Elementary Algebra

Lumen Learning, NROC

CONTENTS

Module 8: Roots and Rational Exponents

MODULE 0: REVIEW

INTRODUCTION

Why review topics from earlier math classes?

You have been in class all day, running around, picking up your kids from daycare and after a long, brutal commute you get home to realize you haven't thought about what to fix for dinner. You go to the fridge and all you see are leftovers, a jumble of unrelated bits and pieces from meals past. Despite your fatigue, you manage to cobble together a meal that everyone will eat willingly and to get on with your evening.

In this section, you may feel like the topics are a cobbled-together meal of leftovers. The goal is to give you a reminder of some of the skills and topics from earlier math classes that, like the box of leftovers in the back of your fridge, are easily forgotten by most people. Because sometimes we just forget how these concepts work . . . until we need them again.

Leftovers again?

Learning Outcomes

- **Arithmetic With Fractions**
  - Add and subtract fractions
  - Multiply fractions
  - Divide fractions
- **Arithmetic of Real Numbers**
  - Add and subtract real numbers
  - Multiply and divide real numbers
  - Simplify expression with real numbers
  - Simplify compound expression with real numbers
- **Percent**
  - Explain the basics of percents
  - Solve equations containing percents
  - Solve percent change and interest problems

As you work through the rest of the sections of this course, please return to this review if you feel like you need a reminder of the topics covered. These topics were chosen because they are often forgotten and are widely used throughout the course. Don't worry, just like ketchup, these concepts have a long shelf life.

FRACTIONS

Learning Objectives

- Add and subtract fractions
  - Find the common denominator of two or more fractions
  - Use the common denominator to add or subtract fractions
  - Simplify a fraction to its lowest terms
- Multiply fractions
  - Multiply two or more fractions
  - Multiply a fraction by a whole number
- Divide fractions
  - Find the reciprocal of a number
  - Divide a fraction by a whole number
  - Divide a fraction by a fraction

Introduction

Before we get started, here is some important terminology that will help you understand the concepts about working with fractions in this section.

- **product:** the result of multiplication
- **factor:** something being multiplied – for $3 \cdot 2 = 6$, both 3 and 2 are factors of 6
- **numerator:** the top part of a fraction – the numerator in the fraction $\frac{2}{3}$ is 2
- **denominator:** the bottom part of a fraction – the denominator in the fraction $\frac{2}{3}$ is 3

Note About Instructions

Many different words are used by math textbooks and teachers to provide students with instructions on what they are to do with a given problem. For example, you may see instructions such as "Find" or "Simplify" in the example in this module. It is important to understand what these words mean so you can successfully work through the problems in this course. Here is a short list of the words you may see that can help you know how to work through the problems in this module.

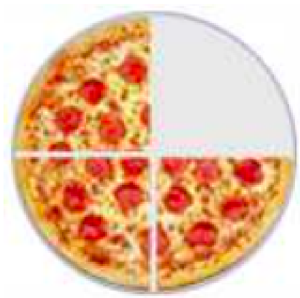
Instruction	Interpretation
Find	Perform the indicated mathematical operations which may include addition, subtraction, multiplication, division.
Simplify	1) Perform the indicated mathematical operations including addition, subtraction, multiplication, division 2) Write a mathematical statement in smallest terms so there are no other mathematical operations that can be performed—often found in problems related to fractions and the order of operations
Evaluate	Perform the indicated mathematical operations including addition, subtraction, multiplication, division
Reduce	Write a mathematical statement in smallest or lowest terms so there are no other mathematical operations that can be performed—often found in problems related to fractions or division

Adding and Subtracting Fractions

Adding Fractions

When you need to add or subtract fractions, you will need to first make sure that the fractions have the same denominator. The denominator tells you how many pieces the whole has been broken into, and the numerator tells you how many of those pieces you are using.

The “parts of a whole” concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now, $\frac{1}{4}$ of the pizza is gone and $\frac{3}{4}$ remains. Note that both of these fractions have a denominator of 4, which refers to the number of slices the whole pizza has been cut into. What if you have another pizza that had been cut into 8 equal parts and 3 of those parts were gone, leaving $\frac{5}{8}$?



How can you describe the total amount of pizza that is left with one number rather than two different fractions? You need a common denominator, technically called the **least common multiple.** Remember that if a number is a multiple of another, you can divide them and have no remainder.

One way to find the least common multiple of two or more numbers is to first multiply each by 1, 2, 3, 4, etc. For example, find the least common multiple of 2 and 5.

First, list all the multiples of 2:	Then list all the multiples of 5:
$2 \cdot 1 = 2$	$5 \cdot 1 = 5$
$2 \cdot 2 = 4$	$5 \cdot 2 = 10$
$2 \cdot 3 = 6$	$5 \cdot 3 = 15$
$2 \cdot 4 = 8$	$5 \cdot 4 = 20$
$2 \cdot 5 = 10$	$5 \cdot 5 = 25$

The smallest multiple they have in common will be the common denominator for the two!

Example

Describe the amount of pizza left using common terms.
Show Solution

To add fractions with unlike denominators, first rewrite them with like denominators. Then, you know what to do! The steps are shown below.

Adding Fractions with Unlike Denominators

1. Find a common denominator.
2. Rewrite each fraction using the common denominator.
3. Now that the fractions have a common denominator, you can add the numerators.
4. Simplify by canceling out all common factors in the numerator and denominator.

Simplifying a Fraction

Often, if the answer to a problem is a fraction, you will be asked to write it in lowest terms. This is a common convention used in mathematics, similar to starting a sentence with a capital letter and ending it with a period. In this course, we will not go into great detail about methods for reducing fractions because there are many. The process of simplifying a fraction is often called *reducing the fraction*. We can simplify by canceling (dividing) the common factors in a fraction's numerator and denominator. We can do this because a fraction represents division.

For example, to simplify $\frac{6}{9}$ you can rewrite 6 and 9 using the smallest factors possible as follows:

$\frac{6}{9}=\frac{2\cdot3}{3\cdot3}$

Since there is a 3 in both the numerator and denominator, and fractions can be considered division, we can divide the 3 in the top by the 3 in the bottom to reduce to 1.

$\frac{6}{9}=\frac{2\cdot\cancel{3}}{3\cdot\cancel{3}}=\frac{2\cdot1}{3}=\frac{2}{3}$

Rewriting fractions with the smallest factors possible is often called prime factorization.

In the next example you are shown how to add two fractions with different denominators, then simplify the answer.

Example

Add $\frac{2}{3}+\frac{1}{5}$. Simplify the answer.
Show Solution

You can find a common denominator by finding the common multiples of the denominators. The least common multiple is the easiest to use.

Example

Add $\frac{3}{7}+\frac{2}{21}$. Simplify the answer.
Show Solution

In the following video you will see an example of how to add two fractions with different denominators.

Watch this video online: https://youtu.be/zV4q7j1-89I

You can also add more than two fractions as long as you first find a common denominator for all of them. An example of a sum of three fractions is shown below. In this example, you will use the prime factorization method to find the LCM.

Think About It

Add $\frac{3}{4}+\frac{1}{6}+\frac{5}{8}$. Simplify the answer and write as a mixed number.

What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would add three fractions with different denominators together.

Show Solution

Subtracting Fractions

When you subtract fractions, you must think about whether they have a common denominator, just like with adding fractions. Below are some examples of subtracting fractions whose denominators are not alike.

Example

Subtract $\frac{1}{5} - \frac{1}{6}$. Simplify the answer.
Show Solution

The example below shows how to use multiples to find the least common multiple, which will be the least common denominator.

Example

Subtract $\frac{5}{6} - \frac{1}{4}$. Simplify the answer.
Show Solution

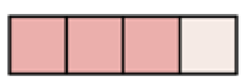
In the following video you will see an example of how to subtract fractions with unlike denominators.

Watch this video online: https://youtu.be/RpHtOMjeI7g

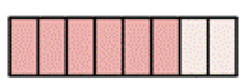
Multiply Fractions

Just as you add, subtract, multiply, and divide when working with whole numbers, you also use these operations when working with fractions. There are many times when it is necessary to multiply fractions. A model may help you understand multiplication of fractions.

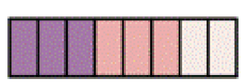
When you multiply a fraction by a fraction, you are finding a "fraction of a fraction." Suppose you have $\frac{3}{4}$ of a candy bar and you want to find $\frac{1}{2}$ of the $\frac{3}{4}$:

$\frac{3}{4}$ 

By dividing each fourth in half, you can divide the candy bar into eighths.

$\frac{6}{8}$ 

Then, choose half of those to get $\frac{3}{8}$.

$\frac{3}{8}$ 

In both of the above cases, to find the answer, you can multiply the numerators together and the denominators together.

Multiplying Two Fractions

$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{\text{product of the numerators}}{\text{product of the denominators}}$

Multiplying More Than Two Fractions

$\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = \frac{a \cdot c \cdot e}{b \cdot d \cdot f}$

Example

Multiply $\frac{2}{3} \cdot \frac{4}{5}$.
Show Solution

To review: if a fraction has common factors in the numerator and denominator, we can reduce the fraction to its simplified form by removing the common factors.

For example,

- Given $\frac{8}{15}$, the factors of 8 are: 1, 2, 4, 8 and the factors of 15 are: 1, 3, 5, 15. $\frac{8}{15}$ is simplified because there are no common factors of 8 and 15.
- Given $\frac{10}{15}$, the factors of 10 are: 1, 2, 5, 10 and the factors of15 are: 1, 3, 5, 15. $\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

You can simplify first, before you multiply two fractions, to make your work easier. This allows you to work with smaller numbers when you multiply.

In the following video you will see an example of how to multiply two fractions, then simplify the answer.

Watch this video online: https://youtu.be/f_L-EFC8Z7c

Think About It

Multiply $\frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{5}$. Simplify the answer.
What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would multiply three fractions together.

Show Solution

Divide Fractions

There are times when you need to use division to solve a problem. For example, if painting one coat of paint on the walls of a room requires 3 quarts of paint and you have a bucket that contains 6 quarts of paint, how many coats of paint can you paint on the walls? You divide 6 by 3 for an answer of 2 coats. There will also be times when you need to divide by a fraction. Suppose painting a closet with one coat only required $\frac{1}{2}$ quart of paint. How many coats could be painted with the 6 quarts of paint? To find the answer, you need to divide 6 by the fraction, $\frac{1}{2}$.

Before we begin dividing fractions, let's cover some important terminology.

- **reciprocal:** two fractions are reciprocals if their product is 1 (Don't worry; we will show you examples of what this means.)
- **quotient:** the result of division

Dividing fractions requires using the reciprocal of a number or fraction. If you multiply two numbers together and get 1 as a result, then the two numbers are reciprocals. Here are some examples of reciprocals:

Original number	Reciprocal	Product
$\frac{3}{4}$	$\frac{4}{3}$	$\frac{3}{4}\cdot\frac{4}{3}=\frac{3\cdot 4}{4\cdot 3}=\frac{12}{12}=1$
$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{2}\cdot\frac{2}{1}=\frac{1\cdot}{2\cdot 1}=\frac{2}{2}=1$
$3=\frac{3}{1}$	$\frac{1}{3}$	$\frac{3}{1}\cdot\frac{1}{3}=\frac{3\cdot 1}{1\cdot 3}=\frac{3}{3}=1$
$2\frac{1}{3}=\frac{7}{3}$	$\frac{3}{7}$	$\frac{7}{3}\cdot\frac{3}{7}=\frac{7\cdot 3}{3\cdot 7}=\frac{21}{21}=1$

Sometimes we call the reciprocal the "flip" of the other number: flip $\frac{2}{5}$ to get the reciprocal $\frac{5}{2}$.

Division by Zero

You know what it means to divide by 2 or divide by 10, but what does it mean to divide a quantity by 0? Is this even possible? Can you divide 0 by a number? Consider the fraction

$$\frac{0}{8}$$

We can read it as, "zero divided by eight." Since multiplication is the inverse of division, we could rewrite this as a multiplication problem.

$$?\cdot 8=0.$$

We can infer that the unknown must be 0 since that is the only number that will give a result of 0 when it is multiplied by 8.

Now let's consider the reciprocal of $\frac{0}{8}$ which would be $\frac{8}{0}$. If we rewrite this as a multiplication problem, we will have

$$?\cdot 0=8.$$

This doesn't make any sense. There are no numbers that you can multiply by zero to get a result of 8. The reciprocal of $\frac{8}{0}$ is undefined, and in fact, all division by zero is undefined.

> Caution! Division by zero is undefined and so is the reciprocal of any fraction that has a zero in the numerator. For any real number a, $\frac{a}{0}$ is undefined. Additionally, the reciprocal of $\frac{0}{a}$ will always be undefined.

Divide a Fraction by a Whole Number

When you divide by a whole number, you are multiplying by the reciprocal. In the painting example where you need 3 quarts of paint for a coat and have 6 quarts of paint, you can find the total number of coats that can be painted by dividing 6 by 3, $6 \div 3 = 2$. You can also multiply 6 by the reciprocal of 3, which is $\frac{1}{3}$, so the multiplication problem becomes

$\frac{6}{1} \cdot \frac{1}{3} = \frac{6}{3} = 2.$

Dividing is Multiplying by the Reciprocal

For all division, you can turn the operation into multiplication by using the reciprocal. Dividing is the same as multiplying by the reciprocal.

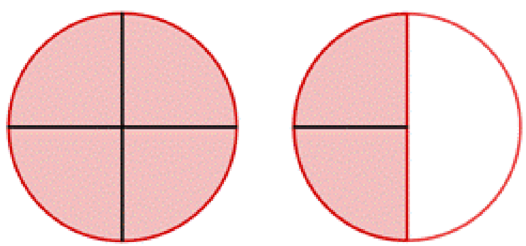
The same idea will work when the divisor (the thing being divided) is a fraction. If you have $\frac{3}{4}$ of a candy bar and need to divide it among 5 people, each person gets $\frac{1}{5}$ of the available candy:

$\frac{1}{5} \text{ of } \frac{3}{4} = \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20}$

Each person gets $\frac{3}{20}$ of a whole candy bar.

If you have a recipe that needs to be divided in half, you can divide each ingredient by 2, or you can multiply each ingredient by $\frac{1}{2}$ to find the new amount.

For example, dividing by 6 is the same as multiplying by the reciprocal of 6, which is $\frac{1}{6}$. Look at the diagram of two pizzas below. How can you divide what is left (the red shaded region) among 6 people fairly?



$$\frac{3}{2} \text{ divided by } 6 = \frac{3}{2} \times \frac{1}{6}$$

$$\frac{3}{2} \times \frac{1}{6} = \frac{1}{4}$$

Each person gets one piece, so each person gets $\frac{1}{4}$ of a pizza.

Dividing a fraction by a whole number is the same as multiplying by the reciprocal, so you can always use multiplication of fractions to solve division problems.

Example

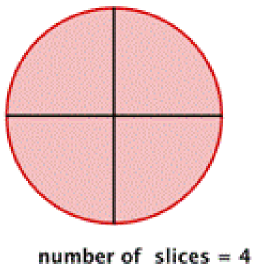
Find $\frac{2}{3} \div 4$.
Show Solution

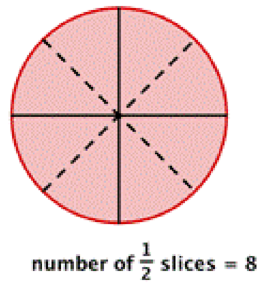
Example

Divide. $9 \div \frac{1}{2}$.
Show Solution

Divide a Fraction by a Fraction

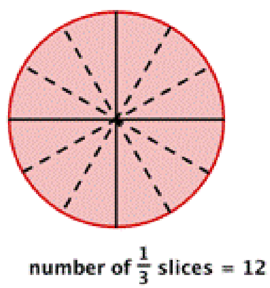
Sometimes you need to solve a problem that requires dividing by a fraction. Suppose you have a pizza that is already cut into 4 slices. How many $\frac{1}{2}$ slices are there?





There are 8 slices. You can see that dividing 4 by $\frac{1}{2}$ gives the same result as multiplying 4 by 2.

What would happen if you needed to divide each slice into thirds?



You would have 12 slices, which is the same as multiplying 4 by 3.

Dividing with Fractions

1. Find the reciprocal of the number that follows the division symbol.
2. Multiply the first number (the one before the division symbol) by the reciprocal of the second number (the one after the division symbol).

Any easy way to remember how to divide fractions is the phrase “keep, change, flip.” This means to **KEEP** the first number, **CHANGE** the division sign to multiplication, and then **FLIP** (use the reciprocal) of the second number.

Example

Divide $\frac{2}{3} \div \frac{1}{6}$.
Show Solution

Example

Divide $\frac{3}{5} \div \frac{2}{3}$.
Show Solution

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Learning Outcomes

Properties of Real Numbers

- Define real numbers and subsets of real numbers
- Define properties of real numbers and sue them to evaluate algebraic expressions

Multi-Step Equations

- Use properties of real numbers to solve multi-step linear equations
- Define and use the distributive property to solve linear equations
- Classify solutions to equations

Problem Solving

- Set up equations from descriptions of problems
- Use formulas to solve application problems

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REAL NUMBERS

Learning Objectives

- Add and subtract real numbers
  - Add real numbers with the same and different signs
  - Subtract real numbers with the same and different signs
  - Simplify combinations that require both addition and subtraction of real numbers.
- Multiply and divide real numbers
  - Multiply two or more real numbers.
  - Divide real numbers
  - Simplify expressions with both multiplication and division
  - Simplify expressions with real numbers
- Recognize and combine like terms in an expression
  - Use the order of operations to simplify expressions
- Simplify compound expressions with real numbers
  - Simplify expressions with fraction bars, brackets, and parentheses
  - Use the distributive property to simplify expressions with grouping symbols
  - Simplify expressions containing absolute values

Some important terminology to remember before we begin is as follows:

- **integers:** counting numbers like 1, 2, 3, etc., including negatives and zero
- **real number:** fractions, negative umbers, decimals, integers, and zero are all real numbers
- **absolute value:** a number's distance from zero; it's always positive. $|-7|=7$
- **sign:** this refers to whether a number is positive or negative, we use $+$ for positive (to the right of zero on the number line) and $-$ for negative (to the left of zero on the number line)
- **difference:** the result of subtraction
- **sum:** the result of addition

The ability to work comfortably with negative numbers is essential to success in algebra. For this reason we will do a quick review of adding, subtracting, multiplying and dividing integers. Integers are all the positive whole numbers, zero, and their opposites (negatives). As this is intended to be a review of integers, the descriptions and examples will not be as detailed as a normal lesson.

Adding and Subtracting Real Numbers

When adding integers we have two cases to consider. The first case is whether the signs match (both positive or both negative). If the signs match, we will add the numbers together and keep the sign.

If the signs don't match (one positive and one negative number) we will subtract the numbers (as if they were all positive) and then use the sign from the larger number. This means if the larger number is positive, the answer is positive. If the larger number is negative, the answer is negative.

To add two numbers with the same sign (both positive or both negative)

- *Add* their absolute values (without the $+$ or $-$ sign)
- Give the sum the same sign.

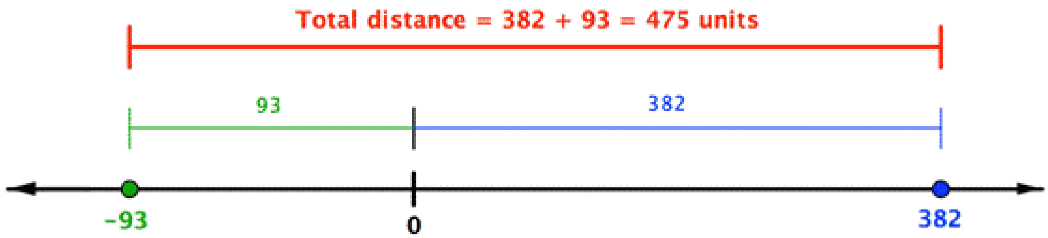
To add two numbers with different signs (one positive and one negative)

- Find the *difference* of their absolute values. (Note that when you find the difference of the absolute values, you always subtract the lesser absolute value from the greater one.)
- Give the sum the same sign as the number with the greater absolute value.

Example

Find $23–73$.
Show Solution

Another way to think about subtracting is to think about the distance between the two numbers on the number line. In the example below, 382 is to the *right* of 0 by 382 units, and -93 is to the *left* of 0 by 93 units. The distance between them is the sum of their distances to 0: $382+93$.



Example

Find $382–(-93)$.
Show Solution

The following video explains how to subtract two signed integers.

Watch this video online: https://youtu.be/ciuIKFCtWWU

Example

Find $-\frac{3}{7}-\frac{6}{7}+\frac{2}{7}$
Show Solution

In the following video you will see an example of how to add three fractions with a common denominator that have different signs.

Watch this video online: https://youtu.be/P972VVbR98k

Example

Evaluate $27.832+(-3.06)$. When you add decimals, remember to line up the decimal points so you are adding tenths to tenths, hundredths to hundredths, and so on.
Show Solution

In the following video are examples of adding and subtracting decimals with different signs.

Watch this video online: https://youtu.be/3FHZQ5iKcpI

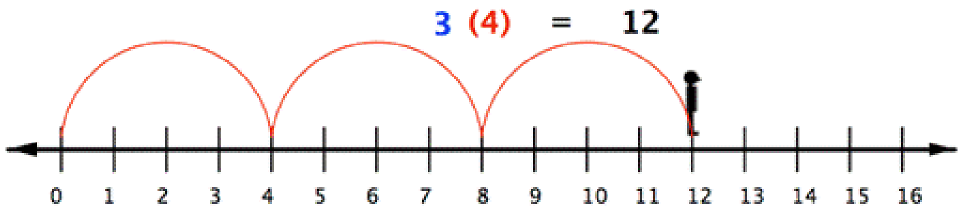
Multiplying and Dividing Real Numbers

Multiplication and division are **inverse operations**, just as addition and subtraction are. You may recall that when you divide fractions, you multiply by the reciprocal. Inverse operations “undo” each other.

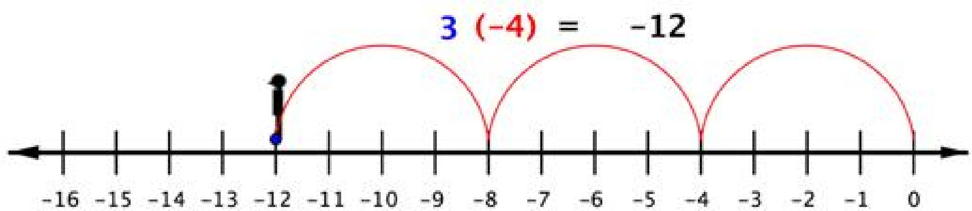
Multiply Real Numbers

Multiplying real numbers is not that different from multiplying whole numbers and positive fractions. However, you haven’t learned what effect a negative sign has on the product.

With whole numbers, you can think of multiplication as repeated addition. Using the number line, you can make multiple jumps of a given size. For example, the following picture shows the product $3 \cdot 4$ as 3 jumps of 4 units each.



So to multiply $3(-4)$, you can face left (toward the negative side) and make three “jumps” forward (in a negative direction).



The product of a positive number and a negative number (or a negative and a positive) is negative.

The Product of a Positive Number and a Negative Number

To multiply a **positive number** and a **negative number**, multiply their absolute values. The product is **negative**.

Example

Find $-3.8(0.6)$.
Show Solution

The following video contains examples of how to multiply decimal numbers with different signs.

Watch this video online: https://youtu.be/7gY0S3LUUyQ

The Product of Two Numbers with the Same Sign (both positive or both negative)

To multiply two **positive numbers**, multiply their absolute values. The product is **positive**.
To multiply two **negative numbers**, multiply their absolute values. The product is **positive**.

Example

Find $\left(-\frac{3}{4}\right)\left(-\frac{2}{5}\right)$
Show Solution

The following video shows examples of multiplying two signed fractions, including simplification of the answer.

Watch this video online: https://youtu.be/yUdJ46pTblo

To summarize:

- **positive** · ***positive*:** The product is **positive**.
- **negative** · ***negative*:** The product is **positive**.
- **negative** · ***positive*:** The product is **negative**.
- **positive** · ***negative*:** The product is **negative**.

You can see that the product of two negative numbers is a positive number. So, if you are multiplying more than two numbers, you can count the number of negative factors.

Multiplying More Than Two Negative Numbers

If there are an **even** number (0, 2, 4, …) of negative factors to multiply, the product is **positive**.
If there are an **odd** number (1, 3, 5, …) of negative factors, the product is **negative**.

Example

Find $3(-6)(2)(-3)(-1)$.
Show Solution

The following video contains examples of multiplying more than two signed integers.

Watch this video online: https://youtu.be/rx8F9SPd0HE

Divide Real Numbers

You may remember that when you divided fractions, you multiplied by the **reciprocal**. *Reciprocal* is another name for the multiplicative inverse (just as *opposite* is another name for additive inverse).

An easy way to find the multiplicative inverse is to just "flip" the numerator and denominator as you did to find the reciprocal. Here are some examples:

- The reciprocal of $\frac{4}{9}$ is $\frac{9}{4}$ because $\frac{4}{9}\left(\frac{9}{4}\right)=\frac{36}{36}=1$.
- The reciprocal of 3 is $\frac{1}{3}$ because $\frac{3}{1}\left(\frac{1}{3}\right)=\frac{3}{3}=1$.
- The reciprocal of $-\frac{5}{6}$ is $\frac{-6}{5}$ because $-\frac{5}{6}\left(-\frac{6}{5}\right)=\frac{30}{30}=1$.
- The reciprocal of 1 is 1 as $1(1)=1$.

When you divided by positive fractions, you learned to multiply by the reciprocal. You also do this to divide **real numbers**.

Think about dividing a bag of 26 marbles into two smaller bags with the same number of marbles in each. You can also say each smaller bag has *one half* of the marbles.

$26 \div 2=26\left(\frac{1}{2}\right)=13$

Notice that 2 and $\frac{1}{2}$ are reciprocals.

Try again, dividing a bag of 36 marbles into smaller bags.

Number of bags	Dividing by number of bags	Multiplying by reciprocal
3	$\frac{36}{3}=12$	$36\left(\frac{1}{3}\right)=\frac{36}{3}=\frac{12(3)}{3}=12$
4	$\frac{36}{4}=9$	$36\left(\frac{1}{4}\right)=\frac{36}{4}=\frac{9(4)}{4}=9$

6	$\frac{36}{6}=6$	$36\left(\frac{1}{6}\right)=\frac{36}{6}=\frac{6(6)}{6}=6$

Dividing by a number is the same as multiplying by its reciprocal. (That is, you use the reciprocal of the **divisor**, the second number in the division problem.)

Example

Find $28\div\frac{4}{3}$
Show Solution

Now let's see what this means when one or more of the numbers is negative. A number and its reciprocal have the same sign. Since division is rewritten as multiplication using the reciprocal of the divisor, and taking the reciprocal doesn't change any of the signs, division follows the same rules as multiplication.

Rules of Division

When dividing, rewrite the problem as multiplication using the reciprocal of the divisor as the second factor.
When one number is **positive** and the other is **negative**, the **quotient** is **negative**.
When *both* numbers are **negative**, the quotient is **positive**.
When *both* numbers are **positive**, the quotient is **positive**.

Example

Find $24\div\left(-\frac{5}{6}\right)$.
Show Solution

Example

Find $4\left(-\frac{2}{3}\right)\div(-6)$
Show Solution

The following video explains how to divide signed fractions.

Watch this video online: https://youtu.be/OPHdadhDJoI

Remember that a fraction bar also indicates division, so a negative sign in front of a fraction goes with the numerator, the denominator, or the whole fraction: $-\frac{3}{4}=\frac{-3}{4}=\frac{3}{-4}$.
In each case, the overall fraction is negative because there's only one negative in the division.

The following video explains how to divide signed fractions.

Watch this video online: https://youtu.be/OPHdadhDJoI

Simplify Expressions With Real Numbers

Some important terminology before we begin:

- **operations/operators:** In mathematics we call things like multiplication, division, addition, and subtraction operations. They are the verbs of the math world, doing work on numbers and variables. The symbols used to denote operations are called operators, such as $+$, $-$, $\times$,$\div$. As you learn more math, you will learn more operators.

- **term:** Examples of terms would be $2x$ and $-\frac{3}{2}$ or a^3. Even lone integers can be a term, like 0.
- **expression: A** mathematical expression is one that connects terms with mathematical operators. For example $\frac{1}{2}+\left(2^2\right)-9\div\frac{6}{7}$ is an expression.

Combining Like Terms

One way we can simplify expressions is to combine like terms. **Like terms** are terms where the variables match exactly (exponents included). Examples of like terms would be $5xy$ and $-3xy$ or $8a^2b$ and a^2b or -3 and 8. If we have like terms we are allowed to add (or subtract) the numbers in front of the variables, then keep the variables the same. As we combine like terms we need to interpret subtraction signs as part of the following term. This means if we see a subtraction sign, we treat the following term like a negative term. The sign always stays with the term.

This is shown in the following examples:

Example

Combine like terms: $5x-2y-8x+7y$
Show Solution

In the following video you will be shown how to combine like terms using the idea of the distributive property. Note that this is a different method than is shown in the written examples on this page, but it obtains the same result.

