min Story Applications

Max/Min **Story Problem Technique**

1. Translate the English statement of the problem line by line into a picture (if that applies) and into math. This is often the hardest step!
2. Identify the **objective function**: the function that you're trying to maximize or minimize. Look for "est" words (greatest, least, highest, farthest, most, etc.).

    1. If you have two or more variables, write a **constraint equation**: an equation that relates two or more of the variables. (Usually there's a number in the constraint equation.)
    2. Solve the constraint equation for one variable and substitute into the objective function. Now you have an equation of one variable.
3. Use calculus to find the optimum values. (Take derivative, find critical points, test for max/min. Don't forget to check the endpoints!)
4. Look back at the question to make sure you answered what was asked.

Marginal Cost, Revenue, Profit

Companies attempting to be successful tend to be concerned with maximizing revenue while minimizing costs. Since Profit = Revenue – Cost, we can say that these companies are interested in maximizing profits. In each of these cases, once you have the function for cost, revenue, or profit, you can maximize or minimize by finding the derivative, solving for the critical points, and testing.

Marginal Cost

The derivative of the cost function is called the **marginal cost**.

Marginal Revenue

The **marginal revenue** is the derivative of the revenue function.

Marginal Profit

The **marginal profit function**, which is the derivative of the profit function, can also be found by MP = MR – MC.

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INTERACTIVE: MAXIMIZING VOLUME

In Isaac Newton's day, one of the biggest problems was poor navigation at sea. Before calculus was developed, the stars were vital for navigation. Shipwrecks occurred because the ship was not where the captain thought it should be. There was not a good enough understanding of how the Earth, stars and planets moved with respect to each other. Calculus (differentiation and integration) was developed to improve this understanding. Differentiation and integration can help us to solve many types of real-world problems in our day to day life. We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.). Derivatives are met in many engineering and science problems, especially when modeling the behavior of moving objects.

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1. A rectangular sheet of tin 7cm x 5cm is to be made into a box without top, by cutting off squares from the corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?
2. A box with a square base has no top. If 100 sq. cm of material is used, what is the maximum possible volume for the box?

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READING: IMPLICIT DIFFERENTIATION

Up to now, the functions we needed to differentiate contained only one variable (or it was possible to solve to get a formula containing only one variable for them). In this section (and occasionally thereafter), we will have an equation relating *x* and *y* which is either too difficult or impossible to solve explicitly for y. In any case, we can still find the derivative $\frac{dy}{dx}$ using **implicit differentiation**.

The key idea behind implicit differentiation is to assume that y is a function of *x* even if we cannot solve for y.

Implicit Differentiation

To determine $\frac{dy}{dx}$, differentiate each side of the equation, treating *y* as a function of *x*, and then algebraically solve for $\frac{dy}{dx}$.

Be careful when you do the derivative. Since you're treating *y* as a function of *x*, every time you come across doing a derivative of *y*, you'll use chain rule. Also, if you have both *x*s and *y*s in the same term, you'll need the product rule.

Related Rates

If several variables or quantities are related to each other, and some of the variables are changing at a known rate (speed), then we can use implicit differentiation to determine how rapidly the other variables must be changing.

When working a related rates problem:

1. Identify the quantities that are changing and assign variables to them.
2. Find an equation that relates those quantities.
3. Differentiate both sides of that equation with respect to time (that is, do "$\frac{d}{dt}$" to both sides of the equation).
4. Plug in any known values for the variables and/or the rates of change.
5. Solve for the desired rate.

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INTEGRALS AND AREAS

WHY IT MATTERS

Motivation

You are a financial advisor and have a client who wants to spend everything he earns. He is just starting out his career. It is your job to convince him to invest 10% of his starting salary with your firm. He makes $100,000 a year to start out with. He is a skeptic and wants you to prove to him (i.e. paper and pencil) that it would be a smart idea to that for the next 30 years. How would you do it? Answer: Using calculus!

Up to now, we've dealt with **differential calculus**. We started with the geometric idea of the **slope of a tangent line** to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This chapter deals with **integral calculus** and starts with the geometric idea of **area**.