Watch this video online: https://youtu.be/JIleqbO8Tf0

Example

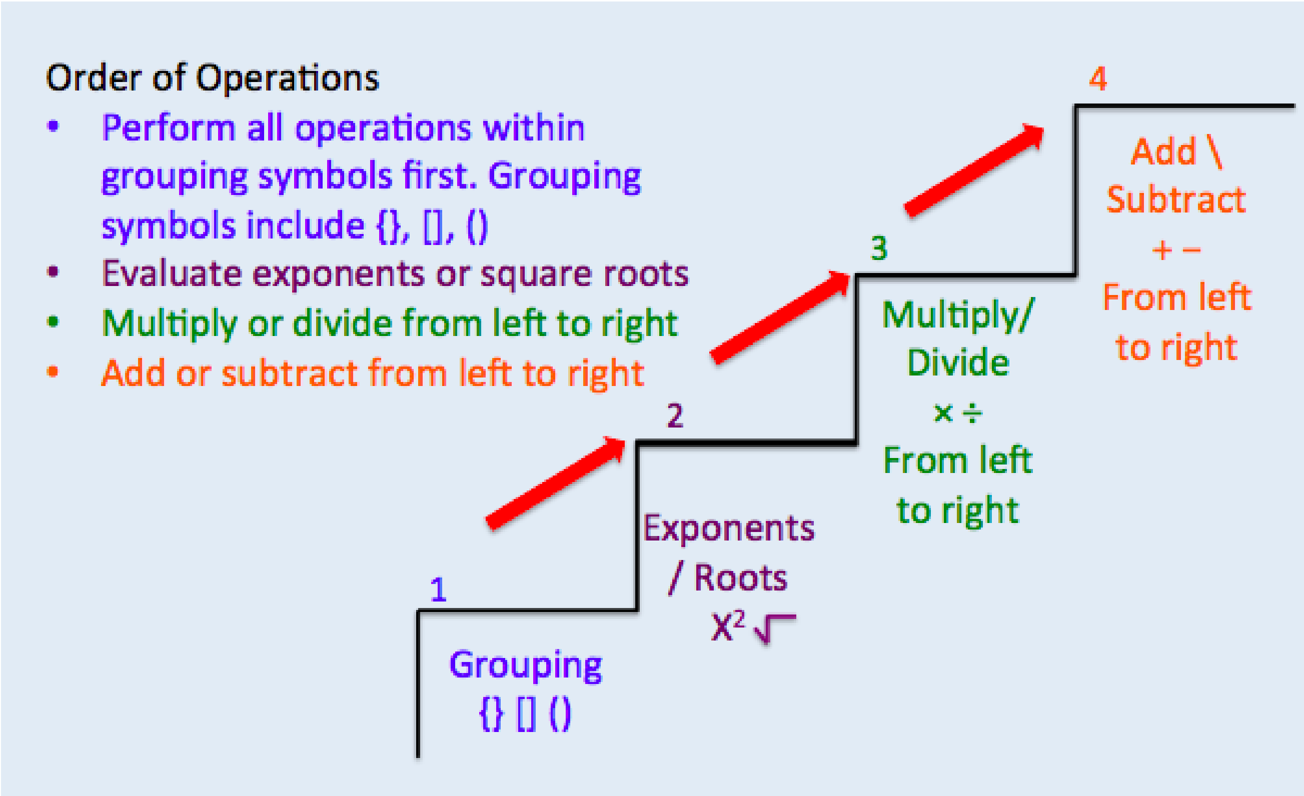
Combine like terms: $x^2-3x+9-5x^2+3x-1$
Show Solution

In the video that follows, you will be shown another example of combining like terms. Pay attention to why you are not able to combine all three terms in the example.

Watch this video online: https://youtu.be/b9-7eu29pNM

Order of Operations

You may or may not recall the order of operations for applying several mathematical operations to one expression. Just as it is a social convention for us to drive on the right-hand side of the road, the order of operations is a set of conventions used to provide order when you are required to use several mathematical operations for one expression. The graphic below depicts the order in which mathematical operations are performed.



Order of operations

Example

Simplify $7-5+3\cdot8$.
Show Solution

In the following example, you will be shown how to simplify an expression that contains both multiplication and subtraction using the order of operations.

Watch this video online: https://youtu.be/yFO_0dIfy-w

When you are applying the order of operations to expressions that contain fractions, decimals, and negative numbers, you will need to recall how to do these computations as well.

Example

Simplify $3\cdot\frac{1}{3}-8\div\frac{1}{4}$.
Show Solution

In the following video you are shown how to use the order of operations to simplify an expression that contains multiplication, division, and subtraction with terms that contain fractions.

Watch this video online: https://youtu.be/yqp06obmcVc

Exponents

When you are evaluating expressions, you will sometimes see exponents used to represent repeated multiplication. Recall that an expression such as 7^2 is **exponential notation** for $7\cdot7$. (Exponential notation has two parts: the **base** and the **exponent** or the **power**. In 7^2, 7 is the base and 2 is the exponent; the exponent determines how many times the base is multiplied by itself.)

Exponents are a way to represent repeated multiplication; the order of operations places it *before* any other multiplication, division, subtraction, and addition is performed.

Example

Simplify $3^2 \cdot 2^3$.
Show Solution

In the video that follows, an expression with exponents on its terms is simplified using the order of operations.

Watch this video online: https://youtu.be/JjBBgV7G_Qw

Grouping Symbols

Grouping symbols such as parentheses ( ), brackets [ ], braces$\{\}$, and fraction bars can be used to further control the order of the four arithmetic operations. The rules of the order of operations require computation within grouping symbols to be completed first, even if you are adding or subtracting within the grouping symbols and you have multiplication outside the grouping symbols. After computing within the grouping symbols, divide or multiply from left to right and then subtract or add from left to right. When there are grouping symbols within grouping symbols, calculate from the inside to the outside. That is, begin simplifying within the innermost grouping symbols first.

Remember that parentheses can also be used to show multiplication. In the example that follows, both uses of parentheses—as a way to represent a group, as well as a way to express multiplication—are shown.

Example

Simplify $(3+4)^2 + (8)(4)$.
Show Solution

Example

Simplify $4 \cdot \frac{3[5+(2+3)^2]}{2}$
Show Solution

In the following video, you are shown how to use the order of operations to simplify an expression with grouping symbols, exponents, multiplication, and addition.

Watch this video online: https://youtu.be/EMch2MKCVdA

Think About It

These problems are very similar to the examples given above. How are they different and what tools do you need to simplify them?
a) Simplify $(1.5+3.5)-2(0.5 \cdot 6)^2$. This problem has parentheses, exponents, multiplication, subtraction, and addition in it, as well as decimals instead of integers.
Use the box below to write down a few thoughts about how you would simplify this expression with decimals and grouping symbols.

Show Solution
b) Simplify $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 \cdot 32$.
Use the box below to write down a few thoughts about how you would simplify this expression with fractions and grouping symbols.

Show Solution

Simplify Compound Expressions With Real Numbers

In this section, we will use the skills from the last section to simplify mathematical expressions that contain many grouping symbols and many operations. We are using the term compound to describe expressions that have many operations and many grouping symbols. More care is needed with these expressions when you apply the order of operations. Additionally, you will see how to handle absolute value terms when you simplify expressions.

Example

Simplify $\frac{5-[3+(2\cdot(-6))]}{3^2+2}$

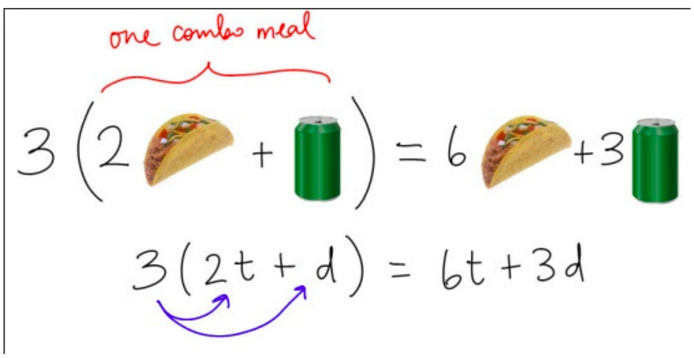
Show Solution

The video that follows contains an example similar to the written one above. Note how the numerator and denominator of the fraction are simplified separately.

Watch this video online: https://youtu.be/xIJLq54jM44

The Distributive Property

Parentheses are used to group or combine expressions and terms in mathematics. You may see them used when you are working with formulas, and when you are translating a real situation into a mathematical problem so you can find a quantitative solution.



Combo Meal Distributive Property

For example, you are on your way to hang out with your friends, and call them to ask if they want something from your favorite drive-through. Three people want the same combo meal of 2 tacos and one drink. You can use the distributive property to find out how many total tacos and how many total drinks you should take to them.

$$\begin{aligned} &3\,(2\text{ tacos } + 1\text{ drink}) \\ &= 3\cdot 2\text{ tacos } + 3\text{ drinks} \\ &= 6\text{ tacos } + 3\text{ drinks}\end{aligned}$$

The distributive property allows us to explicitly describe a total that is a result of a group of groups. In the case of the combo meals, we have three groups of (two tacos plus one drink). The following definition describes how to use the distributive property in general terms.

The Distributive Property of Multiplication

For all real numbers *a, b,* and *c*, $a(b+c)=ab+ac$.
What this means is that when a number multiplies an expression inside parentheses, you can distribute the multiplication to each term of the expression individually.

To simplify $3\,(3+y)-y+9$, it may help to see the expression translated into words:

multiply three by (the sum of three and y), then subtract y, then add 9

To multiply three by the sum of three and y, you use the distributive property –

$$\begin{aligned}&3\left(3+y\right)-y+9\\&=\underbrace{3\cdot 3}+\underbrace{3\cdot y}-y+9\\&=9+3y-y+9\end{aligned}$$

Now you can subtract y from 3y and add 9 to 9.

$$\begin{aligned}&9+3y-y+9\\&=18+2y\end{aligned}$$

The next example shows how to use the distributive property when one of the terms involved is negative.

Example

Simplify $a+2\left(5-a\right)+3\left(a+4\right)$
Show Solution

Absolute Value

Absolute value expressions are one final method of grouping that you may see. Recall that the absolute value of a quantity is always positive or 0.

When you see an absolute value expression included within a larger expression, treat the absolute value like a grouping symbol and evaluate the expression within the absolute value sign first. Then take the absolute value of that expression. The example below shows how this is done.

Example

Simplify $\frac{3+|2-6|}{2|3\cdot 1.5|-(-3)}$.
Show Solution

The following video uses the order of operations to simplify an expression in fraction form that contains absolute value terms. Note how the absolute values are treated like parentheses and brackets when using the order of operations.

Watch this video online: https://youtu.be/6wmCQprxlnU

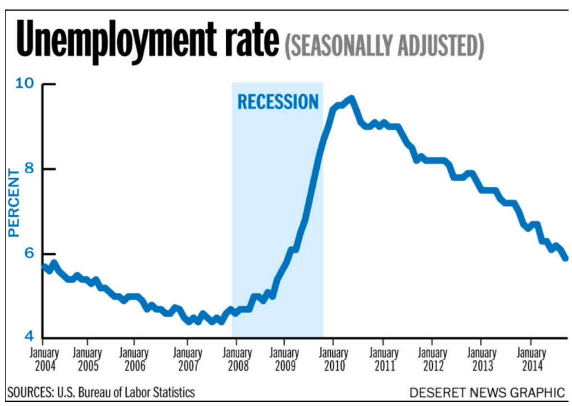
PERCENT

Learning Objectives

- Explain the basics of percents
  - Find a percent of a whole
  - Identify the amount, the base, and the percent in a percent problem
  - Find the unknown in a percent problem
- Solve equations containing percents
  - Identify the unknown in a percent problem
  - Write a percent equation
  - Solve equations with percent
- Solve percent change and interest problems
  - Calculate discounts and markups using percent
  - Calculate interest earned or owed
  - Read and interpret data from pie charts as percents

Percent of a Whole

Percents are the ratio of a number and 100. Percents are used in many different applications. Percents are used widely to describe how something changed. For example, you may have heard that the amount of rainfall this month had decreased by 12% from last year, or that the number of jobless claims has increase by 5% this quarter over last quarter.



Unemployment rate as percent by year between 2004 and 2014.

We regularly use this kind of language to quickly describe how much something increased or decreased over time or between significant events.

Before we dissect the methods for finding percent change of a quantity, let’s learn the basics of finding percent of a whole.

For example, if we knew a gas tank held 14 gallons, and wanted to know how many gallons were in $\frac{1}{4}$ of a tank, we would find $\frac{1}{4}$ of 14 gallons by multiplying:

$$\frac{1}{4}\cdot 14=\frac{1}{4}\cdot\frac{14}{1}=\frac{14}{4}=3\frac{2}{4}=3\frac{1}{2}\ \text{gallons}$$

Likewise, if we wanted to find 25% of 14 gallons, we could find this by multiplying, but first we would need to convert the 25% to a decimal:

$$25\%\ \text{of}\ 14\ \text{gallons}=0.25\cdot 14=3.5\ \text{gallons}$$

Finding a Percent of a Whole

To find a percent of a whole,

- Write the percent as a decimal by moving the decimal two places to the left
- Then multiply the percent by the whole amount

Example

What is 15% of $200?
Show Solution

The following video contains an example that is similar to the one above.

Watch this video online: https://youtu.be/jTM7ZMvAzsc

From the previous example, we can identify three important parts to finding the percent of a whole:

- the **percent**, has the percent symbol (%) or the word "percent"
- the **amount**, the amount is part of the whole
- and the **base,** the base is the whole amount

The following examples show how to identify the three parts: the percent, the base, and the amount.

Example

Identify the percent, amount, and base in this problem.

30 is 20% of what number?
Show Solution

The previous problem states that 30 is a portion of another number. That means 30 is the amount. Note that this problem could be rewritten: 20% of what number is 30?

Example

Identify the percent, amount, and base in this problem.

What percent of 30 is 3?
Show Solution

Example

Identify the percent, amount, and base in this problem.

What is 60% of 45?
Show Solution

The following video provides more examples that describe how to identify the percent, amount, and base in a percent problem.

Watch this video online: https://youtu.be/zwT58-LJCvs

In the next section, you will use the parts of a percent problem to find the percent increase or decrease of a quantity by writing and solving equations.

Percent Equations

Percent problems can be solved by writing **equations**. An equation uses an equal sign (=) to show that two mathematical expressions have the same value.

Percents are fractions, and just like fractions, when finding a percent (or fraction, or portion) of another amount, you multiply.

In the previous section, we identified three important parts to finding the percent of a whole:

- the **percent**, has the percent symbol (%) or the word "percent"
- the **amount**, the amount is part of the whole
- and the **base**, the base is the whole amount

Using these parts, we can define equations that will help us answer percent problems.

The Percent Equation

Percent of the Base is the Amount.
$\text{Percent} \cdot \text{Base} = \text{Amount}$

In the examples below, the unknown is represented by the letter *n.* The unknown can be represented by any letter or a box □, question mark, or even a smiley face □

Example

Write an equation that represents the following problem.

30 is 20% of what number?
Show Solution

The following example shows how to use the percent equation to find the base in a percent equation.

Watch this video online: https://youtu.be/3etjmUw8K3A

Once you have an equation, you can solve it and find the unknown value. For example, to solve
20

you can divide 30 by 20% to find the unknown:

$20\% \cdot n = 30$

You can solve this by writing the percent as a decimal or fraction and then dividing.

$20\% \cdot n = 30$

$n = 30 \div 20\% = 30 \div 0.20 = 150$

Example

What percent of 72 is 9?
Show Solution

In the following video example, you are shown how to use the percent equation to find the base in a percent problem.

Watch this video online: https://www.youtube.com/embed/p2KHHFMhJRs
You can estimate to see if the answer is reasonable. Use 10% and 20%, numbers close to 12.5%, to see if they get you close to the answer.

10% of 72 = 0.1 · 72 = 7.2

20% of 72 = 0.2 · 72 = 14.4

Notice that 9 is between 7.2 and 14.4, so 12.5% is reasonable since it is between 10% and 20%.

Example

What is 110% of 24?
Show Solution

The video that follows shows how top use the percent equation to find the amount in a percent equation.

Watch this video online: https://youtu.be/dO3AaW_c9s0

Percent Change

Percents have a wide variety of applications to everyday life, showing up regularly in taxes, discounts, markups, and interest rates. We will look at several examples of how to use percent to calculate markups, discounts, and interest earned or owed.

Example

Jeff has a coupon at the Guitar Store for 15% off any purchase of $100 or more. He wants to buy a used guitar that has a price tag of $220 on it. Jeff wonders how much money the coupon will take off of the $220 original price.
Show Solution

The example video that follows shows how to use the percent equation to find the amount of a discount from the price of a phone.

Watch this video online: https://youtu.be/bC7GUc1K6LU

You can estimate to see if the answer is reasonable. Since 15% is half way between 10% and 20%, find these numbers.

$$10\% \text{ of } 220 = 0.1 \cdot 220 = 22$$
$$20\% \text{ of } 220 = 0.2 \cdot 220 = 44$$

The answer, 33, is between 22 and 44. So $33 seems reasonable.

There are many other situations that involve percents. Below are just a few.

Example

Evelyn bought some books at the local bookstore. Her total bill was $31.50, which included 5% tax. How much did the books cost before tax?
Show Solution

In the following video example, you will be shown how to find a price before tax using the percent equation.

Watch this video online: https://youtu.be/qgafD4z8OUE

Example

Susana worked 20 hours at her job last week. This week, she worked 35 hours. In terms of a percent, how much more did she work this week than last week?
Show Solution

The video that follows explains how to use the percent equation to determine the percent increase of a given amount.

Watch this video online: https://youtu.be/6YYpqISiF74

Interest

When a person takes out a loan, most lenders charge interest on the loan. **Interest** is a fee or change for borrowing money, typically a percent rate charged per year. We can compute simple interest by finding the interest rate percentage of the amount borrowed, then multiply by the number of years interest is earned.

Simple Interest Equation

$I = p \cdot r \cdot t$
Where:
I is the **interest** paid
p is the **principal**—the original amount of money borrowed
r is the **interest rate**, a per-year rate, written as a decimal
t is the **time** of the loan, expressed in years or portions of a year

Example

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, with a maturity in 2 years. How much interest will you earn?
Show Solution

In the following video, you are shown how to find how much interest is earned on a specified investment amount.

Watch this video online: https://youtu.be/iVmetUIbheY

Example

A friend asks to borrow $240, offering to repay you $250 in 1 month. What annual interest rate is this equivalent to?
Show Solution

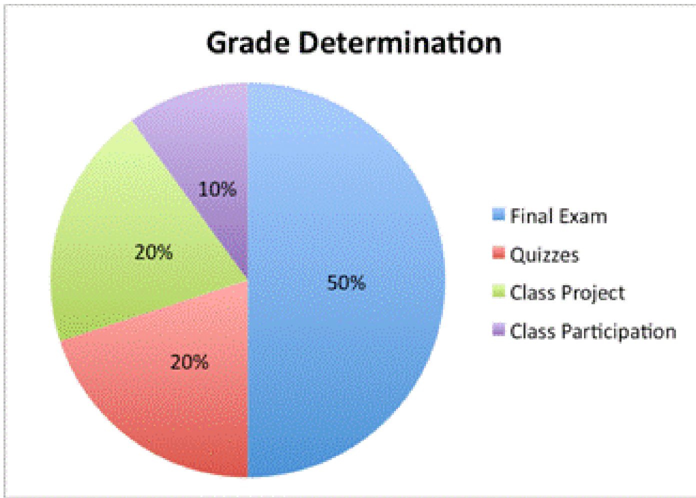
The example video that follows shows how to determine the annual simple interest rate.

Watch this video online: https://youtu.be/SgnE7BJQG10

Pie Charts

Circle graphs, or pie charts, represent data as sections of the circle (or "pieces of the pie"), corresponding to their percentage of the whole. Circle graphs are often used to show how a whole set of data is broken down into individual components.

Here's an example. At the beginning of a semester, a teacher talks about how she will determine student grades. She says, "Half your grade will be based on the final exam and 20% will be determined by quizzes. A class project will also be worth 20% and class participation will count for 10%." In addition to telling the class this information, she could also create a circle graph.



This graph is useful because it relates each part—the final exam, the quizzes, the class project, and the class participation—to the whole.

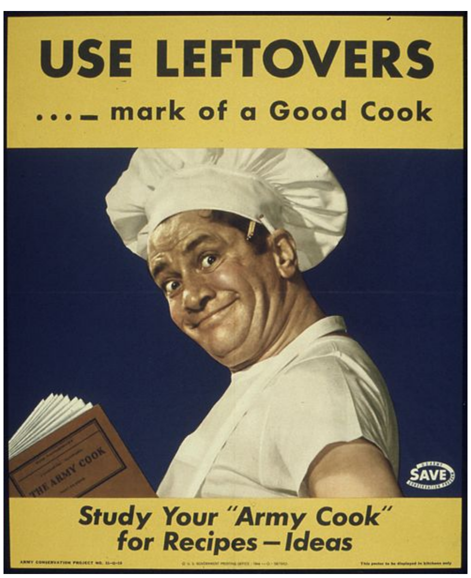
Example
If the total number of points possible in the class is 500, how many points is the final exam worth? Show Solution

In the following video, an example of using a pie chart to determine a percent of a whole is shown.

Watch this video online: https://youtu.be/TAUDMvl8Vg8

Summary

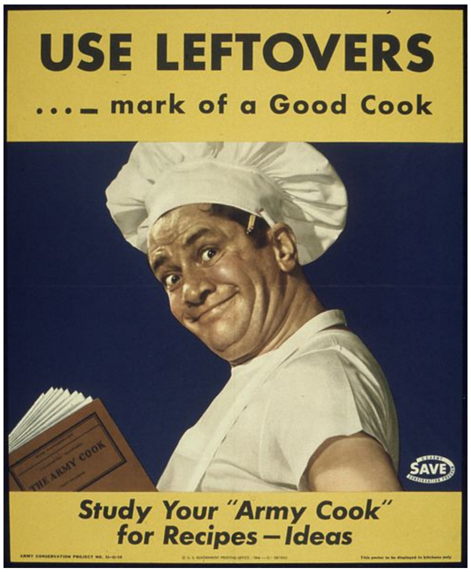
When solving application problems with percents, it is important to be extremely careful in identifying the percent, whole, and amount in the problem. Once those are identified, use the percent equation to solve the problem. Write your final answer back in terms of the original scenario.
Put it Together



While the forgotten food in your fridge may not be what you spend your work day dreaming about devouring when you get home, those jumbled bits and pieces can be used to create something tasty enough to satisfy your hunger. Hopefully this section offered you enough tidbits to keep you full long enough to get to the next section, where you will learn how to put these forgotten skills to work as you continue to grow your mathematical prowess. Remember, you can come back to this review any time throughout the course for practice dividing fractions, simplifying with square roots, or whatever you might need to help you learn the concepts in this course more easily.

Oh, and now you know that $3+5\times 2$ is 13!

CONCLUSION



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MODULE 1: SOLVING EQUATIONS AND INEQUALITIES

INTRODUCTION

Why solve linear equations and inequalities?

You might be surprised to learn that applications of linear equations turn up in many places besides math classrooms. Knowing how to solve them is a basic math skill used in nearly every academic discipline and in many jobs. One of the fundamental principles of solving linear equations is that of reversing or undoing mathematical operations such as addition and subtraction. To see a linear equation in action, let's consider one that's used by forensic scientists to calculate blood alcohol content.

Not surprisingly, blood alcohol content (BAC) is a measurement of how much alcohol is in someone's blood. It's usually measured in grams and as a percentage. For example, a BAC of 0.30% is three-tenths of 1%, and it indicates that there are 3 grams of alcohol for every 1,000 grams of blood—which is actually a lot. A BAC of 0.05% impairs reasoning and the ability to concentrate. A BAC of 0.30% can lead to a blackout, shortness of breath, and loss of bladder control. In most states, the legal limit for driving is a BAC of 0.08%.

BAC is usually determined by the results of a breathalyzer, urinalysis, or blood test. Swedish physician E. M. P. Widmark developed an equation that works well for estimating BAC without using one of those tests. Widmark's formula is widely used by forensic scientists:

$$\text{B} = -0.015t + \left(\frac{2.84N}{Wg}\right)$$

where

- B = percentage of BAC
- t = number of hours since the first drink
- N = number of "standard drinks" (a standard drink is one 12-ounce beer, one 5-ounce glass of wine, or one 1.5-ounce shot of liquor). N should be at least 1.
- W = weight in pounds
- g = gender constant: 0.68 for men and 0.55 for women

In the following table, the progressive effects of alcohol are defined for ranges of blood alcohol content.

Progressive effects of alcohol		
BAC (% by vol.)	**Behavior**	**Impairment**

0.001–0.029	• Average individual appears normal	• Subtle effects that can be detected with special tests
0.030–0.059	• Mild euphoria • Relaxation • Joyousness • Talkativeness • Decreased inhibition	• Concentration
0.060–0.099	• Blunted feelings • Reduced sensitivity to pain • Euphoria • Disinhibition • Extroversion	• Reasoning • Depth perception • Peripheral vision • Glare recovery
0.100–0.199	• Overexpression • Boisterousness • Possibility of nausea and vomiting	• Reflexes • Reaction time • Gross motor control • Staggering • Slurred speech • Temporary erectile dysfunction
0.200–0.299	• Nausea • Vomiting • Emotional swings • Anger or sadness • Partial loss of understanding • Impaired sensations • Decreased libido • Possibility of stupor	• Severe motor impairment • Loss of consciousness • Memory blackout
0.300–0.399	• Stupor • Central nervous system depression • Loss of understanding • Lapses in and out of consciousness • Low possibility of death	• Bladder function • Breathing • Dysequilibrium • Heart rate
0.400–0.500	• Severe central nervous system depression • Coma • Possibility of death	• Breathing • Heart rate • Positional Alcohol Nystagmus
>0.50	• High risk of poisoning • High possibility of death	• Life

Joan likes to party and believes she is “just fine” when it comes to driving. At a party, though, she downs three standard drinks, one after the other, and then decides to leave. If Joan weighs 135 pounds, where would she be on the table of the progressive effects of alcohol after 1.5 hours? Would she be within the legal limit to drive home after this amount of time? Given any amount that she drinks, can you figure out how long she must wait before she can drive safely and legally?

As you’ll discover, these are all questions that can be answered by solving linear equations and inequalities. Read on to learn more. At the end of the module we’ll revisit Joan and see how she fared.

Learning Outcomes

Real Numbers

- Classify a real number
- Define properties of real numbers and use them to evaluate algebraic expressions

Multi-Step Equations

- Use properties of real numbers to solve multi-step equations
- Define and use the distributive property to solve linear equations
- Classify solutions to linear equations

Problem-Solving

- Set up a linear equation from a written description of a problem and solve it
- Use a formula to solve an application problem

SOLVING LINEAR EQUATIONS

Learning Objectives

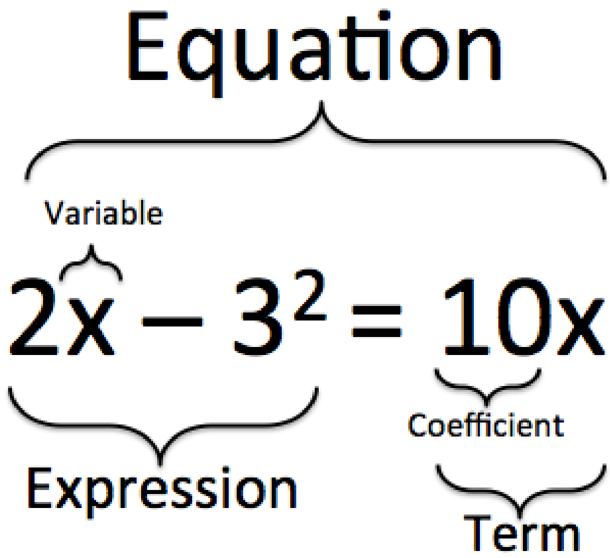
- Use the addition property of equality
  - Solve algebraic equations using the addition property of equality
  - Solve one-step equations containing absolute values with addition
- Use the multiplication property of equality
  - Solve algebraic equations using the multiplication property of equality
  - Solve one-step equations containing absolute values with multiplication

Solve an algebraic equation using the addition property of equality

First, let's define some important terminology:

- **variables:** variables are symbols that stand for an unknown quantity, they are often represented with letters, like *x*, *y*, or *z*.
- **coefficient:** Sometimes a variable is multiplied by a number. This number is called the coefficient of the variable. For example, the coefficient of 3*x* is 3.
- **term:** a single number, or variables and numbers connected by multiplication. -4, 6x and x^2 are all terms
- **expression:** groups of terms connected by addition and subtraction. $2x^2-5$ is an expression
- **equation:** an equation is a mathematical statement that two expressions are equal. An equation will always contain an equal sign with an expression on each side. Think of an equal sign as meaning "the same as." Some examples of equations are $y=mx+b$, $\frac{3}{4}r=v^3-r$, and $2(6-d)+f(3+k)=\frac{1}{4}d$

The following figure shows how coefficients, variables, terms, and expressions all come together to make equations. In the equation $2x-3^2=10x$, the variable is x, a coefficient is 10, a term is $10x$, an expression is $2x-3^2$.