The basic shape we will use is the rectangle; the area of a rectangle is base × height. The only other area formula you'll need to know is for triangles: $A = \frac{1}{2}bh$.

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READING: THE DEFINITE INTEGRAL

Approximating Area with Rectangles

How do we approximate the area between a function and the x-axis if the curve is, well, curvy? We could use rectangles and triangles, but that can get really awkward if the shape is irregular. It turns out to be more useful (and easier) to simply use rectangles. The more rectangles we use, the better our approximation is.

Riemann Sum

A **Riemann sum** for a function f(x) over an interval [a, b] is a sum of areas of rectangles that approximates the area under the curve. Start by dividing the interval [a, b] into *n* subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width: Δx. The height of each rectangle comes from the function evaluated at some point in its subinterval. Then the Riemann sum is:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

Sigma Notation

The upper-case Greek letter Sigma Σ is used to stand for "add up all of the following things." Sigma notation is a way to compactly represent a sum of many similar terms.

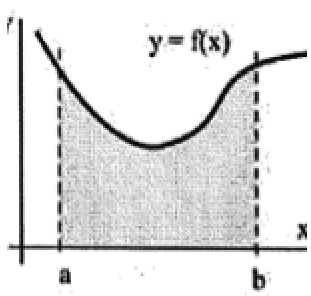
Using the Sigma notation, the Riemann sum can be written $\sum_{i=1}^{n}(f(x_i)\Delta x)$.

This is read aloud as "the sum from i equals 1 to n of f of x sub i delta x."

Definition of the Definite Integral

Because the area under a curve is so important, it has a special vocabulary and notation.

- The **definite integral** of a positive function *f*(*x*) from *a* to *b* is the area between *f* (at the top), the *x*-axis (at the bottom), and the vertical lines *x* = *a* (on the left) and *x* = *b* (on the right).



The shaded region is the area described by a definite integral.

Notation for the Definite Integral

The definite integral of *f* from *a* to *b* is written $\int_a^b f(x)dx$

The ∫ symbol is called an **integral sign**; it's an elongated letter S, standing for sum. (The ∫ is actually the Σ from the Riemann sum, written in Roman letters instead of Greek letters.)

The *dx* on the end must be included; you can think of ∫ and *dx* as left and right parentheses. The *dx* tells what the variable is—in this example, the variable is *x*. (The *dx* is actually the Δ*x* from the Riemann sum, written in Roman letters instead of Greek letters.)

The function *f* is called the **integrand**.

The *a* and *b* are called the **limits of integration**.

Using the Concept in English

We **integrate**, or **find the definite integral** of a function. This process is called **integration**.

Signed Area

Normally, area is always positive. If the “height” (from the function) is a negative number, then multiplying it by the width doesn’t give us actual area, it gives us the area with a negative sign.

But now we’ll need the idea of negative height (and therefore negative areas). For example, if you’re mapping a cross-country trip, you may talk about your height above or below sea level. Above sea level, your height would be a positive number, and below sea level, height would be negative.

The Definite Integral and Signed Area

If the function is above the x-axis everywhere between the limits of integration, the area is positive.

If the function dips below the x-axis, the areas of the regions below the x-axis will have a negative sign. These negative areas take away from the definite integral.

$\int_a^b f(x)dx$ = (Area above x-axis) – (Area below x-axis).

Negative rates indicate that an amount is decreasing. For example, if $f(t)$ is the velocity of a car in one direction along a straight line at time t (miles/hour) , then negative values of f indicate that the car is traveling in the opposite direction: backwards. The definite integral of f is the change in position of the car during the time interval. If the velocity is positive, positive distance accumulates. If the velocity is negative, distance in the negative direction accumulates.

This is true of any rate. For example, if $f(t)$ is the rate of population change (people/year) for a town, then negative values of f would indicate that the population of the town was getting smaller, and the definite integral would be the change in the population, a decrease, during the time interval.