Equation made of coefficients, variables, terms and expressions.

Using the Addition Property of Equality

An important property of equations is one that states that you can add the same quantity to both sides of an equation and still maintain an equivalent equation. Sometimes people refer to this as keeping the equation “balanced.” If you think of an equation as being like a balance scale, the quantities on each side of the equation are equal, or balanced.

Let’s look at a simple numeric equation, $3+7=10$, to explore the idea of an equation as being balanced.

The expressions on each side of the equal sign are equal, so you can add the same value to each side and maintain the equality. Let’s see what happens when 5 is added to each side.

$3+7+5=10+5$

Since each expression is equal to 15, you can see that adding 5 to each side of the original equation resulted in a true equation. The equation is still “balanced.”

On the other hand, let’s look at what would happen if you added 5 to only one side of the equation.

$$\begin{aligned} 3+7&=10\\ 3+7+5&=10\\ 15&\neq 10 \end{aligned}$$

Adding 5 to only one side of the equation resulted in an equation that is false. The equation is no longer "balanced," and it is no longer a true equation!

Addition Property of Equality

For all real numbers *a*, *b*, and *c*: If $a = b$, then $a + c = b + c$.
If two expressions are equal to each other, and you add the same value to both sides of the equation, the equation will remain equal.

Solve algebraic equations using the addition property of equality

When you solve an equation, you find the value of the variable that makes the equation true. In order to solve the equation, you **isolate the variable**. Isolating the variable means rewriting an equivalent equation in which the variable is on one side of the equation and everything else is on the other side of the equation.

When the equation involves addition or subtraction, use the inverse operation to "undo" the operation in order to isolate the variable. For addition and subtraction, your goal is to change any value being added or subtracted to 0, the additive identity.

In the following simulation, you can adjust the quantity being added or subtracted to each side of an equation to see how important it is to perform the same operation on both sides of an equation when you are solving.

Examples

Solve $x - 6 = 8$.
Show Solution

Solve $x + 5 = 27$.
Show Solution

In the following video two examples of using the addition property of equality are shown.

Visit this page in your course online to check your understanding.

Since subtraction can be written as addition (adding the opposite), the **addition property of equality** can be used for subtraction as well. So just as you can add the same value to each side of an equation without changing the meaning of the equation, you can subtract the same value from each side of an equation.

Examples

Solve $x + 10 = -65$. Check your solution.
Show Solution

Solve $x - 4 = -32$. Check your solution.
Show Solution

It is always a good idea to check your answer whether you are requested to or not.

The following video presents two examples of using the addition property of equality when there are negative integers in the equation.

Watch this video online: https://youtu.be/D3T8eCT5U_w

Think About It

Can you determine what you would do differently if you were asked to solve equations like these?

a) Solve $12.5 + t = -7.5$.

What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would solve this equation with decimals.

Show Solution

b) Solve $\frac{1}{4}+y=3$. What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would solve this equation with a fraction.

Show Solution

In the following video, two examples of using the addition property of equality with decimal numbers are shown.

Watch this video online: https://youtu.be/D8wKGlxf6bM

The next video shows how to use the addition property of equality to solve equations with fractions.

Watch this video online: https://youtu.be/O7SPM7Cs8Ds

The examples above are sometimes called **one-step equations** because they require only one step to solve. In these examples, you either added or subtracted a **constant** from both sides of the equation to isolate the variable and solve the equation.

With any equation, you can check your solution by substituting the value for the variable in the original equation. In other words, you evaluate the original equation using your solution. If you get a true statement, then your solution is correct.

Writing and solving algebraic equations is an important part of mathematics. Equations can be used to describe economic, cultural, physical, and biological processes. They help business people make decisions, and help doctors and scientists find ways to heal and help people. Without mathematical equations we would not have the physical infrastructure that rely on every day for transportation and clean water.

Equations can help you model situations and solve problems in which quantities are unknown (like how long Joan should wait before she drives home). The simplest type of algebraic equation is a linear equation that has just one variable.

When you follow the steps to solve an equation, you try to isolate the variable. The variable is a quantity we don't know yet. You have a solution when you get the equation *x* = some value.

Solving One-Step Equations Containing Absolute Values with Addition

The **absolute value** of a number or expression describes its distance from 0 on a number line. Since the absolute value expresses only the distance, not the direction of the number on a number line, it is always expressed as a positive number or 0.

For example, -4 and 4 both have an absolute value of 4 because they are each 4 units from 0 on a number line—though they are located in opposite directions from 0 on the number line.

When solving absolute value **equations** and **inequalities**, you have to consider both the behavior of absolute value and the properties of equality and inequality.

Because both positive and negative values have a positive absolute value, solving absolute value equations means finding the solution for both the positive and the negative values.

Let's first look at a very basic example.

$|x|=5$

This equation is read "the absolute value of *x* is equal to five." The solution is the value(s) that are five units away from 0 on a number line.

You might think of 5 right away; that is one solution to the equation. Notice that -5 is also a solution because -5 is 5 units away from 0 in the opposite direction. So, the solution to this equation $|x|=5$ is $x=-5$ or $x=5$.

Solving Equations of the Form $|x|=a$

For any positive number *a*, the solution of $|x|=a$ is
$x=a$ or $x=-a$
x can be a single variable or any algebraic expression.

You can solve a more complex absolute value problem in a similar fashion.

Example

Solve $|x+5|=15$.
Show Solution

The following video provides worked examples of solving linear equations with absolute value terms.

Visit this page in your course online to check your understanding.

Solve algebraic equations using the multiplication property of equality

Just as you can add or subtract the same exact quantity on both sides of an equation, you can also multiply both sides of an equation by the same quantity to write an equivalent equation. Let's look at a numeric equation, $5\cdot 3=15$, to start. If you multiply both sides of this equation by 2, you will still have a true equation.

$$\begin{aligned} 5\cdot 3&=15\\ 5\cdot 3\cdot 2&=15\cdot 2\\ 30&=30 \end{aligned}$$

This characteristic of equations is generalized in the **multiplication property of equality**.

Multiplication Property of Equality

For all real numbers *a*, *b*, and *c*: If *a* = *b*, then $a\cdot c=b\cdot c$ (or *ab* = *ac*).
If two expressions are equal to each other and you multiply both sides by the same number, the resulting expressions will also be equivalent.

When the equation involves multiplication or division, you can "undo" these operations by using the inverse operation to isolate the variable. When the operation is multiplication or division, your goal is to change the coefficient to 1, the multiplicative identity.

Example

Solve $3x=24$. When you are done, check your solution.
Show Solution

You can also multiply the coefficient by the multiplicative inverse (reciprocal) in order to change the coefficient to 1.

Example

Solve $\frac{1}{2}x=8$ for x.
Show Solution

In the video below you will see examples of how to use the multiplication property of equality to solve one-step equations with integers and fractions.

Visit this page in your course online to check your understanding.

In the next example, we will solve a one-step equation using the multiplication property of equality. You will see that the variable is part of a fraction in the given equation, and using the multiplication property of equality allows us to remove the variable from the fraction. Remember that fractions imply division, so you can think of this as the variable *k* is being divided by 10. To “undo” the division, you can use multiplication to isolate *k*. Lastly, note that there is a negative term in the equation, so it will be important to think about the sign of each term as you work through the problem. Stop after each step you take to make sure all the terms have the correct sign.

Example

Solve $-\frac{7}{2}=\frac{k}{10}$ for *k*.
Show Solution

In the following video you will see examples of using the multiplication property of equality to solve a one-step equation involving negative fractions.

Visit this page in your course online to check your understanding.

Solving One-Step Equations Containing Absolute Values With Multiplication

In the last section, we saw examples of solving equations with absolute values where the only operation was addition or subtraction. Now we will see how to solve equations with absolute value that include multiplication.

Remember that absolute value refers to the distance from zero. You can use the same technique of first isolating the absolute value, then setting up and solving two equations to solve an absolute value equation involving multiplication.

Example

Solve $|2x|=6$.
Show Solution

Example

Solve $\frac{1}{3}|k|=12$.
Show Solution

In the following video you will see two examples of how to solve an absolute value equation, one with integers and one with fractions.

Visit this page in your course online to check your understanding.

MULTI-STEP LINEAR EQUATIONS

Learning Objectives

- Solve multi-step equations
  - Use properties of equality to isolate variables and solve algebraic equations
  - Solve equations containing absolute values
- Use the distributive property
  - Use the properties of equality and the distributive property to solve equations containing parentheses
  - Clear fractions and decimals from equations to make them easier to solve
- Classify solutions to linear equations
  - Solve equations that have one solution, no solution, or an infinite number of solutions
  - Recognize when a linear equation that contains absolute value does not have a solution

Use properties of equality to isolate variables and solve algebraic equations

Steps With an End In Sight

There are some **equations** that you can solve in your head quickly. For example—what is the value of *y* in the equation $2y = 6$? Chances are you didn't need to get out a pencil and paper to calculate that $y = 3$. You only needed to do one thing to get the answer: divide 6 by 2.

Other equations are more complicated. Solving $4\left(\frac{1}{3}t + \frac{1}{2}\right) = 6$ without writing anything down is difficult! That's because this equation contains not just a **variable** but also fractions and **terms** inside parentheses. This is a **multi-step equation**, one that takes several steps to solve. Although multi-step equations take more time and more operations, they can still be simplified and solved by applying basic algebraic rules.

Remember that you can think of an equation as a balance scale, with the goal being to rewrite the equation so that it is easier to solve but still balanced. The **addition property of equality** and the **multiplication property of equality** explain how you can keep the scale, or the equation, balanced. Whenever you perform an operation to one side of the equation, if you perform the same exact operation to the other side, you'll keep both sides of the equation equal.

If the equation is in the form $ax+b=c$, where *x* is the variable, you can solve the equation as before. First "undo" the addition and subtraction, and then "undo" the multiplication and division.

> Example
>
> Solve $3y+2=11$.
> Show Solution

In the following video we show examples of solving two step linear equations.

Visit this page in your course online to check your understanding.

> Example
>
> Solve $3x+5x+4-x+7=88$.
> Show Solution

In the following video, we show an example of solving a linear equation that requires combining like terms.

Watch this video online: https://youtu.be/ez_sP2OTGjU

Some equations may have the variable on both sides of the equal sign, as in this equation: $4x-6=2x+10$.

To solve this equation, we need to "move" one of the variable terms. This can make it difficult to decide which side to work with. It doesn't matter which term gets moved, $4x$ or $2x$, however, to avoid negative coefficients, you can move the smaller term.

> Examples
>
> Solve: $4x-6=2x+10$
> Show Solution

In this video, we show an example of solving equations that have variables on both sides of hte equal sign.

Visit this page in your course online to check your understanding.

Solving Multi-Step Equations With Absolute Value

We can apply the same techniques we used for solving a one-step equation which contains absolute value to an equation that will take more than one step to solve. Let's start with an example where the first step is to write two equations, one equal to positive 26 and one equal to negative 26.

> Example
>
> Solve for *p*. $|2p–4|=26$
> Show Solution

In the next video, we show more examples of solving a simple absolute value equation.

Watch this video online: https://youtu.be/4g-o_-mAFpc

Now let's look at an example where you need to do an algebraic step or two before you can write your two equations. The goal here is to get the absolute value on one side of the equation by itself. Then we can proceed as we did in the previous example.

> Example

Solve for *w*. $3\left|4w{-}1\right|{-}5=10$
Show Solution

In the two videos that follow, we show examples of how to solve an absolute value equation that requires you to isolate the absolute value first using mathematical operations.

Watch this video online: https://youtu.be/-HrOMkIiSfU

Watch this video online: https://youtu.be/2bEA7HoDfpk

The Distributive Property

As we solve linear equations, we often need to do some work to write the linear equations in a form we are familiar with solving. This section will focus on manipulating an equation we are asked to solve in such a way that we can use the skills we learned for solving multi-step equations to ultimately arrive at the solution.

Parentheses can make solving a problem difficult, if not impossible. To get rid of these unwanted parentheses we have the distributive property. Using this property we multiply the number in front of the parentheses by each term inside of the parentheses.

The Distributive Property of Multiplication

For all real numbers *a, b,* and *c*, $a(b+c)=ab+ac$.
What this means is that when a number multiplies an expression inside parentheses, you can distribute the multiplication to each term of the expression individually. Then, you can follow the steps we have already practiced to **isolate the variable** and solve the equation.

Example

Solve for a. $4\left(2a+3\right)=28$
Show Solution

In the video that follows, we show another example of how to use the distributive property to solve a multi-step linear equation.

Watch this video online: https://youtu.be/aQOkD8L57V0

In the next example, you will see that there are parentheses on both sides of the equal sign, so you will need to use the distributive property twice. Notice that you are going to need to distribute a negative number, so be careful with negative signs!

Example

Solve for t. $2\left(4t-5\right)=-3\left(2t+1\right)$
Show Solution

In the following video, we solve another multi-step equation with two sets of parentheses.

Watch this video online: https://youtu.be/StomYTb7Xb8

Sometimes, you will encounter a multi-step equation with fractions. If you prefer not working with fractions, you can use the multiplication property of equality to multiply both sides of the equation by a common denominator of all of the fractions in the equation. This will clear all the fractions out of the equation. See the example below.

Example

Solve $\frac{1}{2}x-3=2-\frac{3}{4}x$ by clearing the fractions in the equation first.
Show Solution

Of course, if you like to work with fractions, you can just apply your knowledge of operations with fractions and solve.

In the following video, we show how to solve a multi-step equation with fractions.

Watch this video online: https://youtu.be/AvJTPeACTY0

Regardless of which method you use to solve equations containing variables, you will get the same answer. You can choose the method you find the easiest! Remember to check your answer by substituting your solution into the original equation.

Sometimes, you will encounter a multi-step equation with decimals. If you prefer not working with decimals, you can use the multiplication property of equality to multiply both sides of the equation by a a factor of 10 that will help clear the decimals. See the example below.

Example

Solve $3y+10.5=6.5+2.5y$ by clearing the decimals in the equation first.
Show Solution

In the following video, we show another example of clearing decimals first to solve a multi-step linear equation.

Watch this video online: https://youtu.be/wtwepTZZnlY

Here are some steps to follow when you solve multi-step equations.

Solving Multi-Step Equations

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term—you may have to move a term with the variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

Classify Solutions to Linear Equations

There are three cases that can come up as we are solving linear equations. We have already seen one, where an equation has one solution. Sometimes we come across equations that don't have any solutions, and even some that have an infinite number of solutions. The case where an equation has no solution is illustrated in the next examples.

Equations with no solutions

Example

Solve for *x*. $12+2x–8=7x+5–5x$
Show Solution

This is *not* a solution! You did *not* find a value for *x*. Solving for *x* the way you know how, you arrive at the false statement $4=5$. Surely 4 cannot be equal to 5!

This may make sense when you consider the second line in the solution where like terms were combined. If you multiply a number by 2 and add 4 you would never get the same answer as when you multiply that same number by 2 and add 5. Since there is no value of *x* that will ever make this a true statement, the solution to the equation above is *"no solution."*

Be careful that you do not confuse the solution $x=0$ with "no solution." The solution $x=0$ means that the value 0 satisfies the equation, so there *is* a solution. "No solution" means that there is no value, not even 0, which would satisfy the equation.

Also, be careful not to make the mistake of thinking that the equation $4 = 5$ means that 4 and 5 are values for *x* that are solutions. If you substitute these values into the original equation, you'll see that they do not satisfy the equation. This is because there is truly *no solution*—there are no values for *x* that will make the equation $12 + 2x – 8 = 7x + 5 – 5x$ true.

Think About It

Try solving these equations. How many steps do you need to take before you can tell whether the equation has no solution or one solution?

a) Solve $8y = 3(y + 4) + y$

Use the textbox below to record how many steps you think it will take before you can tell whether there is no solution or one solution.

Show Solution

b) Solve $2\left(3x - 5\right) - 4x = 2x + 7$

Use the textbox below to record how many steps you think it will take before you can tell whether there is no solution or one solution.

Show Solution

Algebraic Equations with an Infinite Number of Solutions

You have seen that if an equation has no solution, you end up with a false statement instead of a value for *x*. It is possible to have an equation where any value for *x* will provide a solution to the equation. In the example below, notice how combining the terms $5x$ and $-4x$ on the left leaves us with an equation with exactly the same terms on both sides of the equal sign.

Example

Solve for *x*. $5x + 3 – 4x = 3 + x$
Show Solution

You arrive at the true statement "$3 = 3$." When you end up with a true statement like this, it means that the solution to the equation is "all real numbers." Try substituting $x = 0$ into the original equation—you will get a true statement! Try $x = -\frac{3}{4}$, and it also will check!

This equation happens to have an infinite number of solutions. Any value for *x* that you can think of will make this equation true. When you think about the context of the problem, this makes sense—the equation $x + 3 = 3 + x$ means "some number plus 3 is equal to 3 plus that same number." We know that this is always true—it's the commutative property of addition!

In the following video, we show more examples of attempting to solve a linear equation with either no solution or many solutions.

Watch this video online: https://youtu.be/iLkZ3o4wVxU

Example

Solve for *x*. $3\left(2x - 5\right) = 6x - 15$
Show Solution

In this video, we show more examples of solving linear equations with either no solutions or many solutions.

Watch this video online: https://youtu.be/iLkZ3o4wVxU

In the following video, we show more examples of solving linear equations with parentheses that have either no solution or many solutions.

Watch this video online: https://youtu.be/EU_NEo1QBJ0

Absolute value equations with no solutions

As we are solving absolute value equations it is important to be aware of special cases. An absolute value is defined as the distance from 0 on a number line, so it must be a positive number. When an absolute value expression is equal to a negative number, we say the equation has no solution, or DNE. Notice how this happens in the next two examples.

Example

Solve for *x*. $7+|2x-5|=4$
Show Solution

Example

Solve for *x*. $-\frac{1}{2}|x+3|=6$
Show Solution

In this last video, we show more examples of absolute value equations that have no solutions.

Watch this video online: https://youtu.be/T-z5cQ58I_g

Summary

Equations are mathematical statements that combine two expressions of equal value. An algebraic equation can be solved by isolating the variable on one side of the equation using the properties of equality. To check the solution of an algebraic equation, substitute the value of the variable into the original equation.

Complex, multi-step equations often require multi-step solutions. Before you can begin to isolate a variable, you may need to simplify the equation first. This may mean using the distributive property to remove parentheses or multiplying both sides of an equation by a common denominator to get rid of fractions. Sometimes it requires both techniques. If your multi-step equation has an absolute value, you will need to solve two equations, sometimes isolating the absolute value expression first.

We have also seen that solutions to equations can fall into three categories:

- One solution
- No solution, DNE (does not exist)
- Many solutions, also called infinitely many solutions or All Real Numbers

And sometimes, we don’t need to do much algebra to see what the outcome will be.

PROBLEM SOLVING

Learning Objectives

- Define a process for problem solving
  - Translate words into algebraic expressions and equations
  - Define a process for solving word problems
- Solve problems containing rates
  - Apply the steps for solving word problems to distance, rate, and time problems
  - Apply the steps for solving word problems to interest rate problems
  - Evaluate a formula using substitution
  - Rearrange formulas to isolate specific variables
  - Identify an unknown given a formula
- Solve additional applications of linear equations
  - Apply the steps for solving word problems to geometry problems
  - Use the formula for converting between Fahrenheit and Celsius
  - Evaluate a formula using substitution
  - Rearrange formulas to isolate specific variables
  - Identify an unknown given a formula

Define a Process for Problem Solving

Word problems can be tricky. Often it takes a bit of practice to convert an English sentence into a mathematical sentence, which is one of the first steps to solving word problems. In the table below, words or phrases commonly associated with mathematical operators are categorized. Word problems often contain these or similar words, so it’s good to see what mathematical operators are associated with them.

Addition $+$	**Subtraction** $-$	**Multiplication** $\times$	**Variable ?**	**Equals** $=$
More than	Less than	Double	A number	Is
Together	In the past	Product	Often, a value for which no information is given.	The same as
Sum	slower than	times	After how many hours?	
Total	the remainder of		How much will it cost?	

Addition	Subtraction	Multiplication	Variable ?	Equals
In the future	difference			
faster than				

Some examples follow:

- $x \text{ is } 5$ becomes $x = 5$
- Three more than a number becomes $x + 3$
- Four less than a number becomes $x - 4$
- Double the cost becomes $2 \cdot \text{ cost}$
- Groceries and gas together for the week cost $250 means $\text{groceries } + \text{ gas } = 250$
- The difference of 9 and a number becomes $9 - x$. Notice how 9 is first in the sentence and the expression

Let's practice translating a few more English phrases into algebraic expressions.

Example

Translate the table into algebraic expressions:

some number	the sum of the number and 3	twice the sum of the number and 3
a length	double the length	double the length, decreased by 6
a cost	the difference of the cost and 20	2 times the difference of the cost and 20
some quantity	the difference of 5 and the quantity	the difference of 5 and the quantity, divided by 2
an amount of time	triple the amount of time	triple the amount of time, increased by 5
a distance	the sum of -4 and the distance	the sum of -4 and the twice the distance

Show Solution

In this example video, we show how to translate more words into mathematical expressions.

Watch this video online: https://youtu.be/uD_V5t-6Kzs

The power of algebra is how it can help you model real situations in order to answer questions about them.

Here are some steps to translate problem situations into algebraic equations you can solve. Not *every* word problem fits perfectly into these steps, but they will help you get started.

1. Read and understand the problem.
2. Determine the constants and variables in the problem.
3. Translate words into algebraic expressions and equations.
4. Write an equation to represent the problem.
5. Solve the equation.
6. Check and interpret your answer. Sometimes writing a sentence helps.

Example

Twenty-eight less than five times a certain number is 232. What is the number?
Show Solution

In the video that follows, we show another example of how to translate a sentence into a mathematical expression using a problem solving method.

Watch this video online: https://youtu.be/izIIqOztUyI

Another type of number problem involves consecutive numbers. Consecutive numbers are numbers that come one after the other, such as 3, 4, 5. If we are looking for several consecutive numbers it is important to first identify what they look like with variables before we set up the equation.

For example, let's say I want to know the next consecutive integer after 4. In mathematical terms, we would add 1 to 4 to get 5. We can generalize this idea as follows: the consecutive integer of any number, *x*, is $x+1$. If we continue this pattern we can define any number of consecutive integers from any starting point. The following table shows how to describe four consecutive integers using algebraic notation.

First	x
Second	$x+1$
Third	$x+2$
Fourth	$x+3$

We apply the idea of consecutive integers to solving a word problem in the following example.

Example

The sum of three consecutive integers is 93. What are the integers?
Show Solution

In the following video we show another example of a consecutive integer problem.

Visit this page in your course online to check your understanding.

Rates

There is often a well-known formula or relationship that applies to a word problem. For example, if you were to plan a road trip, you would want to know how long it would take you to reach your destination. $d=rt$ is a well-known relationship that associates distance traveled, the rate at which you travel, and how long the travel takes.

Distance, Rate, and Time

If you know two of the quantities in the relationship $d=rt$, you can easily find the third using methods for solving linear equations. For example, if you know that you will be traveling on a road with a speed limit of $30\frac{\text{miles}}{\text{hour}}$ for 2 hours, you can find the distance you would travel by multiplying rate times time or $\left(30\frac{\text{miles}}{\text{hour}}\right)(2\text{ hours })=60\text{ miles }$.

We can generalize this idea depending on what information we are given and what we are looking for. For example, if we need to find time, we could solve the $d=rt$ equation for *t* using division:

$d=rt$
$\frac{d}{r}=t$

Likewise, if we want to find rate, we can isolate *r* using division:

$d=rt$
$\frac{d}{t}=r$

In the following examples you will see how this formula is applied to answer questions about ultra marathon running.

Ultra marathon running (defined as anything longer than 26.2 miles) is becoming very popular among women even though it remains a male-dominated niche sport. Ann Trason has broken twenty world records in her career. One such record was the American River 50-mile Endurance Run which begins in Sacramento, California, and ends in Auburn, California. ((Note: "Ann Trason." Wikipedia. Accessed May 05, 2016. https://en.wikipedia.org/wiki/Ann_Trason.)) In 1993 Trason finished the run with a time of 6:09:08. The men's record for the same course was set in 1994 by Tom Johnson who finished the course with a time of 5:33:21. ((Note: "American River 50 Mile Endurance Run." Wikipedia. Accessed May 05, 2016.

Ann Trason

https://en.wikipedia.org/wiki/American_River_50_Mile_Endurance_Run.))

In the next examples we will use the $d = rt$ formula to answer the following questions about the two runners.

1. What was each runner's rate for their record-setting runs?
2. By the time Johnson had finished, how many more miles did Trason have to run?
3. How much further could Johnson have run if he had run as long as Trason?
4. What was each runner's time for running one mile?

To make answering the questions easier, we will round the two runners' times to 6 hours and 5.5 hours.

Example

What was each runner's rate for their record-setting runs?
Show Solution

Now that we know each runner's rate we can answer the second question.

Example

By the time Johnson had finished, how many more miles did Trason have to run?
Show Solution

The third question is similar to the second. Now that we know each runner's rate, we can answer questions about individual distances or times.

Examples

How much further could Johnson have run if he had run as long as Trason?
Show Solution

Now we will tackle the last question where we are asked to find a time for each runner.

Example

What was each runner's time for running one mile?
Show Solution

In the following video, we show another example of answering many rate questions given distance and time.

Watch this video online: https://youtu.be/3WLp5mY1FhU

Simple Interest

In order to entice customers to invest their money, many banks will offer interest-bearing accounts. The accounts work like this: a customer deposits a certain amount of money (called the Principal, or *P*), which then grows slowly according to the interest rate (*R*, measured in percent) and the length of time (*T*, usually measured in months) that the money stays in the account. The amount earned over time is called the interest (*I*), which is then given to the customer.

Beware! Interest rates are commonly given as yearly rates, but can also be monthly, quarterly, bimonthly, or even some custom amount of time. It is important that the units of time and the units of the interest rate match. You will see why this matters in a later example.

The simplest way to calculate interest earned on an account is through the formula $I = P \cdot R \cdot T$.

If we know any of the three amounts related to this equation, we can find the fourth. For example, if we want to find the time it will take to accrue a specific amount of interest, we can solve for T using division:

$$I = P \cdot R \cdot T$$

$$\frac{I}{P \cdot R} = \frac{P \cdot R \cdot T}{P \cdot R}$$

$$T = \frac{I}{R \cdot T}$$

Below is a table showing the result of solving for each individual variable in the formula.

Solve For	**Result**
I	$I = P \cdot R \cdot T$
P	$P = \frac{I}{R \cdot T}$
R	$R = \frac{I}{P \cdot T}$
T	$T = \frac{I}{P \cdot R}$

In the next examples, we will show how to substitute given values into the simple interest formula, and decipher which variable to solve for.

Example

If a customer deposits a principal of $2000 at a monthly rate of 0.7%, what is the total amount that she has after 24 months?
Show Solution

The following video shows another example of finding an account balance after a given amount of time, principal invested, and a rate.

Watch this video online: https://youtu.be/XkGgEEMR_00

In the following example you will see why it is important to make sure the units of the interest rate match the units of time when using the simple interest formula.

Example

Alex invests $600 at 3.25% monthly interest for 3 years. What amount of interest has Alex earned?
Show Solution

In the following video we show another example of how to find the amount of interest earned after an investment has been sitting for a given monthly interest.

Visit this page in your course online to check your understanding.

Example

After 10 years, Jodi's account balance has earned $1080 in interest. The rate on the account is 0.09% monthly. What was the original amount she invested in the account?
Show Solution

The last video shows another example of finding the principle amount invested based on simple interest.

Visit this page in your course online to check your understanding.

In the next section we will apply our problem-solving method to problems involving dimensions of geometric shapes.

Further Applications of Linear Equations

Formulas come up in many different areas of life. We have seen the formula that relates distance, rate, and time and the formula for simple interest on an investment. In this section we will look further at formulas and see examples of formulas for dimensions of geometric shapes as well as the formula for converting temperature between Fahrenheit and Celsius.

Geometry

There are many geometric shapes that have been well studied over the years. We know quite a bit about circles, rectangles, and triangles. Mathematicians have proven many formulas that describe the dimensions of geometric shapes including area, perimeter, surface area, and volume.

Perimeter

Perimeter is the distance around an object. For example, consider a rectangle with a length of 8 and a width of 3. There are two lengths and two widths in a rectangle (opposite sides), so we add $8+8+3+3=22$. Since there are two lengths and two widths in a rectangle, you may find the perimeter of a rectangle using the formula $P=2\left(L\right)+2\left(W\right)$ where

L = Length

W = Width

In the following example, we will use the problem-solving method we developed to find an unknown width using the formula for the perimeter of a rectangle. By substituting the dimensions we know into the formula, we will be able to isolate the unknown width and find our solution.