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READING: ANTIDERIVATIVE FORMULAS

An **antiderivative of *f*(*x*)** is a function whose *derivative* is *f*(*x*). Usually, we use a capital letter to refer to the antiderivative if we use a lower case letter for the integrand. That is, the antiderivative of *f*(*x*) is *F*(*x*) if we can say *F'*(*x*) = *f*(*x*). (And we'd say G(*x*) is the antiderivative of *g*(*x*) if G'(*x*) = *g*(*x*).)

So we can calculate the definite integral $\int_a^b f(x)dx$ by evaluating $F(b) - F(a)$.

Building Blocks

Antidifferentiation is going backwards through the derivative process. So the easiest antiderivative rules are simply backwards versions of the easiest derivative rules. Recall the derivative rules:

Derivative Rules (Reminder)

In what follows, *f* and *g* are differentiable functions of *x* and *k* and *n* are constants.

1. **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$ except when $n = 0$.
   Special case when $n = 0$: $\frac{d}{dx}(kx^0) = \frac{d}{dx}(k) = 0$ (because $k = kx^0$)
2. **Constant Multiple Rule:** $\frac{d}{dx}(kf) = kf'$
3. **Sum (or Difference) Rule:** $\frac{d}{dx}(f \pm g) = f' \pm g'$
4. **Exponential Functions:** $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(a^x) = a^x \cdot \ln(a)$
5. **Natural Logarithm:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

The corresponding rules for antiderivatives are next—each of the antiderivative rules (almost) is simply rewriting the derivative rule. All of these antiderivatives can be verified by differentiating.

Antiderivative Rules

In what follows, *f* and *g* are integrable functions of *x* and *k*, *n,* and *C* are constants.

1. **Constant Multiple Rule:** $\int kf(x)dx = k\int f(x)dx$
2. **Sum (or Difference) Rule:** $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int f(x)dx$
3. **Power Rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ except when $n = -1$.
   Special case when $n = 0$: $\int kdx = kx + C$ (because $k = kx^0$)
4. **Exponential Functions:** $\int e^x dx = e^x + C$ and $\int a^x dx = \frac{a^x}{\ln a} + C$
5. **Natural Logarithm:** $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$

Note that the antiderivative of $\frac{1}{x}$ is not simply ln(*x*), it's ln|*x*|. The domain of $\frac{1}{x}$ is bigger than the domain of ln(*x*), so using absolute value signs around the *x* means that we can use the same domain for both. We don't have to worry about the sign of the *x*s, so be careful to include those absolute value signs; otherwise, you could end up with domain problems.

Additionally, we had three other rules for derivatives: the product rule, the quotient rule, and the chain rule. We'll see antiderivative rules for each of these in future lessons.

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VIDEO: THE ANTIDERIVATIVE

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READING: OTHER METHODS OF INTEGRATION

Sometimes, we can't find an easy way to get an antiderivative using the rules we've already learned. We'll have to use some algebraic manipulation to get it into something we can integrate. There are many methods that a calculus class learns. Two methods are covered here: integration by substitution and integration by parts.

Integration by Substitution

Integration by Substitution is one way of algebraically manipulating an integrand so that the rules apply. This is a way to unwind the chain rule for derivatives. When you find the derivative of a function using the chain rule, you end up with a product of something like the original function TIMES a derivative. We reverse this idea to find the integral.

Method

The goal is to turn $\int f(g(x))dx$ into $\int f(u)du$ where $f(u)$ is much less complicated than $f(g(x))$ and easier to integrate.

1. Let u be some part of the integrand. A good first attempt is "one step inside the most complicated bit."
2. Compute $du = u'dx$
3. Translate every x in the integral into a u (including the dx). When you're done, you'll have an integral that has only u and du in it. (If you don't get rid of all the xs, then something went wrong—either you missed a piece or you didn't pick the right thing for u. Also, the du **never** goes in the denominator.)
4. Integrate the new u-only integral, if possible. If you still can't integrate, go back to step 1 and try a different choice for u.
5. Substitute back to have all xs in the answer instead of u.

It's a good idea to try integration by substitution if you recognize one part of the integrand as a derivative of another part of the integrand.

Integration by Parts

Integration by Parts is a manipulation of integrating the product rule.

Formula

$\int u \cdot dv = uv - \int v \cdot du$

A helpful hint: first try to call u any logarithmic function (if there are any) or call dv an exponential part of the function (again, if there are any of them).

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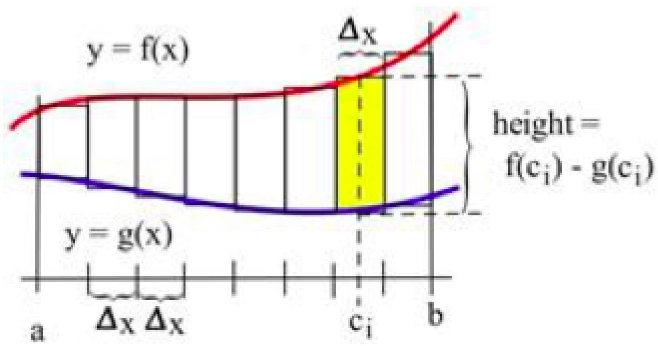
READING: THE DEFINITE INTEGRAL APPLIED TO AREA

Area between Two Functions as a Riemann Sum

We have already used integrals to find the area between the graph of a function and the *x*-axis. Integrals can also be used to find the area between two graphs.

If $f(x) \geq g(x)$ everywhere in an interval, then we can approximate the area between *f* and *g* by partitioning the interval [*a*, *b*] and forming a Riemann sum (see figure below). The height of each rectangle is top minus bottom, $f(c_i) - g(c_i)$ so the area of the *i*th rectangle is (height) × (base) = $\{f(c_i) - g(c_i)\} \times \Delta x$. This approximation of the total area is the Riemann sum $\sum_{i=1}^{n} (f(c_i) - g(c_i))\Delta x$.



The limit of this Riemann sum, as the number of rectangles gets larger and their width gets smaller, is a definite integral:

The Area between Two Functions as a Definite Integral

If you have two functions, $f(x)$ and $g(x)$, and you know $f(x) \geq g(x)$ everywhere between $x = a$ and $x = b$, then the area between the functions is $\int_a^b (f(x) - g(x))dx$.

The integrand is “upper function minus lower function.” Make a graph or plug in test values to be sure which curve is on top and which is on the bottom.

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READING: ACCUMULATION IN REAL LIFE

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if *t* is a measure of time in seconds and f(*t*) is a velocity with units feet/second, then the definite integral has units (feet/second) × (seconds) = feet.

In general, the units for the definite integral $\int_a^b f(x)dx$ are (*y*-units) × (*x*-units). A quick check of the units can help avoid errors in setting up an applied problem.

In previous examples, we looked at functions that represented a rate of travel (in miles per hour). In those examples, the area represents the total distance traveled.

For functions representing other rates such as the production of a factory (which produces items per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (vehicles per minute), or the spread of a disease (new cases per week), the area represents the total amount of something (total items, total gallons, total vehicles, etc.).

If *MR*(*q*) is a company's marginal revenue in dollars/item for selling *q* items, then $\int_0^{150} MR(q)dq$ has units (dollars/item) × (items) = dollars, and represents the accumulated dollars for selling from 0 to 150 items. That is, $\int_0^{150} MR(q)dq = TR(150)$, the total revenue from selling 150 items.

If *r*(*t*), in centimeters per year, represents how the diameter of a tree changes with time. Then $\int_{Time_1}^{Time_2} r(t)dt$ has units of (centimeters per year) × (years) = centimeters, and represents the accumulated growth of the tree's diameter from Time_1 to Time_2. That is, $\int_{Time_1}^{Time_2} r(t)dt$ is the change in the diameter of the tree over this period of time.