Example

You want to make another garden box the same size as the one you already have. You write down the dimensions of the box and go to the lumber store to buy some boards. When you get there you realize you didn't write down the width dimension—only the perimeter and length. You want the exact dimensions so you can have the store cut the lumber for you.

Here is what you have written down:

Perimeter = 16.4 feet
Length = 4.7 feet

Can you find the dimensions you need to have your boards cut at the lumber store? If so, how many boards do you need and what lengths should they be?
Show Solution

This video shows a similar garden box problem.

Watch this video online: https://youtu.be/jlxPgKQfhQs

We could have isolated the *w* in the formula for perimeter before we solved the equation, and if we were going to use the formula many times, it could save a lot of time. The next example shows how to isolate a variable in a formula before substituting known dimensions or values into the formula.

Example

Isolate the term containing the variable, *w,* from the formula for the perimeter of a rectangle*:*
$P = 2\left(L\right) + 2\left(W\right)$.
Show Solution

Area

The area of a triangle is given by $A = \frac{1}{2}bh$ where

A = area
b = the length of the base
h = the height of the triangle

Remember that when two variables or a number and a variable are sitting next to each other without a mathematical operator between them, you can assume they are being multiplied. This can seem frustrating, but you can think of it like mathematical slang. Over the years, people who use math frequently have just made that shortcut enough that it has been adopted as convention.

In the next example we will use the formula for area of a triangle to find a missing dimension, as well as use substitution to solve for the base of a triangle given the area and height.

Example

Find the base (*b)* of a triangle with an area (*A*) of 20 square feet and a height (*h)* of 8 feet.
Show Solution

We can rewrite the formula in terms of *b* or *h* as we did with perimeter previously. This probably seems abstract, but it can help you develop your equation-solving skills, as well as help you get more comfortable with working with all kinds of variables, not just *x*.

Example

Use the multiplication and division properties of equality to isolate the variable *b*.
Show Solution

Use the multiplication and division properties of equality to isolate the variable *h*.
Show Solution

The following video shows another example of finding the base of a triangle given area and height.

Visit this page in your course online to check your understanding.

Temperature

Let's look at another formula that includes parentheses and fractions, the formula for converting from the Fahrenheit temperature scale to the Celsius scale.

$C=(F-32)\cdot\frac{5}{9}$

Example

Given a temperature of $12^{\circ}C$, find the equivalent in $^{\circ}F$.
Show Solution

As with the other formulas we have worked with, we could have isolated the variable F first, then substituted in the given temperature in Celsius.

Example

Solve the formula shown below for converting from the Fahrenheit scale to the Celsius scale for F.
$C=(F-32)\cdot\frac{5}{9}$
Show Solution

Think About It

Express the formula for the surface area of a cylinder, $s=2\pi rh+2\pi r^{2}$, in terms of the height, *h*.
In this example, the variable *h* is buried pretty deeply in the formula for surface area of a cylinder. Using the order of operations, it can be isolated. Before you look at the solution, use the box below to write down what you think is the best first step to take to isolate *h*.

Show Solution

In the last video, we show how to convert from celsius to fahrenheit.

Visit this page in your course online to check your understanding.

SOLVE INEQUALITIES

Learning Objectives

- Describe solutions to inequalities
  - Represent inequalities on a number line
  - Represent inequalities using interval notation
- Solve single-step inequalities
  - Use the addition and multiplication properties to solve algebraic inequalities and express their solutions graphically and with interval notation
  - Solve inequalities that contain absolute value
- Solve multi-step inequalities
  - Combine properties of inequality to isolate variables, solve algebraic inequalities, and express their solutions graphically
  - Simplify and solve algebraic inequalities using the distributive property to clear parentheses and fractions

Represent inequalities on a number line

First, let’s define some important terminology. An inequality is a mathematical statement that compares two expressions using the ideas of greater than or less than. Special symbols are used in these statements. When you read an inequality, read it from left to right—just like reading text on a page. In algebra, inequalities are used to describe large sets of solutions. Sometimes there are an infinite amount of numbers that will satisfy an inequality, so rather than try to list off an infinite amount of numbers, we have developed some ways to describe very large lists in succinct ways.

The first way you are probably familiar with—the basic inequality. For example:

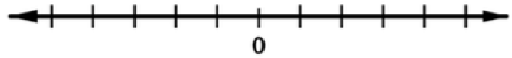
- $x < 9$ indicates the list of numbers that are less than 9. Would you rather write $x < 9$ or try to list all the possible numbers that are less than 9? (hopefully, your answer is no)
- $-5 \leq t$ indicates all the numbers that are greater than or equal to -5.

Note how placing the variable on the left or right of the inequality sign can change whether you are looking for greater than or less than.

For example:

- $x < 5$ means all the real numbers that are less than 5, whereas;
- $5 < x$ means that 5 is less than x, or we could rewrite this with the x on the left: $x > 5$ note how the inequality is still pointing the same direction relative to x. This statement represents all the real numbers that are greater than 5, which is easier to interpret than 5 is less than x.

The second way is with a graph using the number line:



And the third way is with an interval.

We will explore the second and third ways in depth in this section. Again, those three ways to write solutions to inequalities are:

- an inequality
- an interval
- a graph

Inequality Signs

The box below shows the symbol, meaning, and an example for each inequality sign. Sometimes it's easy to get tangled up in inequalities, just remember to read them from left to right.

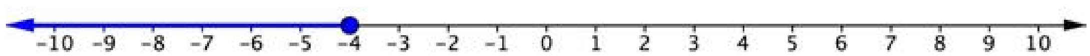
Symbol	**Words**	**Example**
$\neq$	not equal to	$2 \neq 8$, *2* **is not equal** to 8*.*
$>$	greater than	$5 > 1$, *5* **is greater than** *1*
$<$	less than	$2 < 11$, 2 **is less than** *11*
$\geq$	greater than or equal to	$4 \geq 4$, 4 **is greater than or equal to** *4*
$\leq$	less than or equal to	$7 \leq 9$, *7* **is less than or equal to** *9*

The inequality $x > y$ can also be written as $y < x$. The sides of any inequality can be switched as long as the inequality symbol between them is also reversed.

Graphing an Inequality

Inequalities can also be graphed on a number line. Below are three examples of inequalities and their graphs. Graphs are a very helpful way to visualize information – especially when that information represents an infinite list of numbers!

$x \leq -4$. This translates to all the real numbers on a number line that are less than or equal to 4.



$x \geq -3$. This translates to all the real numbers on the number line that are greater than or equal to -3.

Each of these graphs begins with a circle—either an open or closed (shaded) circle. This point is often called the *end point* of the solution. A closed, or shaded, circle is used to represent the inequalities *greater than or equal to* $(\geq)$ or *less than or equal to* $(\leq)$. The point is part of the solution. An open circle is used for *greater than* (>) or *less than* (<). The point is *not* part of the solution.

The graph then extends endlessly in one direction. This is shown by a line with an arrow at the end. For example, notice that for the graph of $x \geq -3$ shown above, the end point is -3, represented with a closed circle since the inequality is *greater than or equal to* -3. The blue line is drawn to the right on the number line because the values in this area are greater than -3. The arrow at the end indicates that the solutions continue infinitely.

Example

Graph the inequality $x \geq 4$
Show Solution

This video shows an example of how to draw the graph of an inequality.

Visit this page in your course online to check your understanding.

Example

Write an inequality describing all the real numbers on the number line that are less than 2, then draw the corresponding graph.
Show Solution

The following video shows how to write an inequality mathematically when it is given in words. We will then graph it.

Visit this page in your course online to check your understanding.

Represent inequalities using interval notation

Another commonly used, and arguably the most concise, method for describing inequalities and solutions to inequalities is called **interval notation.** With this convention, sets are built with parentheses or brackets, each having a distinct meaning. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This method is widely used and will be present in other math courses you may take.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1, 0)$, all real numbers between, but not including -1 and 0; and $(-\infty, 1]$, all real numbers less than and including 1. The table below outlines the possibilities. Remember to read inequalities from left to right, just like text.

The table below describes all the possible inequalities that can occur and how to write them using interval notation, where *a* and *b* are real numbers.

Inequality	Words	Interval Notation
$a < x < b$	all real numbers between a and b, not including a and b	(a, b)
$x > a$	All real numbers greater than a, but not including a	(a, ∞)
$x < b$	All real numbers less than b, but not including b	$(-\infty, b)$
$x \geq a$	All real numbers greater than a, including a	$[a, \infty)$
$x \leq b$	All real numbers less than b, including b	$(-\infty, b]$
$a \leq x < b$	All real numbers between a and b, including a	$[a, b)$
$a < x \leq b$	All real numbers between a and b, including b	$(a, b]$
$a \leq x \leq b$	All real numbers between a and b, including a and b	$[a, b]$

Inequality	Words	Interval Notation
$x < a$ or $x > b$	All real numbers less than a or greater than b	$(-\infty, a) \cup (b, \infty)$
All real numbers	All real numbers	$(-\infty, \infty)$

Example

Describe the inequality $x \geq 4$ using interval notation
Show Solution

In the following video we show another example of using interval notation to describe an inequality.

Visit this page in your course online to check your understanding.

Example

Use interval notation to indicate all real numbers greater than or equal to -2.
Show Solution

In the following video we show another example of translating words into an inequality and writing it in interval notation, as well as drawing the graph.

Visit this page in your course online to check your understanding.

Think About It

In the previous examples you were given an inequality or a description of one with words and asked to draw the corresponding graph and write the interval. In this example you are given an interval and asked to write the inequality and draw the graph.
Given $(-\infty, 10)$, write the associated inequality and draw the graph.
In the box below, write down whether you think it will be easier to draw the graph first or write the inequality first.

Show Solution

In the following video, you will see examples of how to draw a graph given an inequality in interval notation.

Watch this video online: https://youtu.be/lkhILNEPbfk

And finally, one last video that shows how to write inequalities using a graph, with interval notation and as an inequality.

Watch this video online: https://youtu.be/X0xrHKgbDT0

Solve Single-Step Inequalities

Solve inequalities with addition and subtraction

You can solve most inequalities using inverse operations as you did for solving equations. This is because when you add or subtract the same value from both sides of an inequality, you have maintained the inequality. These properties are outlined in the box below.

Addition and Subtraction Properties of Inequality

If $a>b$, then $a+c>b+c$.
If $a>b$, then $a-c>b-c$.

Because inequalities have multiple possible solutions, representing the solutions graphically provides a helpful visual of the situation, as we saw in the last section. The example below shows the steps to solve and graph an inequality and express the solution using interval notation.

Example

Solve for *x.*

$x+3<5$
Show Solution

The line represents *all* the numbers to which you can add 3 and get a number that is less than 5. There's a lot of numbers that solve this inequality!

Just as you can check the solution to an equation, you can check a solution to an inequality. First, you check the end point by substituting it in the related equation. Then you check to see if the inequality is correct by substituting any other solution to see if it is one of the solutions. Because there are multiple solutions, it is a good practice to check more than one of the possible solutions. This can also help you check that your graph is correct.

The example below shows how you could check that $x<2$ is the solution to $x+3<5$.

Example

Check that $x<2$ is the solution to $x+3<5$.
Show Solution

The following examples show inequality problems that include operations with negative numbers. The graph of the solution to the inequality is also shown. Remember to check the solution. This is a good habit to build!

Example

Solve for *x*: $x-10\leq-12$
Show Solution

Check the solution to $x-10\leq-12$
Show Solution

Example

Solve for *a*. $a-17>-17$
Show Solution
Check the solution to $a-17>-17$
Show Solution

The previous examples showed you how to solve a one-step inequality with the variable on the left hand side. The following video provides examples of how to solve the same type of inequality.

Watch this video online: https://youtu.be/1Z22Xh66VFM

What would you do if the variable were on the right side of the inequality? In the following example, you will see how to handle this scenario.

Example

Solve for *x*: $4 \geq x+5$
Show Solution
Check the solution to $4 \geq x+5$
Show Solution

The following video show examples of solving inequalities with the variable on the right side.

Visit this page in your course online to check your understanding.

Solve inequalities with multiplication and division

Solving an inequality with a variable that has a coefficient other than 1 usually involves multiplication or division. The steps are like solving one-step equations involving multiplication or division EXCEPT for the inequality sign. Let's look at what happens to the inequality when you multiply or divide each side by the same number.

Let's start with the true statement: $10>5$	Let's try again by starting with the same true statement: $10>5$
Next, multiply both sides by the same positive number: $10 \cdot 2 > 5 \cdot 2$	This time, multiply both sides by the same negative number: $10 \cdot -2 > 5 \cdot -2$
20 is greater than 10, so you still have a true inequality: $20>10$	Wait a minute! -20 is *not* greater than -10, so you have an untrue statement. $-20>-10$
When you multiply by a positive number, leave the inequality sign as it is!	You must "reverse" the inequality sign to make the statement true: $-20<-10$

Caution! When you multiply or divide by a negative number, "reverse" the inequality sign. Whenever you multiply or divide both sides of an inequality by a negative number, the inequality sign must be reversed in order to keep a true statement. These rules are summarized in the box below.

Multiplication and Division Properties of Inequality

Start With	Multiply By	Final Inequality

$a > b$	c	$ac > bc$
$a > b$	$-c$	$ac < bc$

Start With	**Divide By**	**Final Inequality**
$a > b$	c	$\frac{a}{c} > \frac{b}{c}$
$a > b$	$-c$	$\frac{a}{c} < \frac{b}{c}$

Keep in mind that you only change the sign when you are multiplying and dividing by a *negative* number. If you *add or subtract* by a negative number, the inequality stays the same.

Example

Solve for *x*. $3x > 12$
Show Solution

There was no need to make any changes to the inequality sign because both sides of the inequality were divided by *positive* 3. In the next example, there is division by a negative number, so there is an additional step in the solution!

Example

Solve for *x*. $-2x > 6$
Show Solution

The following video shows examples of solving one step inequalities using the multiplication property of equality where the variable is on the left hand side.

Watch this video online: https://youtu.be/IajiD3R7U-0

Think About It

Before you read the solution to the next example, think about what properties of inequalities you may need to use to solve the inequality. What is different about this example from the previous one? Write your ideas in the box below.
Solve for *x*. $-\frac{1}{2} > -12x$

Show Solution

The following video gives examples of how to solve an inequality with the multiplication property of equality where the variable is on the right hand side.

Watch this video online: https://youtu.be/s9fJOnVTHhs

Combine properties of inequality to solve algebraic inequalities

A popular strategy for solving equations, isolating the variable, also applies to solving inequalities. By adding, subtracting, multiplying and/or dividing, you can rewrite the inequality so that the variable is on one side and everything else is on the other. As with one-step inequalities, the solutions to multi-step inequalities can be graphed on a number line.

Example

Solve for *p*. $4p+5<29$
Show Solution

Check the solution.
Show Solution

Example

Solve for *x*: $3x–7\geq 41$
Show Solution
Check the solution.
Show Solution

When solving multi-step equations, pay attention to situations in which you multiply or divide by a negative number. In these cases, you must reverse the inequality sign.

Example

Solve for *p*. $-58>14-6p$
Show Solution

Check the solution.
Show Solution

In the following video, you will see an example of solving a linear inequality with the variable on the left side of the inequality, and an example of switching the direction of the inequality after dividing by a negative number.

Watch this video online: https://youtu.be/RB9wvIogoEM

In the following video, you will see an example of solving a linear inequality with the variable on the right side of the inequality, and an example of switching the direction of the inequality after dividing by a negative number.

Watch this video online: https://youtu.be/9D2g_FaNBkY

Simplify and solve algebraic inequalities using the distributive property

As with equations, the distributive property can be applied to simplify expressions that are part of an inequality. Once the parentheses have been cleared, solving the inequality will be straightforward.

Example

Solve for *x*. $2\left(3x–5\right)\leq 4x+6$
Show Solution

Check the solution.
Show Solution

In the following video, you are given an example of how to solve a multi-step inequality that requires using the distributive property.

Visit this page in your course online to check your understanding.

Think About It

In the next example, you are given an inequality with a term that looks complicated. If you pause and think about how to use the order of operations to solve the inequality, it will hopefully seem like a straightforward problem. Use the textbox to write down what you think is the best first step to take.

Solve for a. $\frac{2a-4}{6}<2$

Show Solution
Check the solution.
Show Solution

Summary

Solving inequalities is very similar to solving equations, except you have to reverse the inequality symbols when you multiply or divide both sides of an inequality by a negative number. There are three ways to represent solutions to inequalities: an interval, a graph, and an inequality. Because there is usually more than one solution to an inequality, when you check your answer you should check the end point and one other value to check the direction of the inequality.

Inequalities can have a range of answers. The solutions are often graphed on a number line in order to visualize all of the solutions. Multi-step inequalities are solved using the same processes that work for solving equations with one exception. When you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality symbol. The inequality symbols stay the same whenever you add or subtract *either positive or negative* numbers to both sides of the inequality.

SOLVE COMPOUND INEQUALITIES

Learning Objectives

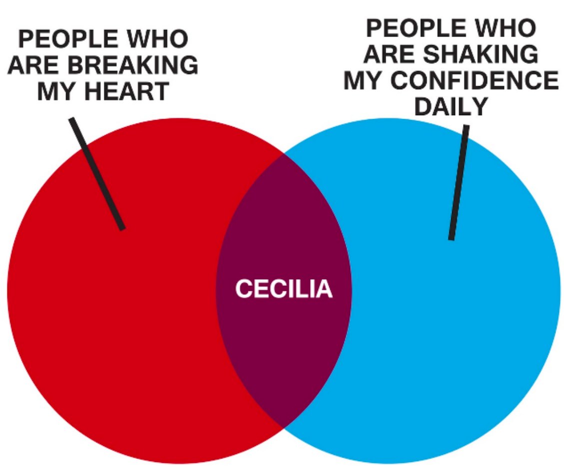
- Describe sets as intersections or unions
  - Use interval notation to describe intersections and unions
  - Use graphs to describe intersections and unions
- Solve compound inequalities—OR
  - Solve compound inequalities in the form of *or* and express the solution graphically and with an interval
- Solve compound inequalities—AND
  - Express solutions to inequalities graphically and with interval notation
  - Identify solutions for compound inequalities in the form $a < x < b$, including cases with no solution
- Solve absolute value inequalities
  - Solve single- and multi-step inequalities containing absolute values
  - Identify cases where there are no solutions to absolute value inequalities

Use interval notation to describe sets of numbers as intersections and unions

When two inequalities are joined by the word *and*, the solution of the compound inequality occurs when *both* inequalities are true at the same time. It is the overlap, or intersection, of the solutions for each inequality. When the two inequalities are joined by the word *or*, the solution of the compound inequality occurs when *either* of the inequalities is true. The solution is the combination, or union, of the two individual solutions.

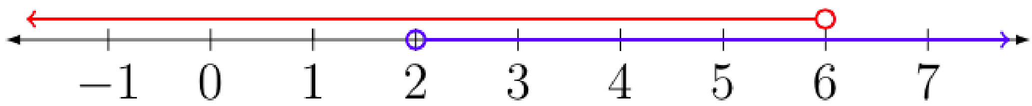
In this section we will learn how to solve compound inequalities that are joined with the words AND and OR. First, it will help to see some examples of inequalities, intervals, and graphs of compound inequalities. This will help you describe the solutions to compound inequalities properly.

Venn diagrams use the concept of intersections and unions to show how much two or more things share in common. For example, this Venn diagram shows the intersection of people who are breaking your heart and those who are shaking your confidence daily. Apparently Cecilia has both of these qualities; therefore she is the intersection of the two.

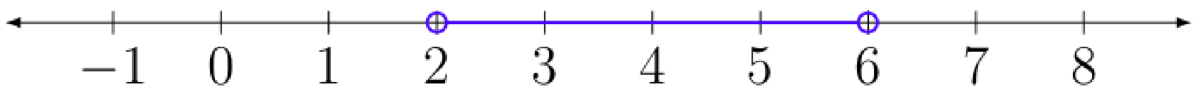


In mathematical terms, consider the inequality $x < 6$ and $x > 2$. How would we interpret what numbers *x* can be, and what would the interval look like?

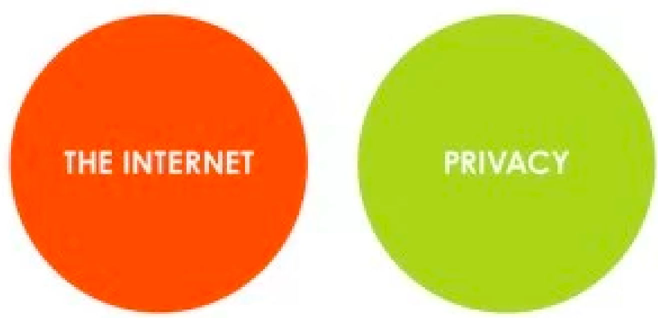
In words, *x* must be less than 6 *and at the same time,* it must be greater than 2, much like the Venn diagram above, where Cecilia is at once breaking your heart and shaking your confidence daily. Let’s look at a graph to see what numbers are possible with these constraints.



The numbers that are shared by both lines on the graph are called the intersection of the two inequalities $x < 6$ and $x > 2$. This is called a *bounded* inequality and is written as $2 < x < 6$. Think about that one for a minute. *x* must be less than 6 and greater than two—the values for *x* will fall *between two numbers.* In interval notation, this looks like $(2, 6)$. The graph would look like this:

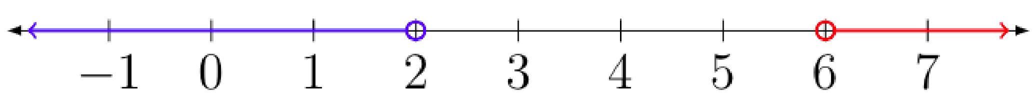


On the other hand, if you need to represent two things that don't share any common elements or traits, you can use a union. The following Venn diagram shows two things that share no similar traits or elements but are often considered in the same application, such as online shopping or banking.



In mathematical terms, for example, $x > 6$ *or* $x < 2$ is an inequality joined by the word *or*. Using interval notation, we can describe each of these inequalities separately:

$x > 6$ is the same as $(6, \infty)$ and $x < 2$ is the same as $(\infty, 2)$. If we are describing solutions to inequalities, what effect does the *or* have? We are saying that solutions are either real numbers less than two *or* real numbers greater than 6. Can you see why we need to write them as two separate intervals? Let's look at a graph to get a clear picture of what is going on.



When you place both of these inequalities on a graph, we can see that they share no numbers in common. This is what we call a union, as mentioned above. The interval notation associated with a union is a big U, so instead of writing *or*, we join our intervals with a big U, like this:

$(\infty, 2) \cup (6, \infty)$

It is common convention to construct intervals starting with the value that is furthest left on the number line as the left value, such as $(2, 6)$, where 2 is less than 6. The number on the right should be greater than the number on the left.

Example

Draw the graph of the compound inequality $x > 3$ *or* $x \leq 4$ and describe the set of *x*-values that will satisfy it with an interval.
Show Solution

In the following video you will see two examples of how to express inequalities involving OR graphically and as an interval.

Visit this page in your course online to check your understanding.

Examples

Draw a graph of the compound inequality: $x < 5$ ***and*** $x \geq -1$, and describe the set of *x*-values that will satisfy it with an interval.
Show Solution

Examples

Draw the graph of the compound inequality $x < -3$ and $x > 3$, and describe the set of *x*-values that will satisfy it with an interval.
Show Solution

The following video presents two examples of how to draw inequalities involving AND, as well as write the corresponding intervals.

Watch this video online: https://youtu.be/LP3fsZNjJkc

Solve compound inequalities in the form of *or*

As we saw in the last section, the solution of a compound inequality that consists of two inequalities joined with the word *or* is the union of the solutions of each inequality. Unions allow us to create a new set from two that may or may not have elements in common.

In this section you will see that some inequalities need to be simplified before their solution can be written or graphed.

In the following example, you will see an example of how to solve a one-step inequality in the OR form. Note how each inequality is treated independently until the end where the solution is described in terms of both inequalities. You will use the same properties to solve compound inequalities that you used to solve regular inequalities.

Example

Solve for *x*. $3x–1 < 8$ or $x–5 > 0$
Show Solution

Remember to apply the properties of inequality when you are solving compound inequalities. The next example involves dividing by a negative to isolate a variable.

Example

Solve for *y*. $2y + 7 < 13 \text{ or } -3y–2 < 10$
Show Solution

In the last example, the final answer included solutions whose intervals overlapped, causing the answer to include all the numbers on the number line. In words, we call this solution "all real numbers." Any real number will produce a true statement for either $y < 3 \text{ or } y \geq -4$, when it is substituted for *x*.

Example

Solve for *z*. $5z–3 > −18$ or $−2z–1 > 15$
Show Solution

The following video contains an example of solving a compound inequality involving OR, and drawing the associated graph.

Watch this video online: https://youtu.be/oRIJ8G7trR8

In the next section you will see examples of how to solve compound inequalities containing *and*.

Solve compound inequalities in the form of *and* and express the solution graphically

The solution of a compound inequality that consists of two inequalities joined with the word *and* is the intersection of the solutions of each inequality. In other words, both statements must be true at the same time. The solution to an *and* compound inequality are all the solutions that the two inequalities have in common. As we saw in the last sections, this is where the two graphs overlap.

In this section we will see more examples where we have to simplify the compound inequalities before we can express their solutions graphically or with an interval.

Example

Solve for *x*. $1-4x\leq21\,\,\text{ and }\,\,5x+2\geq22$
Show Solution

Example

Solve for *x*: $5x-2\leq3 \text{ and } 4x+7>3$
Show Solution

Compound inequalities in the form $a<x<b$

Rather than splitting a compound inequality in the form of $a<x<b$ into two inequalities $x<b$ *and* $x>a$, you can more quickly to solve the inequality by applying the properties of inequality to all three segments of the compound inequality.

Example

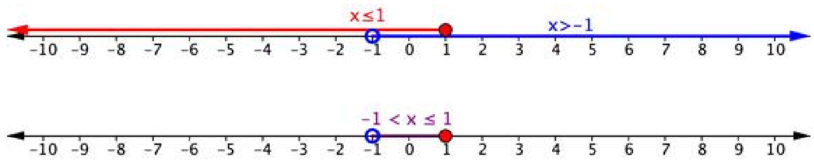
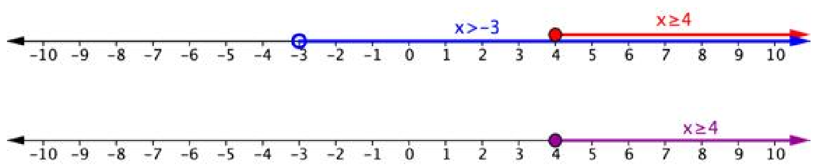
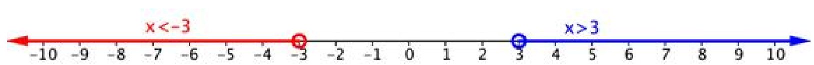
Solve for *x*. $3<2x+3\leq7$
Show Solution

In the video below, you will see another example of how to solve an inequality in the form $a<x<b$

Watch this video online: https://youtu.be/UU_KJI59_08

To solve inequalities like $a < x < b$, use the addition and multiplication properties of inequality to solve the inequality for *x*. Whatever operation you perform on the middle portion of the inequality, you must also perform to each of the outside sections as well. Pay particular attention to division or multiplication by a negative.

The solution to a compound inequality with *and* is always the overlap between the solution to each inequality. There are three possible outcomes for compound inequalities joined by the word *and*:

Case 1:	
Description	The solution could be all the values between two endpoints
Inequalities	$x \leq 1$ and $x > -1$, or as a bounded inequality: $-1 < x \leq 1$
Interval	$(-1, 1]$
Graphs	
Case 2:	
Description	The solution could begin at a point on the number line and extend in one direction.
Inequalities	$x > 3$ and $x \geq 4$
Interval	$[4, \infty)$
Graphs	
Case 3:	
Description	In cases where there is no overlap between the two inequalities, there is no solution to the compound inequality
Inequalities	$x < -3$ and $x > 3$
Intervals	$(-\infty, -3)$ and $(3, \infty)$
Graph	

In the example below, there is no solution to the compound inequality because there is no overlap between the inequalities.

Example

Solve for *x*. $x+2>5$ and $x+4<5$
Show Solution

Solve inequalities containing absolute values

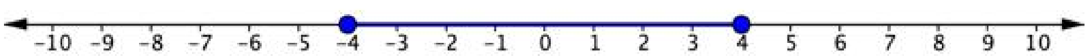
Let’s apply what you know about solving equations that contain absolute values and what you know about inequalities to solve inequalities that contain absolute values. Let’s start with a simple inequality.

$|x| \leq 4$

This inequality is read, “the absolute value of *x* is less than or equal to 4.” If you are asked to solve for *x*, you want to find out what values of *x* are 4 units or less away from 0 on a number line. You could start by thinking about the number line and what values of *x* would satisfy this equation.

4 and -4 are both four units away from 0, so they are solutions. 3 and -3 are also solutions because each of these values is less than 4 units away from 0. So are 1 and -1, 0.5 and -0.5, and so on—there are an infinite number of values for *x* that will satisfy this inequality.

The graph of this inequality will have two closed circles, at 4 and -4. The distance between these two values on the number line is colored in blue because all of these values satisfy the inequality.