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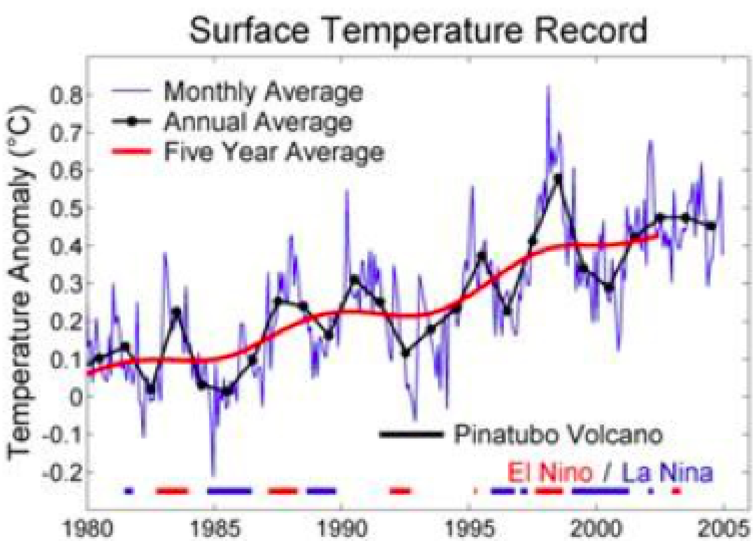
READING: AVERAGE VALUE

We know the average of n numbers, a_1, a_2, . . . , a_n, is their sum divided by n. But what if we need to find the average temperature over a day's time—there are too many possible temperatures to add them up. This is a job for the definite integral.

The **average value of a function** $f(x)$ on the interval [a, b] is given by $\frac{1}{b-a}\int_a^b f(x)dx$

In general, the average value of a function will have the same units as the integrand.

Function averages, involving means and more complicated averages, are used to "smooth" data so that underlying patterns are more obvious and to remove high frequency "noise" from signals. In these situations, the original function f is replaced by some "average of f." For example, the graph below shows the graphs of a Monthly Average (rather "noisy" data) of surface temperature data, an Annual Average (still rather "jagged), and a Five Year Average (a much smoother function). Typically the average function reveals the pattern much more clearly than the original data. The use of a "moving average" value for "noisy" data (weather information, stock prices, etc.) is very common.



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READING: CONTINUOUS INCOME STREAM

In a previous class, you learned about compound interest in a really simple situation where you made a single deposit into an interest-bearing account and let it sit undisturbed, earning interest, for some period of time.

Compound Interest Formulas

Let P = the principal (initial investment), r = the annual interest rate expressed as a decimal, and let $A(t)$ be the amount in the account at the end of t years.

- **Compounding n times per year:** $A(t) = P(1 + \frac{r}{n})^{nt}$
- **Compounded continuously:** $A(t) = Pe^{rt}$

You may also have learned somewhat more complicated annuity formulas to deal with slightly more complicated situations—where you make equal deposits equally spaced in time.

But real life is not usually so neat.

Calculus allows us to handle situations where "deposits" are flowing continuously into an account that earns interest. As long as we can model the flow of income with a function, we can use a definite integral to calculate the value of a continuous income stream. The idea is that each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then we'll add them all up (a definite integral).

Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year), that applies between year 0 and year T.

Then the present value of that income stream is given by $PV = \int_0^T F(t)e^{-rt}dt$.

The future value can be computed by the ordinary compound interest formula $FV = PVe^{rt}$.

This is a useful way to compare two investments—find the present value of each to see which is worth more today.

Text:

http://www.opentextbookstore.com/appcalc/Chapter3-7.pdf

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PUTTING IT TOGETHER

Summary

Skeptics abound in this world, but even Einstein was "perplexed by the miracle of compound interest." Your task is simply a future value problem.

Examples

If you assume a continuous compounding interest (to simplify your calculations), your equation is:

$$\int_0^N Ke^{r(N-t)}dt$$

Where you invest K dollars per year for N years at interest rate r compounded continuously. You do not want to overplay your hand, so you use a modest rate of return of 8%.

$$\int_0^{30} 10000e^{0.08(30-t)}dt$$

$$= \frac{10000}{-0.08}e^{0.08(30-t)}\Big|_1^9$$

$$= \frac{10000}{-0.08}\left(e^{0.08(30-30)} - e^{0.08(30-0)}\right)$$

$$= \$1252897.05$$

To strengthen your case for him to invest, you can redo these calculations for a 10% rate of return which would yield $1,908,553.69. Show him the math and let it speak for itself.