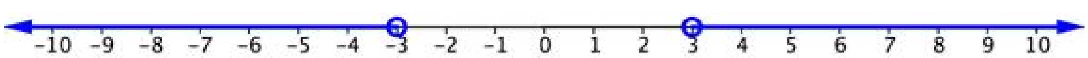


The solution can be written this way:

Inequality: $-4 \leq x \leq 4$

Interval: $[-4,4]$

The situation is a little different when the inequality sign is “greater than” or “greater than or equal to.” Consider the simple inequality $|x|>3$. Again, you could think of the number line and what values of *x* are greater than 3 units away from zero. This time, 3 and -3 are not included in the solution, so there are open circles on both of these values. 2 and -2 would not be solutions because they are not more than 3 units away from 0. But 5 and -5 would work, and so would all of the values extending to the left of -3 and to the right of 3. The graph would look like the one below.



The solution to this inequality can be written this way:

Inequality*:* $x<-3$ or $x>3$.

Interval: $(-\infty,-3) \cup (3,\infty)$

In the following video, you will see examples of how to solve and express the solution to absolute value inequalities involving both AND and OR.

Watch this video online: https://youtu.be/0cXxATY2S-k

Writing Solutions to Absolute Value Inequalities

For any positive value of *a* and *x,* a single variable, or any algebraic expression:

Absolute Value Inequality	**Equivalent Inequality**	**Interval Notation**

$\lvert x\rvert \leq a$	$-a \leq x \leq a$	$[-a, a]$
$\lvert x\rvert < a$	$-a < x < a$	$(-a, a)$
$\lvert x\rvert \geq a$	$x \leq -\text{a}$ or $x \geq a$	$(-\infty, -a] \cup [a, \infty)$
$\lvert x\rvert > \text{a}$	$x < -\text{a}$ or $x > a$	$(-\infty, -a) \cup (a, \infty)$

Let's look at a few more examples of inequalities containing absolute values.

Example

Solve for *x*. $|x+3| > 4$
Show Solution

Example

Solve for *y*. $3\,|2\text{y}+6| - 9 < 27$
Show Solution

In the following video, you will see an example of solving multi-step absolute value inequalities involving an OR situation.

Visit this page in your course online to check your understanding.

In the following video you will see an example of solving multi-step absolute value inequalities involving an AND situation.

Visit this page in your course online to check your understanding.

In the last video that follows, you will see an example of solving an absolute value inequality where you need to isolate the absolute value first.

Visit this page in your course online to check your understanding.

Identify cases of inequalities containing absolute values that have no solutions

As with equations, there may be instances in which there is no solution to an inequality.

Example

Solve for *x*. $|2x+3| + 9 \leq 7$
Show Solution

Summary

A compound inequality is a statement of two inequality statements linked together either by the word *or* or by the word *and*. Sometimes, an *and* compound inequality is shown symbolically, like $a < x < b$, and does not even need the word *and*. Because compound inequalities represent either a union or intersection of the individual inequalities, graphing them on a number line can be a helpful way to see or check a solution. Compound inequalities can be manipulated and solved in much the same way any inequality is solved, by paying attention to the properties of inequalities and the rules for solving them.

Absolute inequalities can be solved by rewriting them using compound inequalities. The first step to solving absolute inequalities is to isolate the absolute value. The next step is to decide whether you are working with an OR inequality or an AND inequality. If the inequality is greater than a number, we will use OR. If the inequality is less than a number, we will use AND. Remember that if we end up with an absolute value greater than or less than a negative number, there is no solution.

CONCLUSION

At the start of this module, you were presented with a formula commonly used by forensic scientists to calculate blood alcohol content:

$$B = -0.015t + \left(\frac{2.84N}{Wg}\right)$$

where

- B = percentage of BAC
- t = number of hours since the first drink
- N = number of “standard drinks” (a standard drink is one 12-ounce beer, one 5-ounce glass of wine, or one 1.5-ounce shot of liquor). N should be at least 1.
- W = weight in pounds
- g = gender constant: 0.68 for men and 0.55 for women

Remember Joan? When we left her, she was at a party. She had three drinks and then wanted to leave. This was worrisome. We posed the following questions:

1. Where would she fall on the table of the progressive effects of alcohol after 1.5 hours?
2. Would she be within the legal limit to drive after this amount of time?

The table showing the progressive effects of alcohol appears at the bottom of the page.

The information needed to answer the first question includes BAC, N, W, t and g. Of these, we know N, W, t, and g—we will need to solve for BAC.

- N = 3 standard drinks
- W = 135 pounds
- t = 1.5 hours
- g = 0.55

Using these values, we can calculate Joan’s BAC, which is the first step in answering question 1.

$$\mathrm{B} = -0.015t + \left(\frac{2.84N}{Wg}\right)$$
$$\mathrm{B} = -0.015(1.5) + \left(\frac{2.84(3)}{(135)(0.55)}\right)$$
$$\mathrm{B} = -0.0225 + \left(\frac{8.52}{(74.25)}\right)$$
$$\mathrm{B} = -0.0225 + (0.1147)$$
$$\mathrm{B} = 0.092$$

The units of B are a percentage, so in this scenario after 1.5 hours and after consuming three standard drinks, Joan’s BAC is 0.092%.

How is Joan doing? From the table below, we can predict that she’s experiencing blunted feelings, reduced sensitivity to pain, euphoria, disinhibition, and extroversion. In addition, her reasoning, depth perception, peripheral vision, and glare recovery are probably impaired.

Question 2 asks whether she would be within the legal limit to drive. If we assume that the legal limit is 0.08%, then NO, she would not be within the legal limit to drive. One of her friends should take her car keys and help her get home.

Progressive effects of alcohol		
BAC (% by vol.)	**Behavior**	**Impairment**
0.001–0.029	• Average individual appears normal	• Subtle effects that can be detected with special tests
0.030–0.059	• Mild euphoria • Relaxation • Joyousness • Talkativeness • Decreased inhibition	• Concentration
0.060–0.099	• Blunted feelings • Reduced sensitivity to pain • Euphoria • Disinhibition • Extroversion	• Reasoning • Depth perception • Peripheral vision • Glare recovery
0.100–0.199	• Over-expression • Boisterousness • Possibility of nausea and vomiting	• Reflexes • Reaction time • Gross motor control • Staggering • Slurred speech • Temporary erectile dysfunction

0.200–0.299	• Nausea • Vomiting • Emotional swings • Anger or sadness • Partial loss of understanding • Impaired sensations • Decreased libido • Possibility of stupor	• Severe motor impairment • Loss of consciousness • Memory blackout
0.300–0.399	• Stupor • Central nervous system depression • Loss of understanding • Lapses in and out of consciousness • Low possibility of death	• Bladder function • Breathing • Dysequilibrium • Heart rate
0.400–0.500	• Severe central nervous system depression • Coma • Possibility of death	• Breathing • Heart rate • Positional Alcohol Nystagmus
>0.50	• High risk of poisoning • High possibility of death	• Life

MODULE 2: GRAPHING

INTRODUCTION

Why explain the use of graphs and create graphs using linear equations?

Much as the road sign at the right can help you quickly understand a “falling rocks” hazard ahead, graphs are a useful way of conveying information visually. Graphs of linear equations are especially effective for representing relationships between things that change at a constant rate, and they often do a better job than words or mathematical equations alone.

Suppose your manager has asked you to create and present a financial report at the next company meeting. She wants to know how the amount of revenue your team brought to the company changed during the last year. Your job is to get the information and choose the best method for communicating it.

You start with a table and add the values:

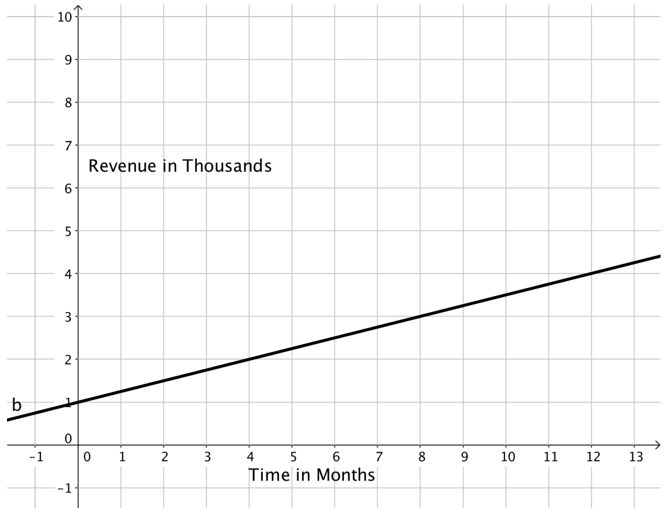
Month	Revenue
0	1000
1	1250
3	1750
5	2250
7	2750
9	3250
11	3750

The table is nicely organized, and its values are very precise, but it doesn’t readily show how the values change over time. So you study the numbers and try to describe this change in words:

> Our team started the year with $1000 per month in revenue. For each additional month in the year 2015, our team produced an increase in $250 of revenue.

Next, you try expressing the information as a linear equation: $y = 250x + 1000$

Again, both are very accurate, but neither reading two sentences out loud nor putting an equation on a poster board will go over very well with your manager or team members. You haven’t given them a way to picture the information. So, you try graphing the equation, which gives you the following:



Graph showing the linear relationship between time and the change in revenue.

All four—the table, the words, the equation, and the graph—are representations of the same thing. And, depending on your purpose, one representation may get your point across more clearly than another. As you can see, graphs are especially good at conveying information about the relationship between things (in this case, *time* and *revenue*) with very little explanation. Your graph shows that the revenue your team produced during the last year steadily increased—a very encouraging trend! You decide that the graph is definitely what you'll present at the meeting.

In this module, you'll learn how to create Cartesian graphs (like the one above) and why graphs are an especially useful means of conveying information.

Learning Outcomes

- **The Coordinate Plane**
  - Plot ordered pairs
  - Identify quadrants on the coordinate plane
- **Graphing Linear Equations**
  - Graph an equation using ordered pairs
  - Graph linear equations in different forms
  - Graph an equation using intercepts
- **Slope**
  - Find the slope from a graph
  - Find the slope from two points
  - Find the slope of horizontal and vertical lines
- **Equations of Lines**
  - Write the equation and draw the graph of a line using slope and y-intercept
  - Write and solve equations of lines using slope and a point on the line
  - Write and solve equations of lines using two points on the line
- **Parallel and Perpendicular Lines**
  - Identify slopes of parallel and perpendicular lines
  - Write equations of parallel and perpendicular lines
- **Applications of Graphs**
  - Interpret slope in equations and graphs
  - Interpret the y-intercept of a linear equation and use that equation to make predictions
- **Graphing Linear Inequalities**
  - Classify solutions and graphs as equations or inequalities
  - Graph an inequality in two variables

THE COORDINATE PLANE

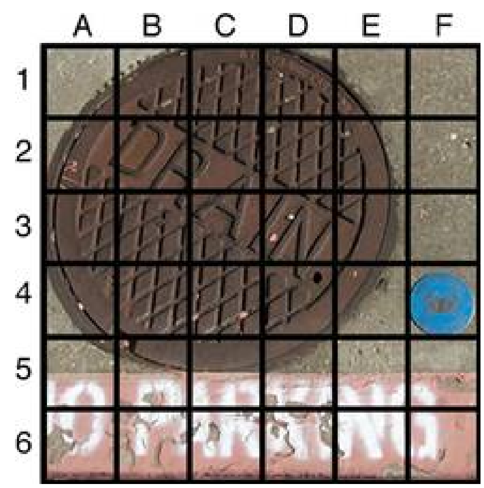
Learning Objectives

- Plot Ordered Pairs
  - Identify the components of the coordinate plane
  - Plot ordered pairs on the coordinate plane
- Identify Quadrants on the Coordinate Plane
  - Identify the four quadrants of a coordinate plane
  - Given an ordered pair, determine its quadrant

The **coordinate plane** was developed centuries ago (in 1637, to be exact) and refined by the French mathematician René Descartes. In his honor, the system is sometimes called the Cartesian coordinate system. The coordinate plane can be used to plot points and graph lines. This system allows us to describe algebraic relationships in a visual sense, and also helps us create and interpret algebraic concepts.

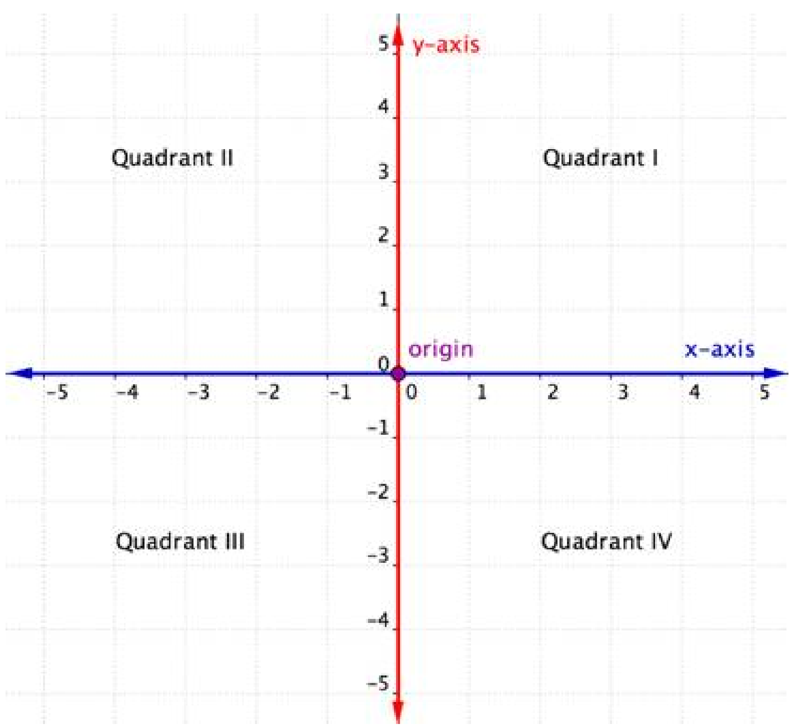
The Components of the Coordinate Plane

You have likely used a coordinate plane before. For example, have you ever used a gridded overlay to map the position of an object? (This is often done with road maps, too.)



This "map" uses a horizontal and vertical grid to convey information about an object's location. Notice that the letters A–F are listed along the top, and the numbers 1–6 are listed along the left edge. The general location of any item on this map can be found by using the letter and number of its grid square. For example, you can find the item that exists at square "4F" by moving your finger along the horizontal to letter F and then straight down so you are in line with the 4. You'll find a blue disc is at this location on the map.

The coordinate plane has similar elements to the grid shown above. It consists of a horizontal **axis** and a vertical axis, number lines that intersect at right angles. (They are perpendicular to each other.)

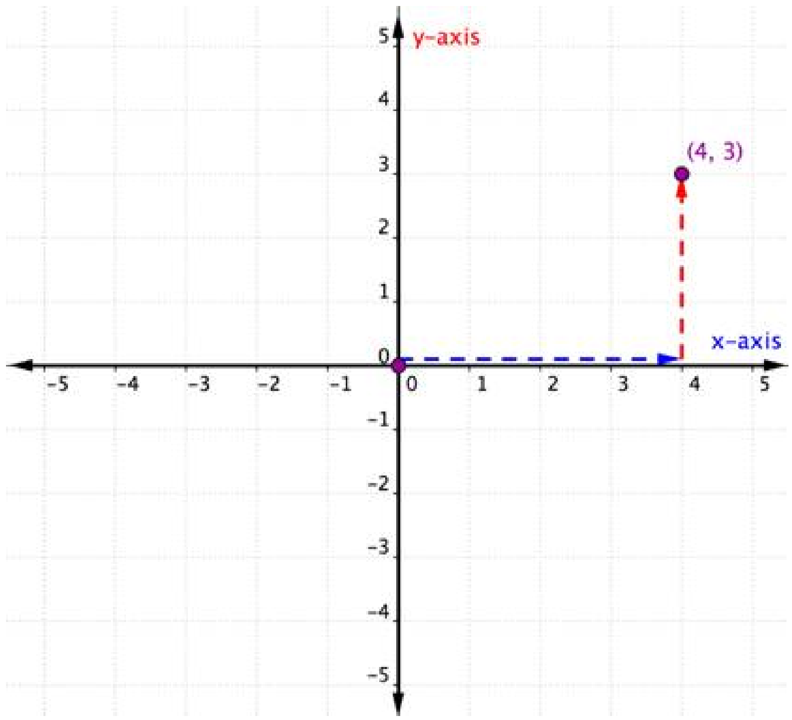


The horizontal axis in the coordinate plane is called the **x-axis**. The vertical axis is called the **y-axis**. The point at which the two axes intersect is called the **origin**. The origin is at 0 on the *x*-axis and 0 on the *y*-axis.

Locations on the coordinate plane are described as **ordered pairs**. An ordered pair tells you the location of a point by relating the point's location along the *x*-axis (the first value of the ordered pair) and along the *y*-axis (the second value of the ordered pair).

In an ordered pair, such as (*x*, *y*), the first value is called the **x-coordinate** and the second value is the **y-coordinate**. Note that the *x*-coordinate is listed before the *y*-coordinate. Since the origin has an *x*-coordinate of 0 and a *y*-coordinate of 0, its ordered pair is written (0, 0).

Consider the point below.



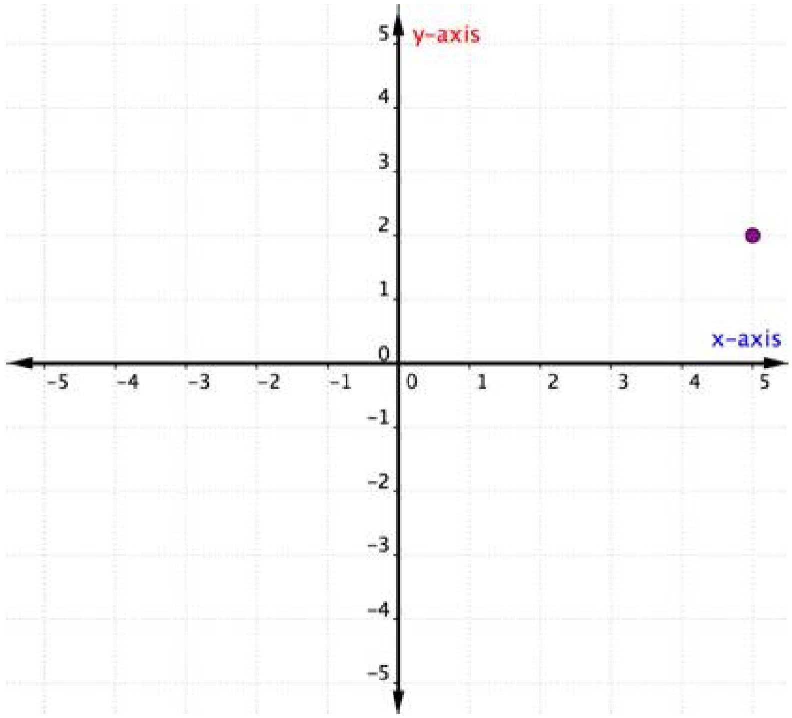
To identify the location of this point, start at the origin (0, 0) and move right along the *x*-axis until you are under the point. Look at the label on the *x*-axis. The 4 indicates that, from the origin, you have traveled four units to the right along the *x*-axis. This is the *x*-coordinate, the first number in the ordered pair.

From 4 on the *x*-axis move up to the point and notice the number with which it aligns on the *y*-axis. The 3 indicates that, after leaving the *x*-axis, you traveled 3 units up in the vertical direction, the direction of the *y*-axis. This number is the *y*-coordinate, the second number in the ordered pair. With an *x*-coordinate of 4 and a *y*-coordinate of 3, you have the ordered pair (4, 3).

Let's look at another example.

Example

Describe the point shown as an ordered pair.



Show Solution

Describe the point shown as an ordered pair

Watch this video online: https://youtu.be/c9WVU34MY5Q

Plotting Points in the Coordinate Plane

Now that you know how to use the *x*- and *y*-axes, you can plot an ordered pair as well. Just remember, both processes start at the origin—the beginning! The example that follows shows how to graph the ordered pair (1,3).

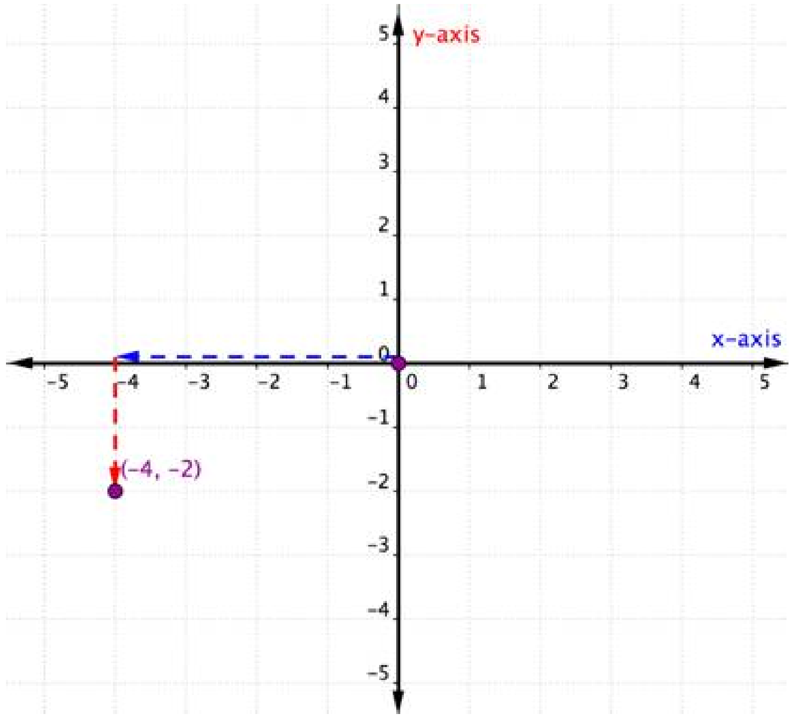
Example

Plot the point (1, 3).
Show Solution

In the previous example, both the *x*- and *y*-coordinates were positive. When one (or both) of the coordinates of an ordered pair is negative, you will need to move in the negative direction along one or both axes. Consider the example below in which both coordinates are negative.

Example

Plot the point $(-4, -2)$.



The *x*-coordinate is -4 because it comes first in the ordered pair. Start at the origin and move 4 units in a negative direction (left) along the *x*-axis.

The *y*-coordinate is -2 because it comes second in the ordered pair. Now move 2 units in a negative direction (down). If you look over to the *y*-axis, you should be lined up with -2 on that axis.
Show Answer

The steps for plotting a point are summarized below.

Steps for Plotting an Ordered Pair (*x*, *y*) in the Coordinate Plane

- Determine the *x*-coordinate. Beginning at the origin, move horizontally, the direction of the *x*-axis, the distance given by the *x*-coordinate. If the *x*-coordinate is positive, move to the right; if the *x*-coordinate is negative, move to the left.
- Determine the *y*-coordinate. Beginning at the *x*-coordinate, move vertically, the direction of the *y*-axis, the distance given by the *y*-coordinate. If the *y*-coordinate is positive, move up; if the *y*-coordinate is negative, move down.
- Draw a point at the ending location. Label the point with the ordered pair.

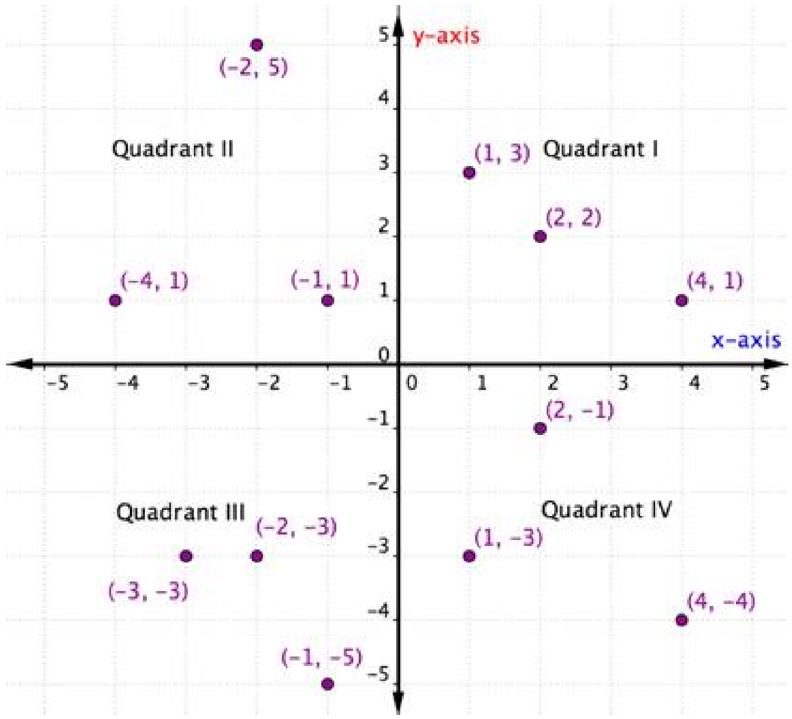
Plotting Points on the Coordinate Plane

Watch this video online: https://youtu.be/p_MESleS3mw

Identify quadrants and use them to plot points

The intersecting *x*- and *y*-axes of the coordinate plane divide it into four sections. These four sections are called **quadrants**. Quadrants are named using the Roman numerals I, II, III, and IV beginning with the top right quadrant and moving counter clockwise.

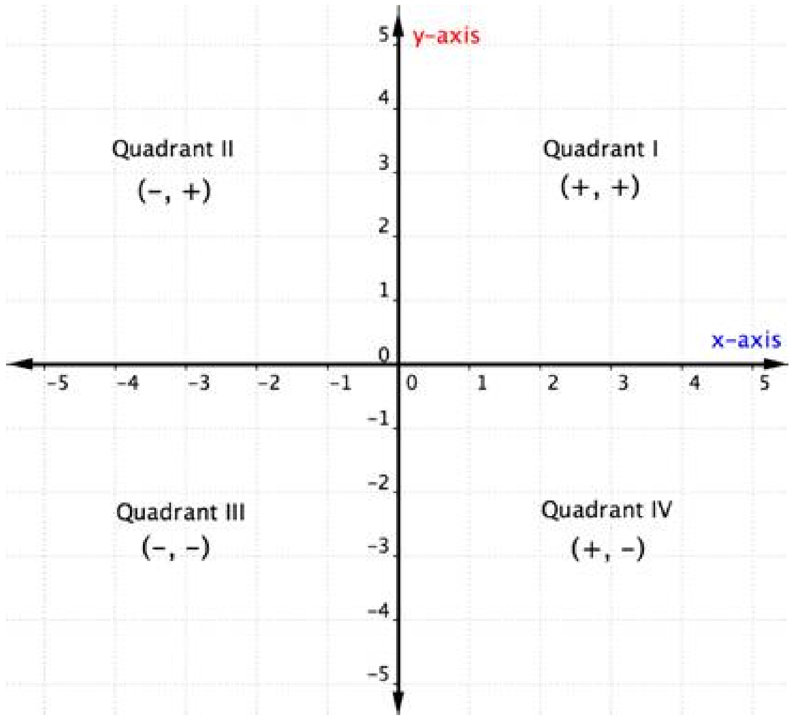
Ordered pairs within any particular quadrant share certain characteristics. Look at each quadrant in the graph below. What do you notice about the signs of the *x*- and *y*-coordinates of the points within each quadrant?



Within each quadrant, the signs of the *x*-coordinates and *y*-coordinates of each ordered pair are the same. They also follow a pattern, which is outlined in the table below.

Quadrant	**General Form of Point in this Quadrant**	**Example**	**Description**
I	$(+,+)$	$(5,4)$	Starting from the origin, go along the *x*-axis in a positive direction (right) and along the *y*-axis in a positive direction (up).
II	$(-,+)$	$(-5,4)$	Starting from the origin, go along the *x*-axis in a negative direction (left) and along the *y*-axis in a positive direction (up).
III	$(-,-)$	$(-5,-4)$	Starting from the origin, go along the *x*-axis in a negative direction (left) and along the *y*-axis in a negative direction (down).
IV	$(+,-)$	$(5,-4)$	Starting from the origin, go along the *x*-axis in a positive direction (right) and along the *y*-axis in a negative direction (down).

Once you know about the quadrants in the coordinate plane, you can determine the quadrant of an ordered pair without even graphing it by looking at the chart above. Here’s another way to think about it.



The example below details how to determine the quadrant location of a point just by thinking about the signs of its coordinates. Thinking about the quadrant location before plotting a point can help you prevent a mistake. It is also useful knowledge for checking that you have plotted a point correctly.

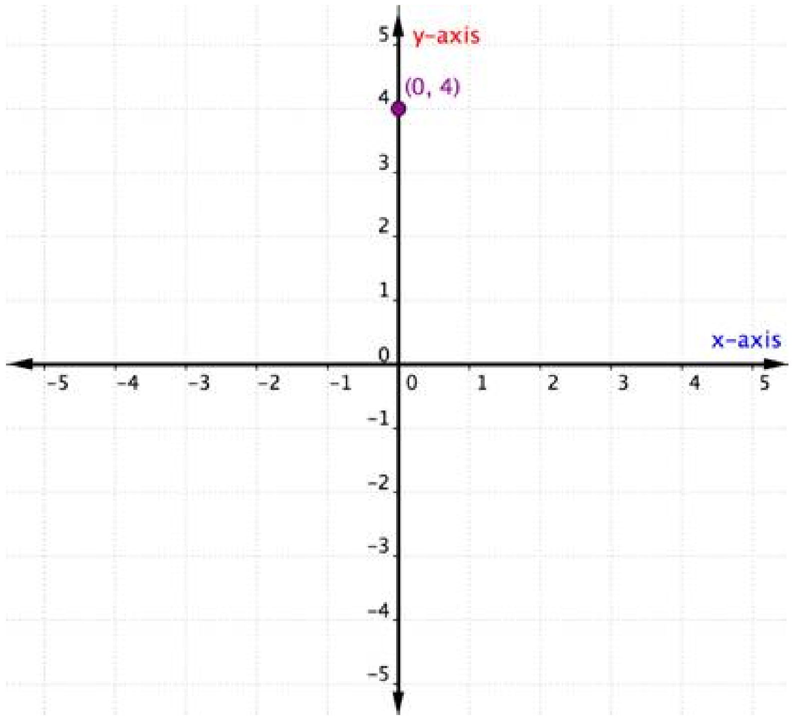
Example

In which quadrant is the point $(-7, 10)$ located?
Show Solution

Example

In which quadrant is the point $(-10, -5)$ located?
Show Solution

What happens if an ordered pair has an *x*– or *y*-coordinate of zero? The example below shows the graph of the ordered pair $(0, 4)$.



A point located on one of the axes is not considered to be in a quadrant. It is simply on one of the axes. Whenever the *x*-coordinate is 0, the point is located on the *y*-axis. Similarly, any point that has a *y*-coordinate of 0 will be located on the *x*-axis.

Identify quadrants and use them to plot points

Watch this video online: https://youtu.be/iTsJsPgcE4E

Summary

The coordinate plane is a system for graphing and describing points and lines. The coordinate plane is comprised of a horizontal (*x*-) axis and a vertical (*y*-) axis. The intersection of these lines creates the origin, which is the point $(0,0)$. The coordinate plane is split into four quadrants. Together, these features of the coordinate system allow for the graphical representation and communication about points, lines, and other algebraic concepts.

GRAPHING LINEAR EQUATIONS

Learning Objectives

- Graph an Equation Using Ordered Pairs
  - Create a table of ordered pairs from a two-variable linear equation
  - Graph a two-variable linear equation using a table of ordered pairs
  - Determine whether an ordered pair is a solution of an equation
- Graph Linear Equations in Different Forms

- Solve for *y*, then graph a two-variable linear equation
- Graph horizontal and vertical lines

- Graph an Equation Using Intercepts
  - Recognize when an ordered pair is a *y*-intercept or an *x*-intercept
  - Graph a linear equation using *x*– and *y*-intercepts

Graphing Using Ordered Pairs

Graphing ordered pairs is only the beginning of the story. Once you know how to place points on a grid, you can use them to make sense of all kinds of mathematical relationships.

You can use a **coordinate plane** to plot points and to map various relationships, such as the relationship between an object's distance and the elapsed time. Many mathematical relationships are **linear relationships**. Let's look at what a linear relationship is.

Plotting points to graph linear relationships

A linear relationship is a relationship between variables such that when plotted on a coordinate plane, the points lie on a line. Let's start by looking at a series of points in Quadrant I on the coordinate plane.

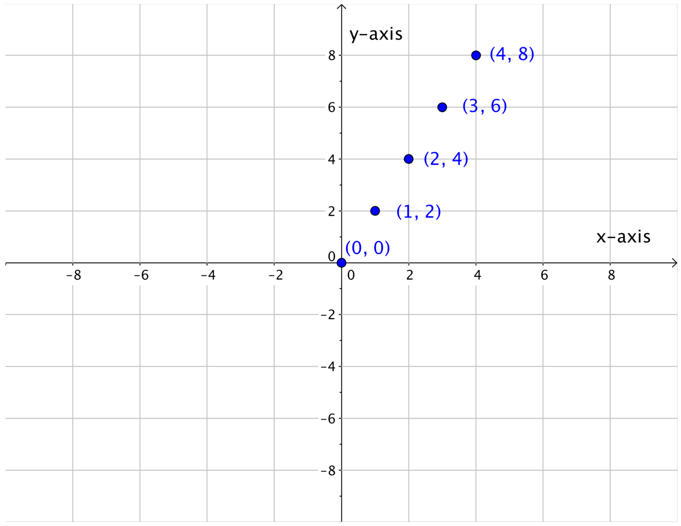
Look at the five **ordered pairs** (and their *x*– and *y*-coordinates) below. Do you see any pattern to the location of the points? If this pattern continued, what other points could be on the line?

You probably identified that if this pattern continued the next ordered pair would be at (5, 10). This makes sense because the point (5, 10) "lines up" with the other points in the series—it is literally on the same line as the others. Applying the same logic, you may identify that the ordered pairs (6, 12) and (7, 14) would also belong if this coordinate plane were larger; they, too, will line up with the other points.

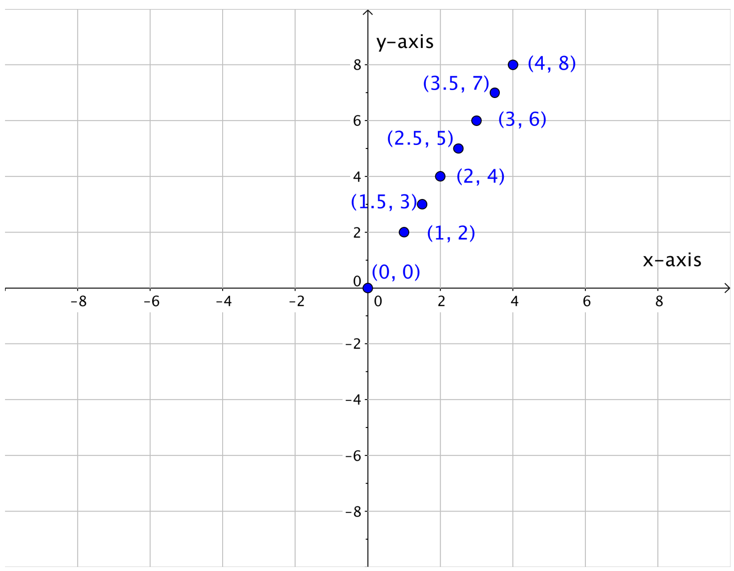
These series of points can also be represented in a table. In the table below, the *x*- and *y*-coordinates of each ordered pair on the graph is recorded.

***x*-coordinate**	***y*-coordinate**
0	0
1	2
2	4
3	6
4	8

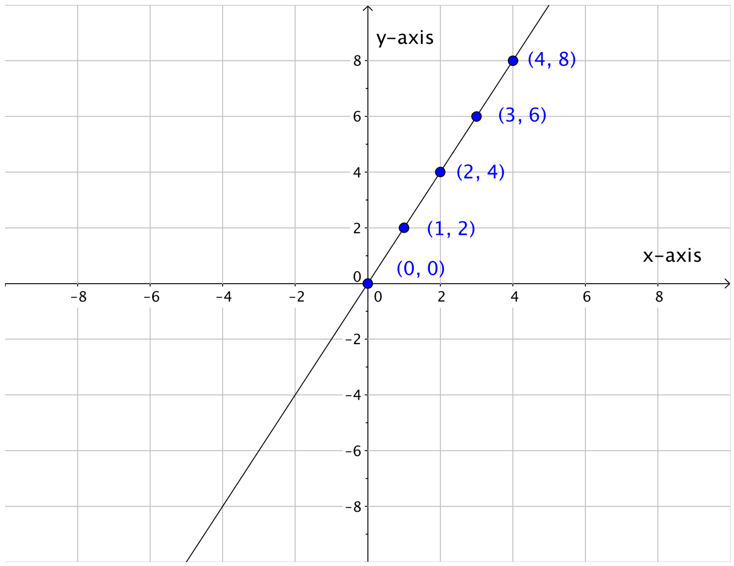
Notice that each *y*-coordinate is twice the corresponding *x*-value. All of these *x*- and *y*-values follow the same pattern, and, when placed on a coordinate plane, they all line up.



Once you know the pattern that relates the *x*- and *y*-values, you can find a *y*-value for any *x*-value that lies on the line. So if the rule of this pattern is that each *y*-coordinate is *twice* the corresponding *x*-value, then the ordered pairs (1.5, 3), (2.5, 5), and (3.5, 7) should all appear on the line too, correct? Look to see what happens.



If you were to keep adding ordered pairs (*x*, *y*) where the *y*-value was twice the *x*-value, you would end up with a graph like this.



Look at how all of the points blend together to create a line. You can think of a line, then, as a collection of an infinite number of individual points that share the same mathematical relationship. In this case, the relationship is that the *y*-value is twice the *x*-value.

There are multiple ways to represent a linear relationship—a table, a linear graph, and there is also a **linear equation**. A linear equation is an equation with two variables whose ordered pairs graph as a straight line.

There are several ways to create a graph from a linear equation. One way is to create a table of values for *x* and *y*, and then plot these ordered pairs on the coordinate plane. Two points are enough to determine a line. However, it's always a good idea to plot more than two points to avoid possible errors.

Then you draw a line through the points to show all of the points that are on the line. The arrows at each end of the graph indicate that the line continues endlessly in both directions. Every point on this line is a solution to the linear equation.

Example

Graph the linear equation $y = -1.5x$.
Show Solution

Graph the linear equation

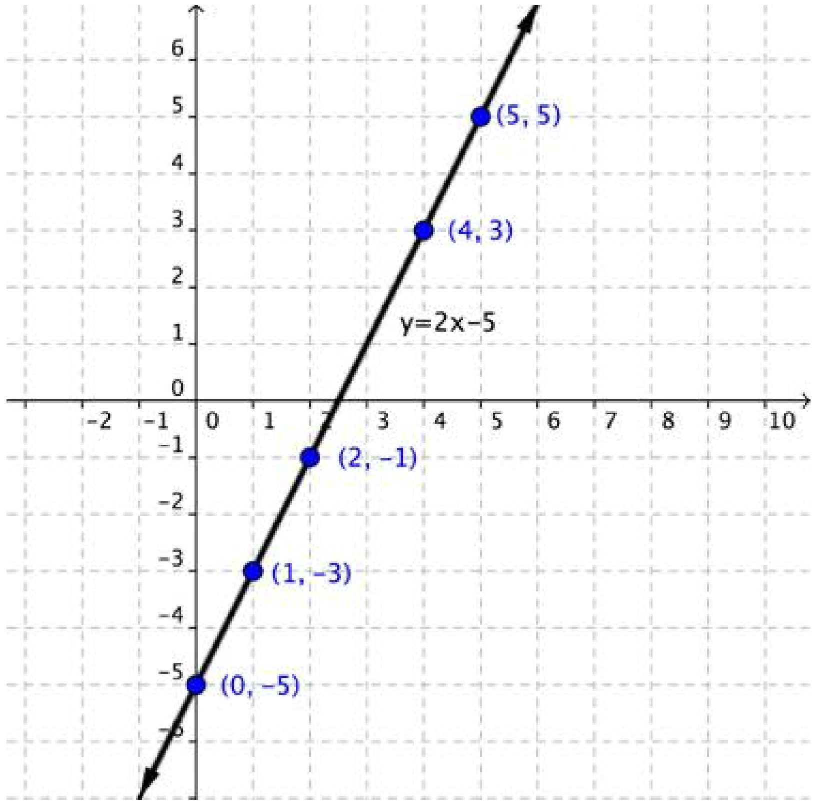
Watch this video online: https://youtu.be/f5yvGPEWpvE

Example

Graph the linear equation $y = 2x + 3$.
Show Solution

Ordered Pairs as Solutions

So far, you have considered the following ideas about lines: a line is a visual representation of a linear equation, and the line itself is made up of an infinite number of points (or ordered pairs). The picture below shows the line of the linear equation $y = 2x - 5$ with some of the specific points on the line.



Every point on the line is a solution to the equation $y = 2x{-}5$. You can try any of the points that are labeled like the ordered pair, $(1, -3)$.

$$\begin{array}{l} y = 2x - 5 \\ -3 = 2(1) - 5 \\ -3 = 2 - 5 \\ -3 = -3 \\ \text{This is true.} \end{array}$$

You can also try ANY of the other points on the line. Every point on the line is a solution to the equation $y = 2x{-}5$. All this means is that determining whether an ordered pair is a solution of an equation is pretty straightforward. If the ordered pair is on the line created by the linear equation, then it is a solution to the equation. But if the ordered pair is not on the line—no matter how close it may look—then it is not a solution to the equation.

Identifying Solutions

To find out whether an ordered pair is a solution of a linear equation, you can do the following:

- Graph the linear equation, and graph the ordered pair. If the ordered pair appears to be on the graph of a line, then it is a possible solution of the linear equation. If the ordered pair does not lie on the graph of a line, then it is not a solution.
- Substitute the (*x*, *y*) values into the equation. If the equation yields a true statement, then the ordered pair is a solution of the linear equation. If the ordered pair does not yield a true statement then it is not a solution.

Example

Determine whether $(-2, 4)$ is a solution to the equation $4y + 5x = 3$.
Show Solution

Determine If an Ordered Pair is a Solution to a Linear Equation

Watch this video online: https://youtu.be/9aWGxt7OnB8

Solve for *y*, then graph a linear equation

The linear equations we have graphed so far are in the form $y = mx + b$ where *m* and *b* are real numbers. In this section we will graph linear equations that appear in different forms than we have seen.

Example

Graph the linear equation $y + 3x = 5$.
Show Solution

Video: Solve for y, then graph a linear equation

Watch this video online: https://youtu.be/6yL3gfPbOt8

Horizontal and Vertical Lines

The linear equations $x = 2$ and $y = -3$ only have one variable in each of them. However, because these are linear equations, then they will graph on a coordinate plane just as the linear equations above do. Just think of the equation $x = 2$ as $x = 0y + 2$ and think of $y = -3$ as $y = 0x - 3$.

Example

Graph $y = -3$.
Show Solution

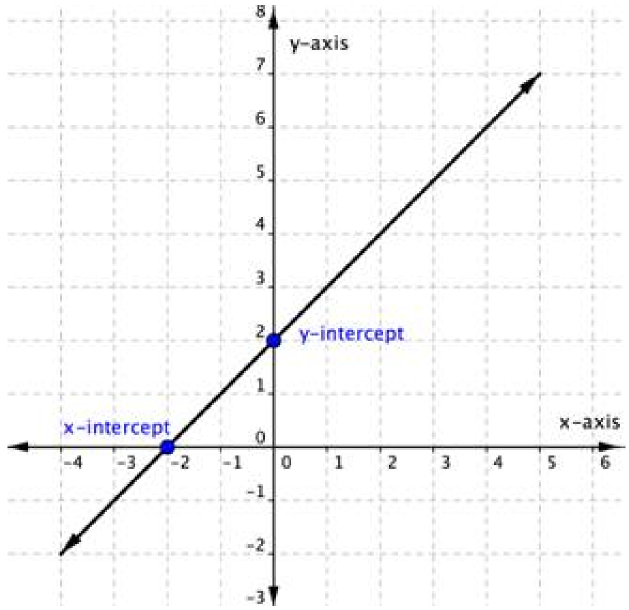
In the following video you will see more examples of graphing horizontal and vertical lines.

Watch this video online: https://youtu.be/2A2fhImjOBc

Intercepts

The intercepts of a line are the points where the line intercepts, or crosses, the horizontal and vertical axes. To help you remember what "intercept" means, think about the word "intersect." The two words sound alike and in this case mean the same thing.

The straight line on the graph below intercepts the two coordinate axes. The point where the line crosses the *x*-axis is called the ***x*-intercept**. The ***y*-intercept** is the point where the line crosses the *y*-axis.



The *x*-intercept above is the point $(-2, 0)$. The *y*-intercept above is the point (0, 2).

Notice that the *y*-intercept always occurs where $x = 0$, and the *x*-intercept always occurs where $y = 0$.

To find the *x*– and *y*-intercepts of a linear equation, you can substitute 0 for *y* and for *x* respectively.

For example, the linear equation $3y + 2x = 6$ has an *x* intercept when $y = 0$, so $3(0) + 2x = 6$.

$$\begin{aligned} 2x &= 6 \\ x &= 3 \end{aligned}$$

The *x*-intercept is $(3, 0)$.

Likewise the *y*-intercept occurs when $x = 0$.

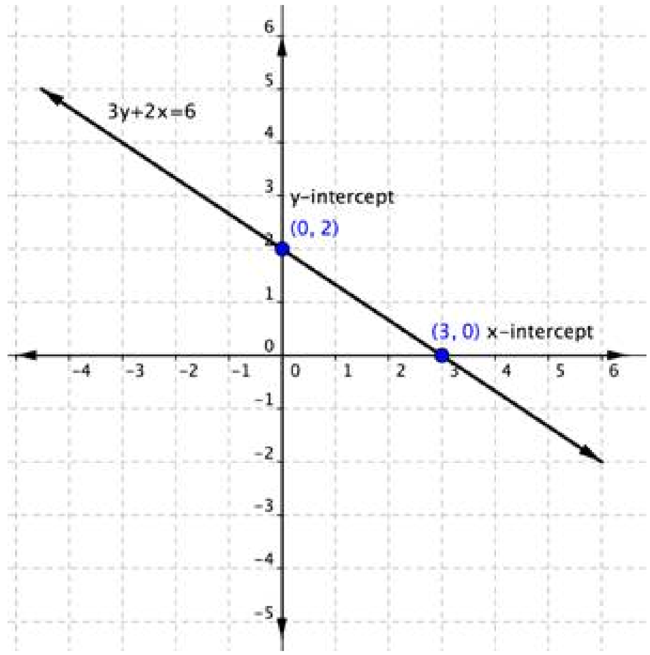
$$\begin{aligned} 3y + 2(0) &= 6 \\ 3y &= 6 \\ y &= 2 \end{aligned}$$

The *y*-intercept is $(0, 2)$.

Using Intercepts to Graph Lines

You can use intercepts to graph linear equations. Once you have found the two intercepts, draw a line through them.

Let's do it with the equation $3y + 2x = 6$. You figured out that the intercepts of the line this equation represents are $(0, 2)$ and $(3, 0)$. That's all you need to know.



Example

Graph $5y+3x=30$ using the *x* and *y*-intercepts.
Show Solution

Watch this video online: https://youtu.be/k8r-q_T6UFk

Example

Graph $y=2x-4$ using the *x* and *y*-intercepts.
Show Solution

Summary

The coordinate plane is a system for graphing and describing points and lines. The coordinate plane is comprised of a horizontal (*x*-) axis and a vertical (*y*-) axis. The intersection of these lines creates the origin, which is the point $(0,0)$. The coordinate plane is split into four quadrants. Together, these features of the coordinate system allow for the graphical representation and communication about points, lines, and other algebraic concepts.

SLOPE OF A LINE

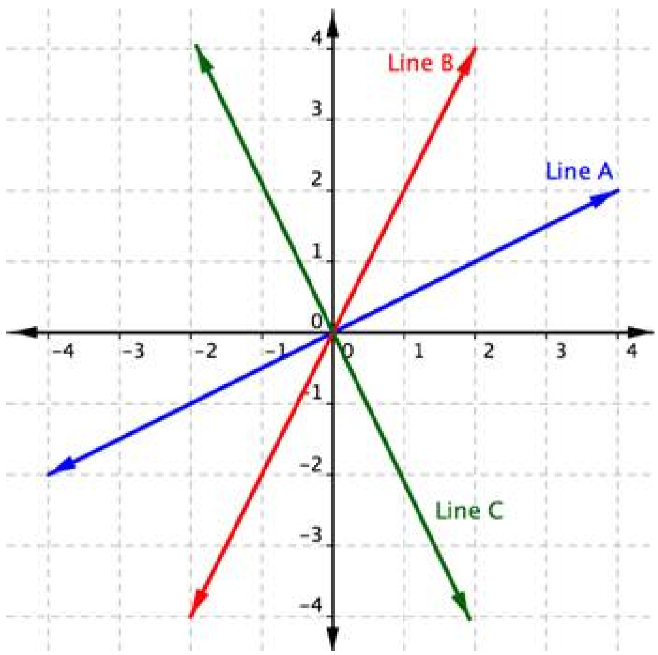
Learning Objectives

- Find the Slope from a Graph

- Identify rise and run from a graph
    - Distinguish between graphs of lines with negative and positive slopes
- Find the Slope from Two Points
    - Use the formula for slope to define the slope of a line through two points
- Find the Slope of Horizontal and Vertical Lines
    - Find the slope of the lines $x = a$ and $y = b$
    - Recognize that horizontal lines have slope = 0
    - Recognize that vertical lines have slopes that are undefined
- Identify slopes of parallel and perpendicular lines
    - Given a line, identify the slope of another line that is parallel to it
    - Given a line, identify the slope of another line that is perpendicular to it
- Interpret slope in equations and graphs
    - Verify the slope of a linear equation given a dataset
    - Interpret the slope of a linear equation as it applies to a real situation

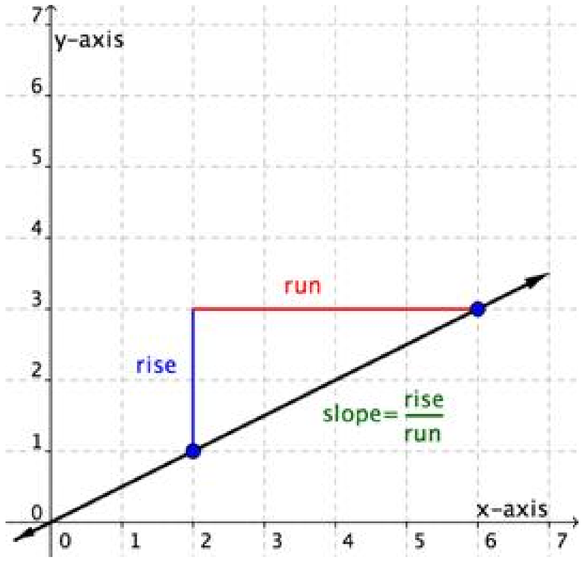
Identify slope from a graph

The mathematical definition of **slope** is very similar to our everyday one. In math, slope is used to describe the steepness and direction of lines. By just looking at the graph of a line, you can learn some things about its slope, especially relative to other lines graphed on the same coordinate plane. Consider the graphs of the three lines shown below:



First, let’s look at lines A and B. If you imagined these lines to be hills, you would say that line B is steeper than line A. Line B has a greater slope than line A.

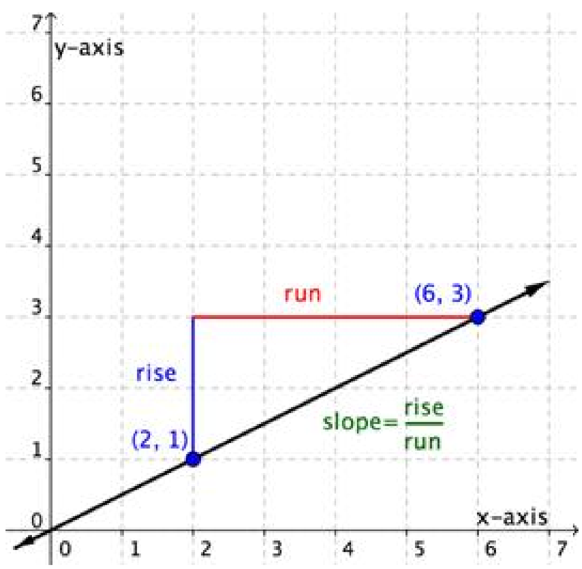
Next, notice that lines A and B slant up as you move from left to right. We say these two lines have a positive slope. Line C slants down from left to right. Line C has a negative slope. Using two of the points on the line, you can find the slope of the line by finding the rise and the run. The vertical change between two points is called the **rise**, and the horizontal change is called the **run**. The slope equals the rise divided by the run: $\text{Slope } = \dfrac{\text{rise}}{\text{run}}$.



You can determine the slope of a line from its graph by looking at the rise and run. One characteristic of a line is that its slope is constant all the way along it. So, you can choose any 2 points along the graph of the line to figure out the slope. Let’s look at an example.

Example

Use the graph to find the slope of the line.



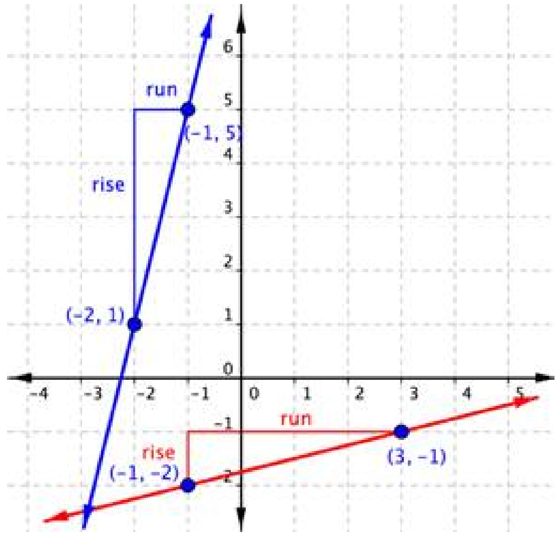
Show Solution

This line will have a slope of $\frac{1}{2}$ no matter which two points you pick on the line. Try measuring the slope from the origin, $(0,0)$, to the point $(6,3)$. You will find that the $\text{rise} = 3$ and the $\text{run} = 6$. The slope is $\frac{\text{rise}}{\text{run}} = \frac{3}{6} = \frac{1}{2}$. It is the same!

Let’s look at another example.

Example

Use the graph to find the slope of the two lines.



Show Solution

When you look at the two lines, you can see that the blue line is steeper than the red line. It makes sense the value of the slope of the blue line, 4, is greater than the value of the slope of the red line, $\frac{1}{4}$. The greater the slope, the steeper the line.

Finding the Slope of a Line From a Graph

Watch this video online: https://youtu.be/29BpBqsiE5w

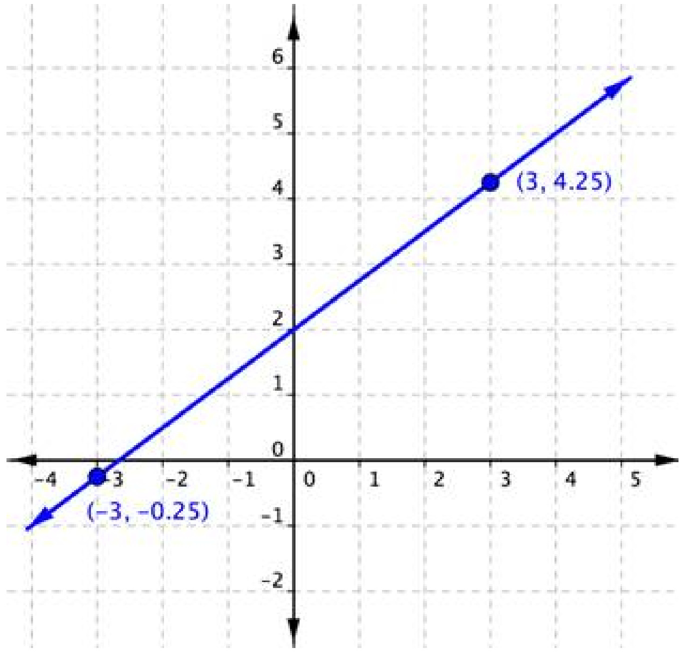
Distinguish between graphs of lines with negative and positive slopes

Direction is important when it comes to determining slope. It's important to pay attention to whether you are moving up, down, left, or right; that is, if you are moving in a positive or negative direction. If you go up to get to your second point, the rise is positive. If you go down to get to your second point, the rise is negative. If you go right to get to your second point, the run is positive. If you go left to get to your second point, the run is negative.

In the following two examples, you will see a slope that is positive and one that is negative.

Example (Advanced)

Find the slope of the line graphed below.

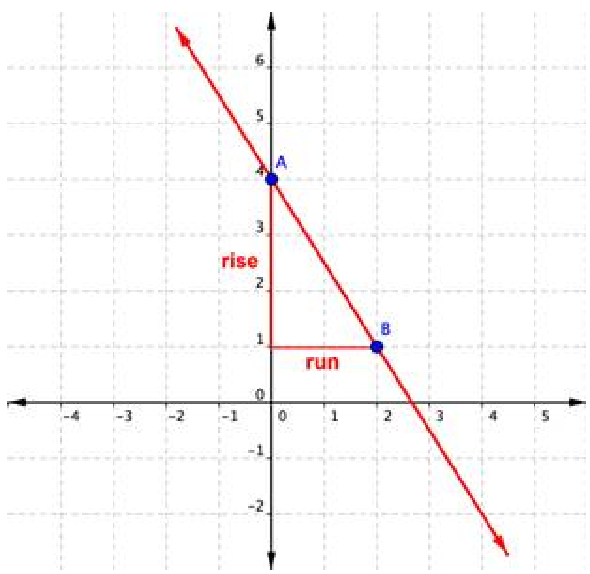


Show Solution

The next example shows a line with a negative slope.

Example

Find the slope of the line graphed below.



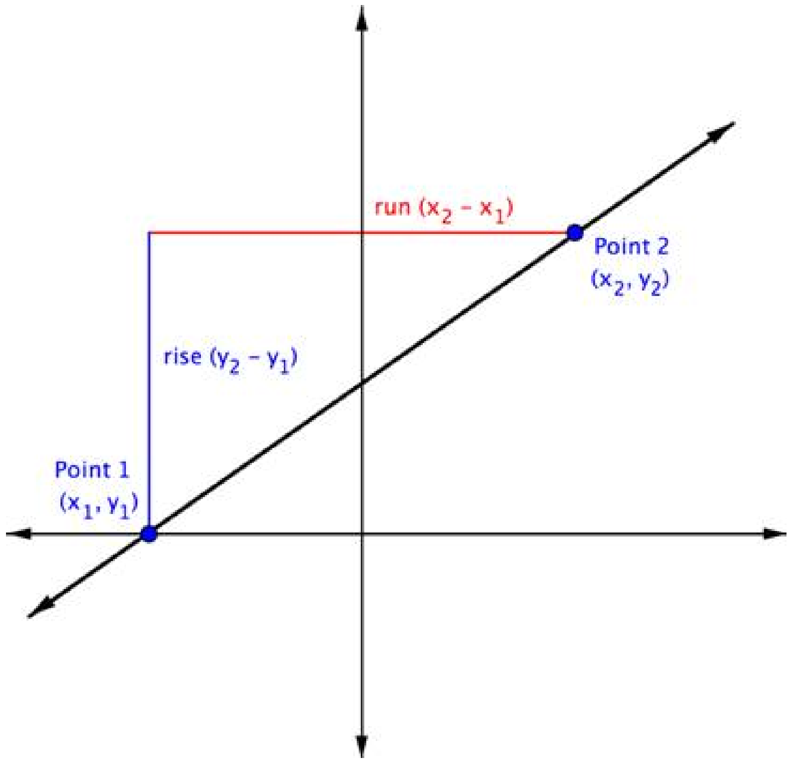
Show Solution

In the example above, you could have found the slope by starting at point B, running -2, and then rising $+3$ to arrive at point A. The result is still a slope of $\frac{\text{rise}}{\text{run}} = \frac{+3}{-2} = -\frac{3}{2}$.

Finding the Slope from Two Points on the Line

You've seen that you can find the slope of a line on a graph by measuring the rise and the run. You can also find the slope of a straight line without its graph if you know the coordinates of any two points on that line. Every point has a set of coordinates: an *x*-value and a *y*-value, written as an ordered pair (*x*, *y*). The *x* value tells you where a point is horizontally. The *y* value tells you where the point is vertically.

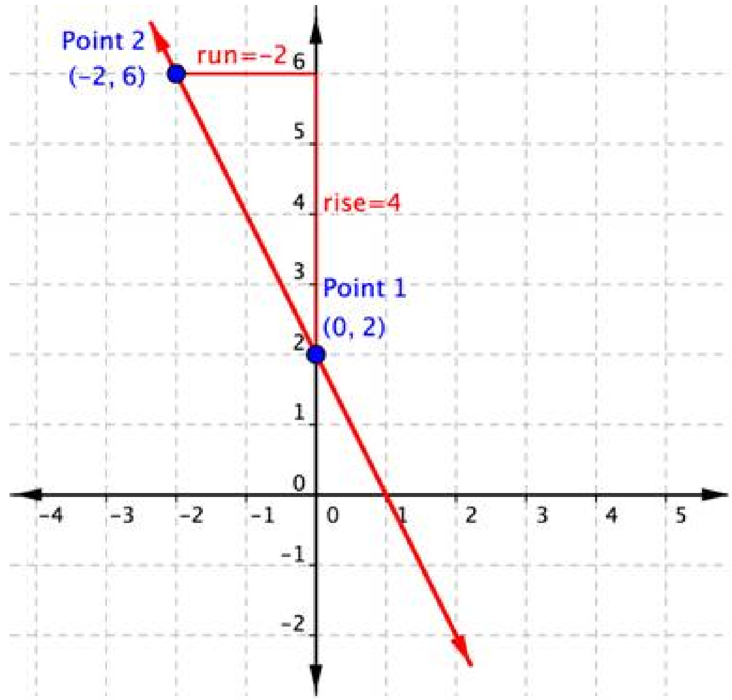
Consider two points on a line—Point 1 and Point 2. Point 1 has coordinates (x_1, y_1) and Point 2 has coordinates (x_2, y_2).



The rise is the vertical distance between the two points, which is the difference between their *y*-coordinates. That makes the rise $(y_2 - y_1)$. The run between these two points is the difference in the *x*-coordinates, or $(x_2 - x_1)$.

So, $\text{Slope} = \dfrac{\text{rise}}{\text{run}}$ or $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

In the example below, you'll see that the line has two points each indicated as an ordered pair. The point $(0, 2)$ is indicated as Point 1, and $(-2, 6)$ as Point 2. So you are going to move from Point 1 to Point 2. A triangle is drawn in above the line to help illustrate the rise and run.



You can see from the graph that the rise going from Point 1 to Point 2 is 4, because you are moving 4 units in a positive direction (up). The run is -2, because you are then moving in a negative direction (left) 2 units. Using the slope formula,

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{4}{-2} = -2.$$

You do not need the graph to find the slope. You can just use the coordinates, keeping careful track of which is Point 1 and which is Point 2. Let's organize the information about the two points:

Name	Ordered Pair	Coordinates
Point 1	$(0, 2)$	$x_1 = 0$ $y_1 = 2$
Point 2	$(-2, 6)$	$x_2 = -2$ $y_2 = 6$

The slope, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{-2-0} = \frac{4}{-2} = -2$. The slope of the line, *m*, is -2.

It doesn't matter which point is designated as Point 1 and which is Point 2. You could have called $(-2, 6)$ Point 1, and $(0, 2)$ Point 2. In that case, putting the coordinates into the slope formula produces the equation $m = \frac{2-6}{0-(-2)} = \frac{-4}{2} = -2$. Once again, the slope is $m = -2$. That's the same slope as before. The important thing is to be consistent when you subtract: you must always subtract in the same order (y_2, y_1) and (x_2, x_1).

Example

What is the slope of the line that contains the points $(5, 5)$ and $(4, 2)$?
Show Solution

The example below shows the solution when you reverse the order of the points, calling $(5, 5)$ Point 1 and $(4, 2)$ Point 2.

Example

What is the slope of the line that contains the points $(5, 5)$ and $(4, 2)$?
Show Solution

Notice that regardless of which ordered pair is named Point 1 and which is named Point 2, the slope is still 3.

Example (Advanced)

What is the slope of the line that contains the points $(3, -6.25)$ and $(-1, 8.5)$?
Show Solution

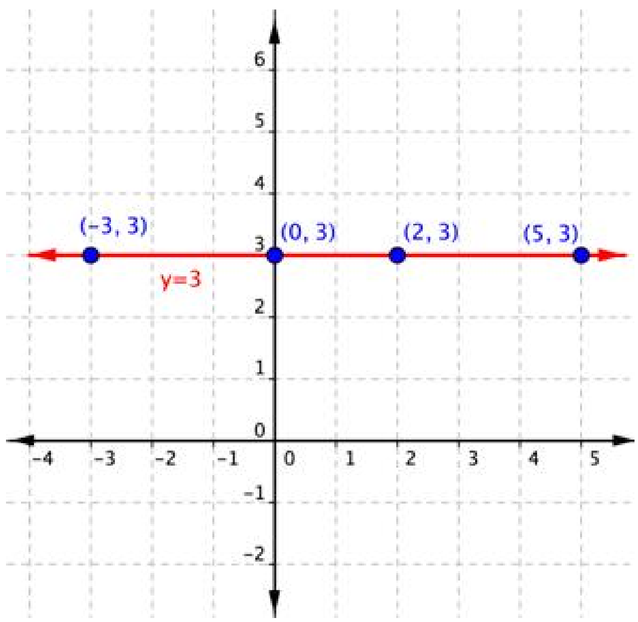
Let's consider a horizontal line on a graph. No matter which two points you choose on the line, they will always have the same *y*-coordinate. The equation for this line is $y = 3$. The equation can also be written as $y = (0)\,x + 3$.

Video: Finding the Slope of a Line Given Two Points on the Line

Watch this video online: https://youtu.be/ZW7rQa8SJSU

Finding the Slopes of Horizontal and Vertical Lines

So far you've considered lines that run "uphill" or "downhill." Their slopes may be steep or gradual, but they are always positive or negative numbers. But there are two other kinds of lines, horizontal and vertical. What is the slope of a flat line or level ground? Of a wall or a vertical line?



Using the form $y = 0x + 3$, you can see that the slope is 0. You can also use the slope formula with two points on this horizontal line to calculate the slope of this horizontal line. Using $(-3, 3)$ as Point 1 and (2, 3) as Point 2, you get:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

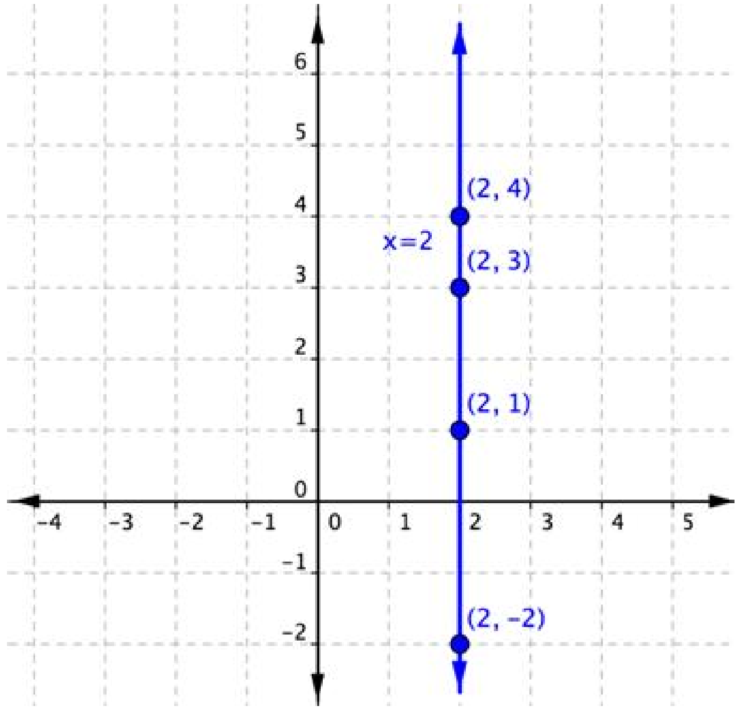
$$m = \frac{3-3}{2-(-3)} = \frac{0}{5} = 0$$

The slope of this horizontal line is 0.

Let's consider any horizontal line. No matter which two points you choose on the line, they will always have the same *y*-coordinate. So, when you apply the slope formula, the numerator will always be 0. Zero divided by any non-zero number is 0, so the slope of any horizontal line is always 0.

The equation for the horizontal line $y = 3$ is telling you that no matter which two points you choose on this line, the *y*-coordinate will always be 3.

How about vertical lines? In their case, no matter which two points you choose, they will always have the same *x*-coordinate. The equation for this line is $x = 2$.



There is no way that this equation can be put in the slope-point form, as the coefficient of *y* is $0\,(x = 0y + 2)$.

So, what happens when you use the slope formula with two points on this vertical line to calculate the slope? Using $(2, 1)$ as Point 1 and $(2, 3)$ as Point 2, you get:

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{3-1}{2-2} = \frac{2}{0}$

But division by zero has no meaning for the set of real numbers. Because of this fact, it is said that the slope of this vertical line is undefined. This is true for all vertical lines—they all have a slope that is undefined.

Example

What is the slope of the line that contains the points $(3, 2)$ and $(-8, 2)$?
Show Solution

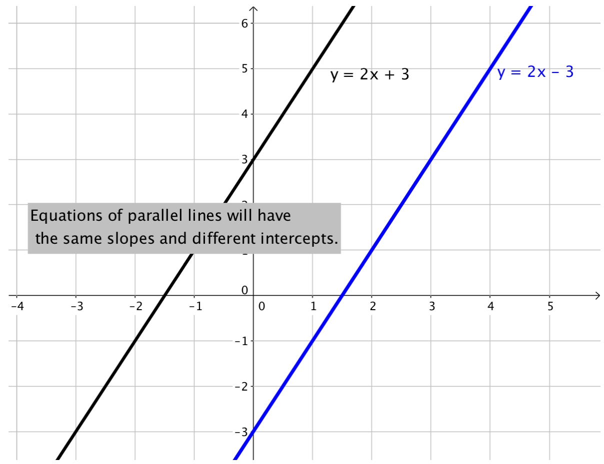
Finding Slopes of Horizontal and Vertical Lines

Watch this video online: https://youtu.be/zoLM3rxzndo

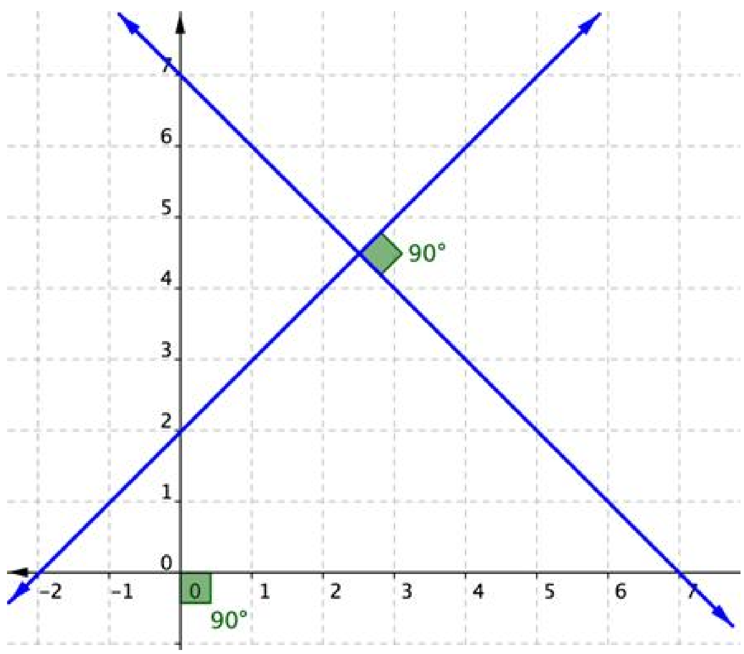
Characterize the slopes of parallel and perpendicular lines

When you graph two or more linear equations in a coordinate plane, they generally cross at a point. However, when two lines in a coordinate plane never cross, they are called **parallel lines**. You will also look at the case where two lines in a coordinate plane cross at a right angle. These are called **perpendicular lines**. The slopes of the graphs in each of these cases have a special relationship to each other.

Parallel lines are two or more lines in a plane that never intersect. Examples of parallel lines are all around us, such as the opposite sides of a rectangular picture frame and the shelves of a bookcase.



Perpendicular lines are two or more lines that intersect at a 90-degree angle, like the two lines drawn on this graph. These 90-degree angles are also known as right angles.



Perpendicular lines are also everywhere, not just on graph paper but also in the world around us, from the crossing pattern of roads at an intersection to the colored lines of a plaid shirt.

Parallel Lines

Two non-vertical lines in a plane are parallel if they have both:

- the same slope
- different *y*-intercepts

Any two vertical lines in a plane are parallel.

Example

Find the slope of a line parallel to the line $y = -3x + 4$.
Show Solution

Example

Determine whether the lines $y = 6x + 5$ and $y = 6x - 1$ are parallel.
Show Solution

Perpendicular Lines

Two non-vertical lines are perpendicular if the slope of one is the negative reciprocal of the slope of the other. If the slope of the first equation is 4, then the slope of the second equation will need to be $-\frac{1}{4}$ for the lines to be perpendicular.

You can also check the two slopes to see if the lines are perpendicular by multiplying the two slopes together. If they are perpendicular, the product of the slopes will be -1. For example, $4 \cdot -\frac{1}{4} = \frac{4}{1} \cdot -\frac{1}{4} = -1$.

Example

Find the slope of a line perpendicular to the line $y = 2x - 6$.
Show Solution

To find the slope of a perpendicular line, find the reciprocal, $\frac{1}{2}$, and then find the opposite of this reciprocal $-\frac{1}{2}$.

Note that the product $2\left(-\frac{1}{2}\right) = \frac{2}{1}\left(-\frac{1}{2}\right) = -1$, so this means the slopes are perpendicular.

In the case where one of the lines is vertical, the slope of that line is undefined and it is not possible to calculate the product with an undefined number. When one line is vertical, the line perpendicular to it will be horizontal, having a slope of zero $(m = 0)$.

Example

Determine whether the lines $y = -8x + 5$ and $y = \frac{1}{8}x - 1$ are parallel, perpendicular, or neither.
Show Solution

The Slope of Parallel and Perpendicular Lines

Watch this video online: https://youtu.be/Ily4N2lAkDs

Verify Slope From a Dataset

Massive amounts of data is being collected every day by a wide range of institutions and groups. This data is used for many purposes including business decisions about location and marketing, government decisions about allocation of resources and infrastructure, and personal decisions about where to live or where to buy food.

In the following example, you will see how a dataset can be used to define the slope of a linear equation.

Example

Given the dataset, verify the values of the slopes of each equation.
Linear equations describing the change in median home values between 1950 and 2000 in Mississippi and Hawaii are as follows:
Hawaii: $y = 3966x + 74,400$
Mississippi: $y = 924x + 25,200$
The equations are based on the following dataset.
x = the number of years since 1950, and y = the median value of a house in the given state.

Year (*x*)	**Mississippi House Value (*y*)**	**Hawaii House Value (*y*)**
0	$25,200	$74,400
50	$71,400	$272,700

The slopes of each equation can be calculated with the formula you learned in the section on slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Mississippi:

Name	**Ordered Pair**	**Coordinates**
Point 1	(0, 25,200)	$x_1 = 0$ $y_1 = 25,200$
Point 2	(50, 71,400)	$x_2 = 50$ $y_2 = 71,400$

$$m = \frac{71,400 - 25,200}{50 - 0} = \frac{46,200}{50} = 924$$

We have verified that the slope $m = 924$ matches the dataset provided.
Hawaii:

Name	**Ordered Pair**	**Coordinates**
Point 1	(0, 74,400)	$x_1 = 1950$ $y_1 = 74,400$
Point 2	(50, 272,700)	$x_2 = 2000$ $y_2 = 272,700$

$$m = \frac{272,700 - 74,400}{50 - 0} = \frac{198,300}{50} = 3966$$

We have verified that the slope $m = 3966$ matches the dataset provided.

Example

Given the dataset, verify the values of the slopes of the equation.
A linear equation describing the change in the number of high school students who smoke, in a group of 100, between 2011 and 2015 is given as:
$y = -1.75x + 16$
And is based on the data from this table, provided by the Centers for Disease Control.
x = the number of years since 2011, and y = the number of high school smokers per 100 students.

Year	Number of High School Students Smoking Cigarettes (per 100)

0	16
4	9

Name	Ordered Pair	Coordinates
Point 1	(0, 16)	$x_1 = 0$ $y_1 = 16$
Point 2	(4, 9)	$x_2 = 4$ $y_2 = 9$

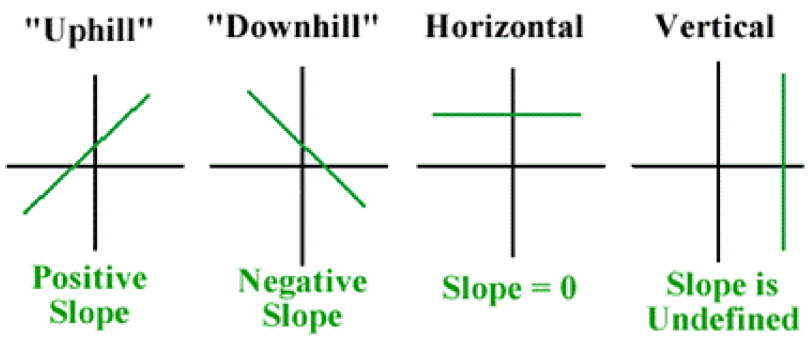
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 16}{4 - 0} = \frac{-7}{4} = -1.75$

We have verified that the slope $m = -1.75$ matches the dataset provided.

Interpret the Slope of Linear Equation

Okay, now we have verified that data can provide us with the slope of a linear equation. So what? We can use this information to describe how something changes using words.

First, let's review the different kinds of slopes possible in a linear equation.



We often use specific words to describe the different types of slopes when we are using lines and equations to represent "real" situations. The following table pairs the type of slope with the common language used to describe it both verbally and visually.

Type of Slope	Visual Description	Verbal Description
positive	uphill	increasing
negative	downhill	decreasing
0	horizontal	constant
undefined	vertical	N/A

Example

Interpret the slope of each equation for house values using words.
Hawaii: $y = 3966x + 74,400$
Mississippi: $y = 924x + 25,200$
Show Solution

Interpret the Meaning of the Slope Given a Linear Equation—Median Home Values

Watch this video online: https://youtu.be/JT0WX5KOkJ8

Example

Interpret the slope of the line describing the change in the number of high school smokers using words.
Apply units to the formula for slope. The *x* values represent years, and the *y* values represent the number of smokers. Remember that this dataset is per 100 high school students.

$$m = \frac{9-16}{2015-2011} = \frac{-7 \text{ smokers}}{4 \text{ year}} = -1.75\frac{\text{smokers}}{\text{year}}$$

The slope of this linear equation is **negative**, so this tells us that there is a **decrease** in the number of high school age smokers each year.
The number of high schoolers that smoke decreases by 1.75 per 100 each year.

Interpret the Meaning of the Slope of a Linear Equation—Smokers

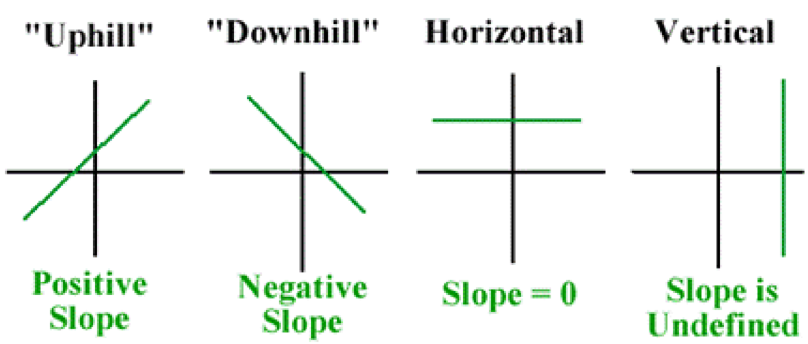
Watch this video online: https://youtu.be/aHLw5FcMjdc

On the next page, we will see how to interpret the *y*-intercept of a linear equation, and make a prediction based on a linear equation.

Summary

Slope describes the steepness of a line. The slope of any line remains constant along the line. The slope can also tell you information about the direction of the line on the coordinate plane. Slope can be calculated either by looking at the graph of a line or by using the coordinates of any two points on a line. There are two common formulas for slope: $\text{Slope } = \frac{\text{rise}}{\text{run}}$ and $m = \frac{y_2 - y_1}{x_2 - x_1}$ where $m = \text{slope}$ and (x_1, y_1) and (x_2, y_2) are two points on the line.

The images below summarize the slopes of different types of lines.



WRITING EQUATIONS OF LINES

Learning Objectives

- Write the equation and draw the graph of a line using slope and y-intercept
  - Write the equation of a line using slope and y-intercept
  - Rearrange a linear equation so it is in slope-intercept form.
  - Graph a line using slope and y-intercept
- Write and solve equations of lines using slope and a point on the line
  - Write the equation of a line given the slope and a point on the line.
  - Identify which parts of a linear equation are given and which parts need to be solved for using algebra
- Write and solve equations of lines using two points on the line
  - Write the equation of a line given two points on the line
  - Identify which parts of a linear equation are given and which parts need to be solved for using algebra.
- Write equations of parallel and perpendicular lines
  - Find a line that is parallel to another line given a point
  - Find a line that is perpendicular to another line given a point
- Interpret the y-intercept of a linear equation and use that equation to make predictions
  - Interpret the *y*-intercept of a linear equation
  - Use a linear equation to make a prediction

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the *y*-intercept of the equation. The slope can be represented by m and the *y*-intercept, where it crosses the axis and $x = 0$, can be represented by $(0, b)$ where *b* is the value where the graph crosses the vertical *y*-axis. Any other point on the line can be represented by (x, y).

In the equation,

$y = mx + b$

$$\begin{array}{rcl} m & = & \text{slope} \\ (x,y) & = & \text{a point on the line} \\ b & = & \text{the y value of the y-intercept} \end{array}$$

This formula is known as the slope-intercept equation. If we know the slope and the *y*-intercept we can easily find the equation that represents the line

Example

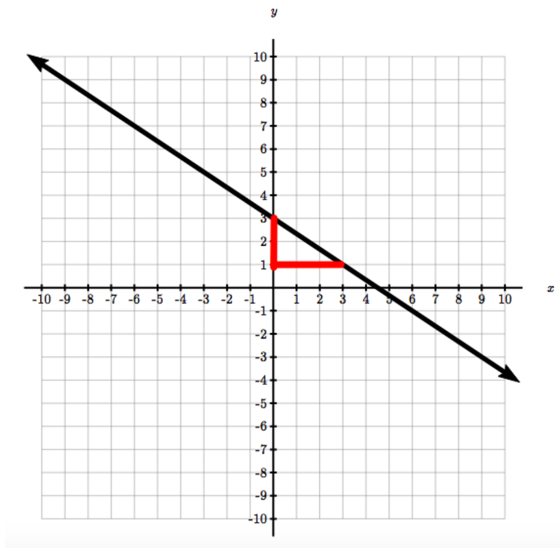
Write the equation of the line that has a slope of $\dfrac{1}{2}$ and a *y*-intercept of -5.

Show Solution

We can also find the equation by looking at a graph and finding the slope and *y*-intercept.

Example

Write the equation of the line in the graph by identifying the slope and *y*-intercept.



Show Solution

We can also move the opposite direction, using the equation identify the slope and *y*-intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for *y* so we can identify the slope and the *y*-intercept.

Example

Write the following equation in slope-intercept form.

$2x + 4y = 6$
Show Solution

Once we have an equation in slope-intercept form we can graph it by first plotting the *y*-intercept, then using the slope, find a second point and connecting the dots.

Example

Graph $y = \frac{1}{2}x - 4$ using the slope-intercept equation.
Show Solution

Slope-Intercept Form of a Line

Watch this video online: https://youtu.be/GIn7vbB5AYo

Find the Equation of a Line Given the Slope and a Point on the Line

Using the slope-intercept equation of a line is possible when you know both the slope (*m*) and the *y*-intercept (*b*), but what if you know the slope and just any point on the line, not specifically the *y*-intercept? Can you still write the equation? The answer is *yes*, but you will need to put in a little more thought and work than you did previously.

Recall that a point is an (*x*, *y*) coordinate pair and that all points on the line will satisfy the linear equation. So, if you have a point on the line, it must be a solution to the equation. Although you don't know the exact equation yet, you know that you can express the line in slope-intercept form, $y = mx + b$.

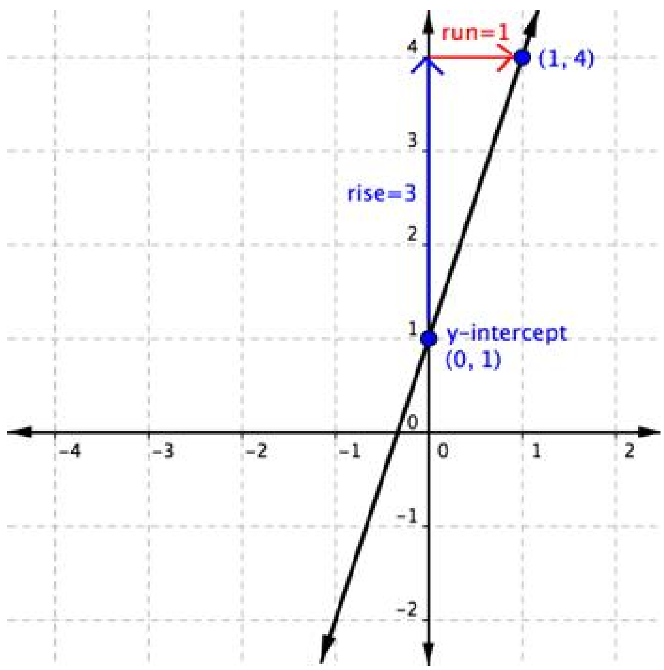
You do know the slope (*m*), but you just don't know the value of the *y*-intercept (*b*). Since point (*x*, *y*) is a solution to the equation, you can substitute its coordinates for *x* and *y* in $y = mx + b$ and solve to find *b*!

This may seem a bit confusing with all the variables, but an example with an actual slope and a point will help to clarify.

Example

Write the equation of the line that has a slope of 3 and contains the point $(1, 4)$.
Show Solution

To confirm our algebra, you can check by graphing the equation $y = 3x + 1$. The equation checks because when graphed it passes through the point $(1, 4)$.

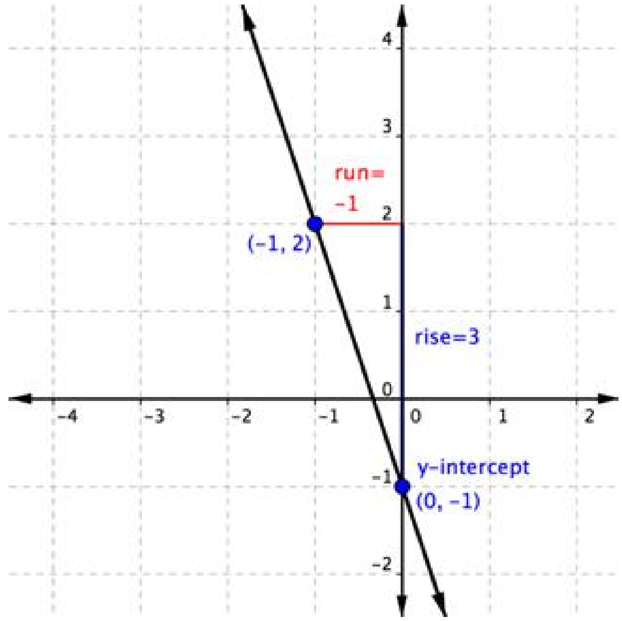


If you know the slope of a line and a point on the line, you can draw a graph. Using an equation in the point-slope form allows you to identify the slope and a point. Consider the equation $y = -3x - 1$. The *y*-intercept is the point on the line where it passes through the *y*-axis. What is the value of *x* at this point?

Reminder: All *y*-intercepts are points in the form $(0, y)$. The *x* value of any *y*-intercept is *always* zero.

Therefore, you can tell from this equation that the *y*-intercept is at $(0, -1)$, check this by replacing *x* with 0 and solving for *y*. To graph the line, start by plotting that point, $(0, -1)$, on a graph.

You can also tell from the equation that the slope of this line is -3. So start at $(0, -1)$ and count up 3 and over -1 (1 unit in the negative direction, left) and plot a second point. (You could also have gone down 3 and over 1.) Then draw a line through both points, and there it is, the graph of $y = -3x - 1$.



Example (Advanced)

Write the equation of the line that has a slope of $\frac{7}{8}$ and contains the point $\left(4, \frac{5}{4}\right)$.
Show Solution

Video: Find the Equation of a Line Given the Slope and a Point on the Line

Watch this video online: https://youtu.be/URYnKqEctgc

Find the Equation of a Line Given Two Points on the Line

Let's suppose you don't know either the slope or the *y*-intercept, but you do know the location of two points on the line. It is more challenging, but you can find the equation of the line that would pass through those two points. You will again use slope-intercept form to help you.

The slope of a linear equation is always the same, no matter which two points you use to find the slope. Since you have two points, you can use those points to find the slope (*m*). Now you have the slope and a point on the line! You can now substitute values for *m*, *x*, and *y* into the equation $y = mx + b$ and find *b*.

Example

Write the equation of the line that passes through the points $(2, 1)$ and $(-1, -5)$.
Show Solution

Notice that is doesn't matter which point you use when you substitute and solve for *b*—you get the same result for *b* either way. In the example above, you substituted the coordinates of the point (2, 1) in the equation $y = 2x + b$. Let's start with the same equation, $y = 2x + b$, but substitute in $(-1, -5)$:

$$
\begin{aligned}
y &= 2x + b \\
-5 &= 2(-1) + b \\
-5 &= -2 + b \\
-3 &= b
\end{aligned}
$$

The final equation is the same: $y = 2x - 3$.

Example (Advanced)

Write the equation of the line that passes through the points $(-4.6, 6.45)$ and $(1.15, 7.6)$.
Show Solution

Video: Find the Equation of a Line Given Two Points on the Line

Watch this video online: https://youtu.be/P1ex_a6iYDo

Write the equations of parallel and perpendicular lines

The relationships between slopes of parallel and perpendicular lines can be used to write equations of parallel and perpendicular lines.

Let's start with an example involving parallel lines.

Example

Write the equation of a line that is parallel to the line $x - y = 5$ and goes through the point $(-2, 1)$.
Show Solution

Determine the Equation of a Line Parallel to Another Line Through a Given Point

Watch this video online: https://youtu.be/TQKz2XHI09E

Determine the Equation of a Line Perpendicular to Another Line Through a Given Point

When you are working with perpendicular lines, you will usually be given one of the lines and an additional point. Remember that two non-vertical lines are perpendicular if the slope of one is the negative reciprocal of the slope of the other. To find the slope of a perpendicular line, find the reciprocal, and then find the opposite of this reciprocal. In other words, flip it and change the sign.

Example

Write the equation of a line that contains the point $(1, 5)$ and is perpendicular to the line $y = 2x - 6$.
Show Solution

Determine the Equation of a Line Perpendicular to a Line in Slope-Intercept Form

Watch this video online: https://youtu.be/QtvtzKjtowA

Example

Write the equation of a line that is parallel to the line $y = 4$ through the point $(0, 10)$.
Show Solution

Example

Write the equation of a line that is perpendicular to the line $y = -3$ through the point $(-2, 5)$.
Show Solution

Find the Equation of a Perpendicular and Horizontal Line to a Horizontal Line

Watch this video online: https://youtu.be/Qpn3f3wMeIs

Interpret the *y*-intercept of a linear equation

Often, when the line in question represents a set of data or observations, the *y*-intercept can be interpreted as a starting point. We will continue to use the examples for house value in Mississippi and Hawaii and high school smokers to interpret the meaning of the *y*-intercept in those equations.

Example

Recall the equations and data for house value:
Linear equations describing the change in median home values between 1950 and 2000 in Mississippi and Hawaii are as follows:
Hawaii: $y = 3966x + 74,400$
Mississippi: $y = 924x + 25,200$
The equations are based on the following dataset.
x = the number of years since 1950, and y = the median value of a house in the given state.

Year (*x*)	**Mississippi House Value (*y*)**	**Hawaii House Value (*y*)**
0	$25,200	$74,400
50	$71,400	$272,700

And the equations and data for high school smokers:
A linear equation describing the change in the number of high school students who smoke, in a group of 100, between 2011 and 2015 is given as:
$y = -1.75x + 16$
And is based on the data from this table, provided by the Centers for Disease Control.
x = the number of years since 2011, and y = the number of high school smokers per 100 students.

Year	Number of High School Students Smoking Cigarettes (per 100)
0	16
4	9

Also recall that the equation of a line in slope-intercept form is as follows:
$y = mx + b$

$$
\begin{aligned}
m &= \text{slope} \\
(x, y) &= \text{a point on the line} \\
b &= \text{the y value of the y-intercept}
\end{aligned}
$$

The examples that follow show how to interpret the y-intercept of the equations used to model house value and the number of high school smokers. Additionally, you will see how to use the equations to make predictions about house value and the number of smokers in future years.

Example

Interpret the *y*-intercepts of the equations that represent the change in house value for Hawaii and Mississippi.
Hawaii: $y = 3966x + 74,400$
Mississippi: $y = 924x + 25,200$
The *y*-intercept of a two-variable linear equation can be found by substituting 0 in for x.

Hawaii

$y = 3966x + 74,400$

$y = 3966(0) + 74,400$

$y = 74,400$

The *y*-intercept is a point, so we write it as (0, 74,400). Remember that *y*-values represent dollars and *x* values represent years. When the year is 0—in this case 0 because that is the first date we have in the dataset—the price of a house in Hawaii was $74,400.

Mississippi

$y = 924x + 25,200$

$y = 924(0) + 25,200$

$y = 25,200$

The *y*-intercept is (0, 25,200). This means that in 1950 the value of a house in Mississippi was $25,200. Remember that *x* represents the number of years since 1950, so if $x = 0$ the year is 1950.

Example

Interpret the y-intercept of the equation that represents the change in the number of high school students who smoke out of 100.
Substitute 0 in for *x*.

$y = -1.75x + 16$

$y = -1.75(0) + 16$

$y = 16$

The y-intercept is $(0, 16)$. The data starts at 2011, so we represent that year as 0. We can interpret the *y*-intercept as follows:
In the year 2011, 16 out of every 100 high school students smoked.

In the following video you will see an example of how to interpret the y- intercept given a linear equation that represents a set of data.

Watch this video online: https://youtu.be/Yhtl28DRqfU

Use a linear equation to make a prediction

Another useful outcome we gain from writing equations from data is the ability to make predictions about what may happen in the future. We will continue our analysis of the house price and high school smokers. In the following examples you will be shown how to predict future outcomes based on the linear equations that model current behavior.

Example

Use the equations for house value in Hawaii and Mississippi to predict house value in 2035.
We are asked to find house value, *y*, when the year, *x*, is 2035. Since the equations we have represent house value increase since 1950, we have to be careful. We can't just plug in 2035 for *x*, because *x* represents the years since 1950.
How many years are between 1950 and 2035? $2035-1950=85$
This is our *x*-value.
For Hawaii:
$y=3966x+74,400$
$y=3966(85)+74,400$
$y=337110+74,400=411,510$
Holy cow! The average price for a house in Hawaii in 2035 is predicted to be $411,510 according to this model. See if you can find the *current* average value of a house in Hawaii. Does the model measure up?
For Mississippi:
$y=924x+25,200$
$y=924(85)+25,200$
$y=78540+25,200=103,740$
The average price for a home in Mississippi in 2035 is predicted to be $103,740 according to the model. See if you can find the *current* average value of a house in Mississippi. Does the model measure up?

In the following video, you will see the example of how to make a prediction with the home value data.

Watch this video online: https://youtu.be/Bw9XjDAI-K0

Example

Use the equation for the number of high school smokers per 100 to predict the year when there will be 0 smokers per 100.
$y=-1.75x+16$
This question takes a little more thinking. In terms of *x* and *y*, what does it mean to have 0 smokers? Since *y* represents the number of smokers and *x* represent the year, we are being asked when y will be 0.
Substitute 0 for *y*.
$y=-1.75x+16$
$0=-1.75x+16$
$-16=-1.75x$
$\frac{-16}{-1.75}=x$
$x=9.14$ years
Again, like the last example, *x* is representing the number of years since the start of the data—which was 2011, based on the table:

Year	Number of High School Students Smoking Cigarettes (per 100)
0	16
4	9

So we are predicting that there will be no smokers in high school by $2011+9.14=2020$. How accurate do you think this model is? Do you think there will ever be 0 smokers in high school?

The following video gives a thorough explanation of making a prediction given a linear equation.

Watch this video online: https://youtu.be/5W0qq8saxO0

Bringing it Together

The last example we will show will include many of the concepts that we have been building up throughout this section. We will interpret a word problem, write a linear equation from it, graph the equation, interpret the y-intercept and make a prediction. Hopefully this example will help you to make connections between the concepts we have presented.

Example

It costs $600 to purchase an iphone, plus $55 per month for unlimited use and data.
Write a linear equation that represents the cost, y, of owning and using the iPhone for x amount of months. When you have written your equation, answer the following questions:

1. What is the total cost you've paid after owning and using your phone for 24 months?
2. If you have spent $2,580 since you purchased your phone, how many months have you used your phone?

iPhone

Show Solution

Read and Understand: We need to write a linear equation that represents the cost of owning and using an iPhone for any number of months. We are to use y to represent cost, and x to represent the number of months we have used the phone.

Define and Translate: We will use the slope-intercept form of a line, $y=mx+b$, because we are given a starting cost and a monthly cost for use. We will need to find the slope and the y-intercept.

Slope: in this case we don't know two points, but we are given a rate in dollars for monthly use of the phone. Our units are dollars per month because slope is $\frac{\Delta y}{\Delta x}$, and y is in dollars and x is in months. The slope will be $\frac{55\text{ dollars}}{1\text{ month}}$.

$m=\frac{55}{1}=55$

Y-Intercept: the y-intercept is defined as a point $(0,b)$. We want to know how much money we have spent, y, after 0 months. We haven't paid for service yet, but we have paid $600 for the phone. The y-intercept in this case is called an initial cost. $b=600$

Write and Solve: Substitute the slope and intercept you defined into the slope=intercept equation.

$$y=mx+b$$
$$y=55x+600$$

Now we will answer the following questions:

1. What is the total cost you've paid after owning and using your phone for 24 months?

Since x represents the number of months you have used the phone, we can substitute x=24 into our equation.

$$y=55x+600$$
$$y=55\left(24\right)+600$$
$$y=1320+600$$
$$y=1920$$

Y represents the cost after x number of months, so in this scenario, after 24 months, you have spent $1920 to own and use an iPhone.

1. If you have spent $2,580 since you purchased your phone, how many months have you used your phone?

We know that y represents cost, and we are given a cost and asked to find the number of months related to having spent that much. We will substitute y=$2,580 into the equation, then use what we know about solving linear equations to isolate x:

$$\begin{array}{rrcl} & y & = & 55x+600 \\ & 2580 & = & 55x+600 \\ \text{subtract 600 from each side} & \underline{-600} & & \underline{-600} \\ & 1980 & = & 55x \\ \text{divide each side by 55} & \frac{1980}{55} & = & \frac{55x}{55} \\ & 36 & = & x \end{array}$$

If you have spent $2,580 then you have been using your iPhone for 36 months, or 3 years.

Summary

The slope-intercept form of a linear equation is written as $y=mx+b$, where *m* is the slope and *b* is the value of *y* at the *y*-intercept, which can be written as $(0,b)$. When you know the slope and the *y*-intercept of a line you can use the slope-intercept form to immediately write the equation of that line. The slope-intercept form can also help you to write the equation of a line when you know the slope and a point on the line or when you know two points on the line.

When lines in a plane are parallel (that is, they never cross), they have the same slope. When lines are perpendicular (that is, they cross at a 90° angle), their slopes are opposite reciprocals of each other. The product of their slopes will be -1, except in the case where one of the lines is vertical causing its slope to be undefined. You can use these relationships to find an equation of a line that goes through a particular point and is parallel or perpendicular to another line.

GRAPHING INEQUALITES

Learning Objectives

- Identify graphs and solutions of equations and inequalities
  - Identify the similarities and differences between solutions for linear equations in two variables and linear inequalities in two variables
  - Identify the similarities and differences between graphs for linear equations in two variables and linear inequalities in two variables
- Graph an inequality in two variables
  - Identify and follow steps for graphing a linear inequality in two variables

Identify the difference between the graph of a linear equation and linear inequality

Recall that solutions to linear inequalities are whole sets of numbers, rather than just one number, like you find with solutions to equalities (equations).

Here is an example from the section on solving linear inequalities:

Solve for *p*. $4p+5<29$

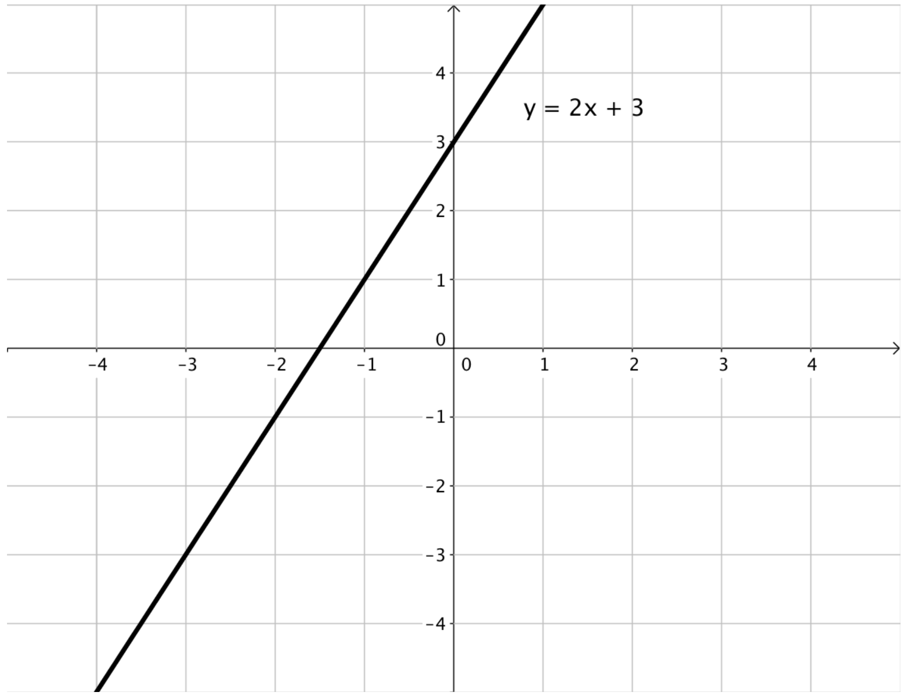
$$\begin{array}{rcl} 4p+5< & 29 \\ \underline{\quad -5 \quad} & \underline{-\,5} \\ \frac{4p}{4} \quad < & \frac{24}{4} \\ p<6 & \end{array}$$

You can interpret the solution as *p* can be any number less than six. Now recall that we can graph equations of lines by defining the outputs, *y*, and the inputs, *x*, and writing an equation.

Previously, we showed how to graph the line described by this equation: $y=2x+3$ and found that we can construct a never-ending table of values that make points on the line—these are some of the solutions to the equation $y=2x+3$.

***x* values**	$2x+3$	***y* values**
0	$2(0)+3$	3
1	$2(1)+3$	5
2	$2(2)+3$	7
3	$2(3)+3$	9

Additionally, we learned how to graph the line that represents all the points that make $y=2x+3$ a true statement.



What if we combined these two ideas—linear inequalities and graphs of lines? First translate the line, $y = 2x + 3$, into words:

You get *y* by multiplying *x* by two and adding three. $y = 2x + 3$

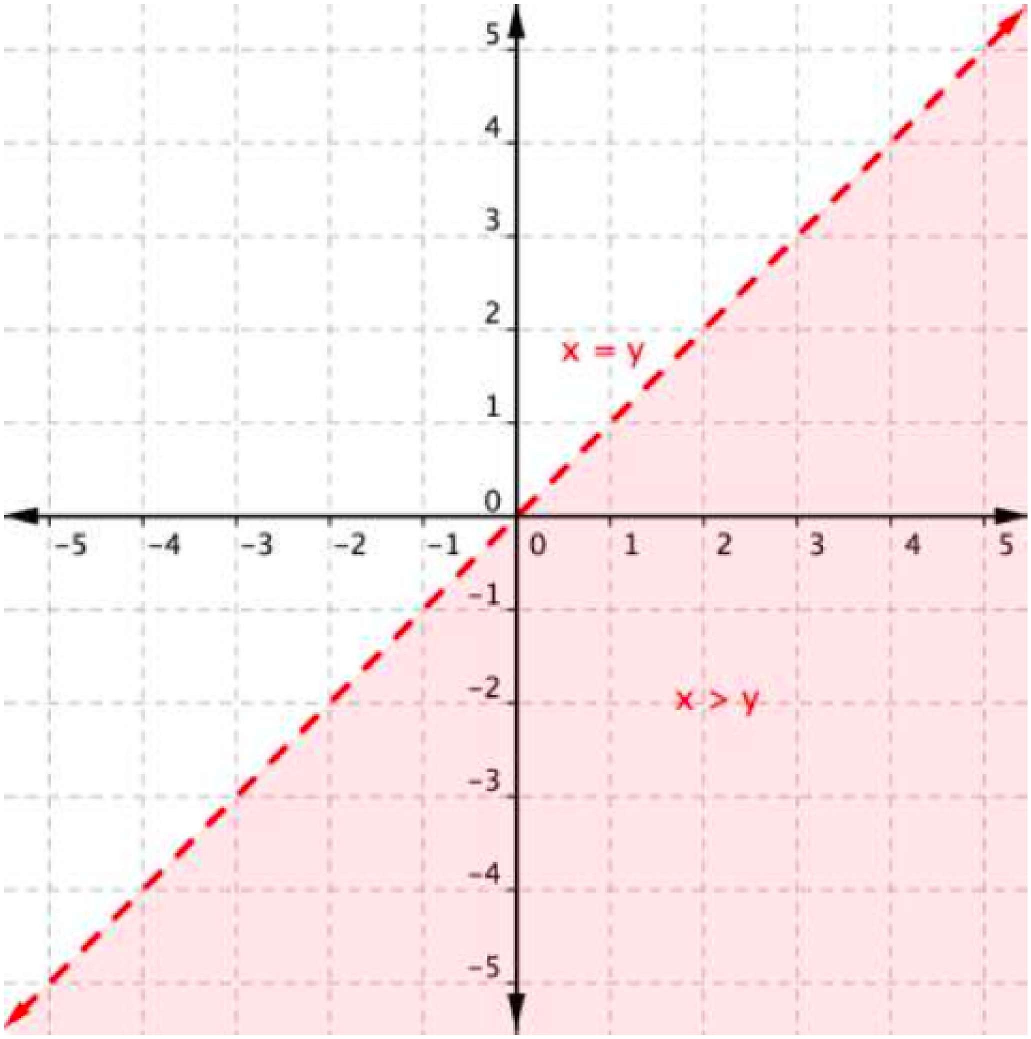
How would you translate this inequality into words? $y < 2x + 3$

For what values of *x* will you get an output, y, that is *less than* 2 times *x* plus three?

WOW, that may seem confusing, but keep reading, we'll help you figure it out.

Linear inequalities are different than linear equations, although you can apply what you know about equations to help you understand inequalities. Inequalities and equations are both math statements that compare two values. Equations use the symbol = ; recall that inequalities are represented by the symbols < , ≤ , > , and ≥.

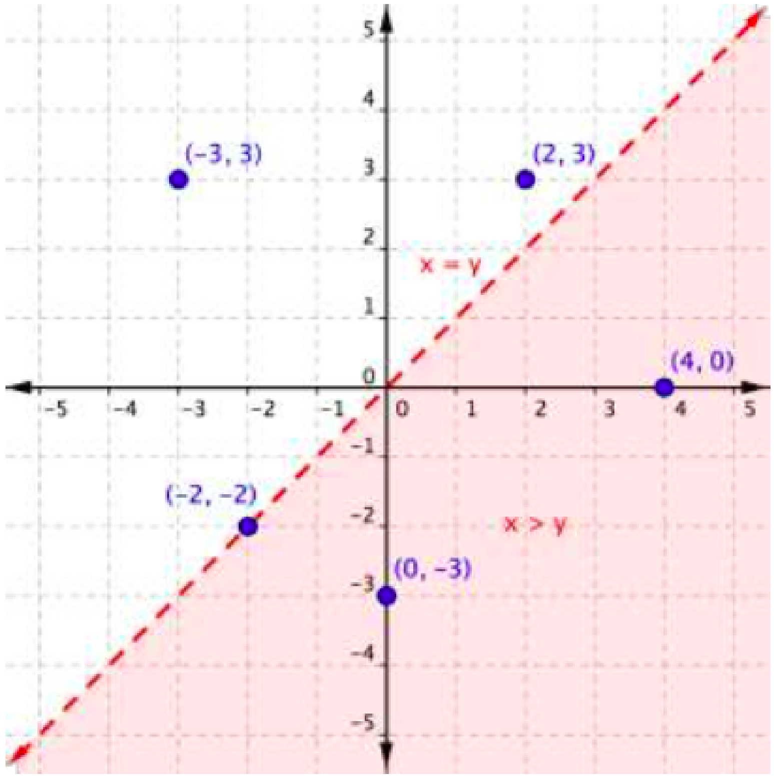
One way to visualize two-variable inequalities is to plot them on a coordinate plane. Here is what the inequality $x > y$ looks like. The solution is a region, which is shaded. This region is made up of lots and lots of ordered pairs that all make the statement $x > y$ true.



There are a few things to notice here. First, look at the dashed red boundary line: this is the graph of the related linear equation $x = y$. Next, look at the light red region that is to the right of the line. This region (excluding the line $x = y$) represents the entire set of solutions for the inequality $x > y$. Remember how all points on a *line* are solutions to the linear equation of the line? Well, all points in a *region* are solutions to the **linear inequality** representing that region.

Let's think about it for a moment—if $x > y$, then a graph of $x > y$ will show all ordered pairs (x, y) for which the *x*-coordinate is greater than the *y*-coordinate.

The graph below shows the region $x > y$ as well as some ordered pairs on the coordinate plane. Look at each ordered pair. Is the *x*-coordinate greater than the *y*-coordinate? Does the ordered pair sit inside or outside of the shaded region?



The ordered pairs $(4, 0)$ and $(0, -3)$ lie inside the shaded region. In these ordered pairs, the *x*-coordinate is larger than the *y*-coordinate. These ordered pairs are in the solution set of the equation $x > y$.

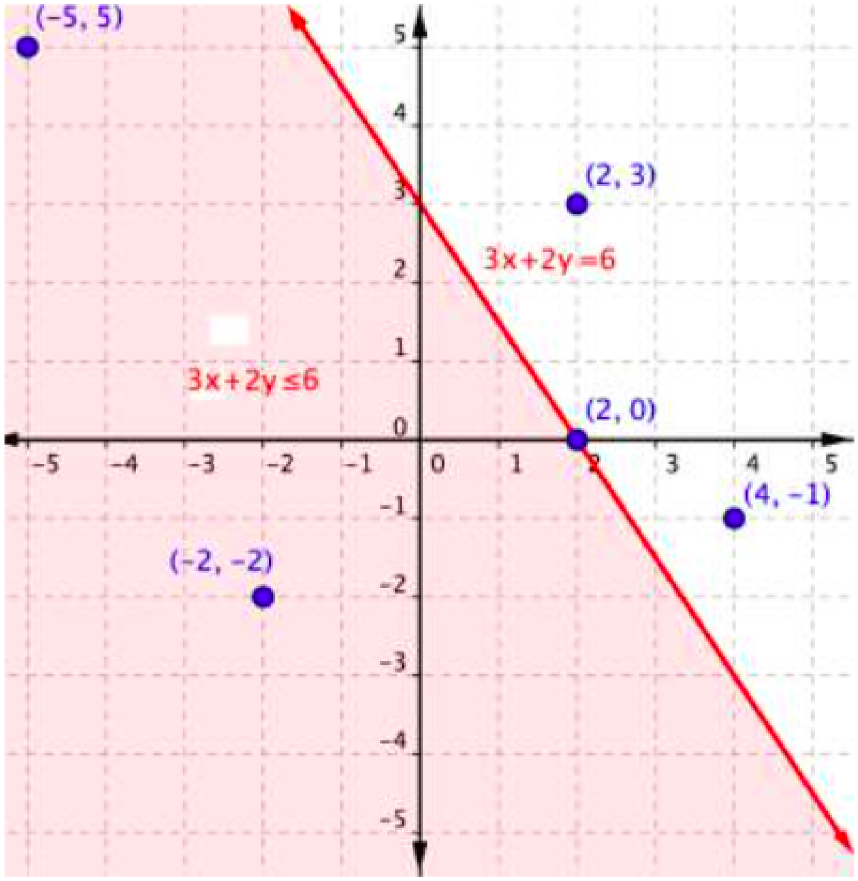
The ordered pairs $(-3, 3)$ and $(2, 3)$ are outside of the shaded area. In these ordered pairs, the *x*-coordinate is *smaller* than the *y*-coordinate, so they are not included in the set of solutions for the inequality.

The ordered pair $(-2, -2)$ is on the boundary line. It is not a solution as -2 is not greater than -2. However, had the inequality been $x \geq y$ (read as "*x* is greater than or equal to *y*"), then $(-2, -2)$ would have been included (and the line would have been represented by a solid line, not a dashed line).

The Difference Between a Linear Equation and Linear Inequality (Two Variables)

Watch this video online: https://youtu.be/EcrLbRJ2zV0

Let's take a look at one more example: the inequality $3x + 2y \leq 6$. The graph below shows the region of values that makes this inequality true (shaded red), the boundary line $3x + 2y = 6$, as well as a handful of ordered pairs. The boundary line is solid this time, because points on the boundary line $3x + 2y = 6$ will make the inequality $3x + 2y \leq 6$ true.



As you did with the previous example, you can substitute the *x*- and *y*-values in each of the (x, y) ordered pairs into the inequality to find solutions. While you may have been able to do this in your head for the inequality $x > y$, sometimes making a table of values makes sense for more complicated inequalities.

Ordered Pair	**Makes the inequality** $3x + 2y \leq 6$ **a true statement**	**Makes the inequality** $3x + 2y \leq 6$ **a false statement**
$(-5, 5)$	$3(-5) + 2(5) \leq 6$ $-15 + 10 \leq 6$ $-5 \leq 6$	
$(-2, -2)$	$3(-2) + 2(-2) \leq 6$ $-6 + (-4) \leq 6$ $-10 \leq 6$	
$(2, 3)$		$3(2) + 2(3) \leq 6$ $6 + 6 \leq 6$ $12 \leq 6$
$(2, 0)$	$3(2) + 2(0) \leq 6$ $6 + 0 \leq 6$ $6 \leq 6$	

Ordered Pair	Makes the inequality a true statement	Makes the inequality a false statement
$(4,-1)$		$3\,(4)+2\,(-1)\leq 6$ $12+(-2)\leq 6$ $10\leq 6$

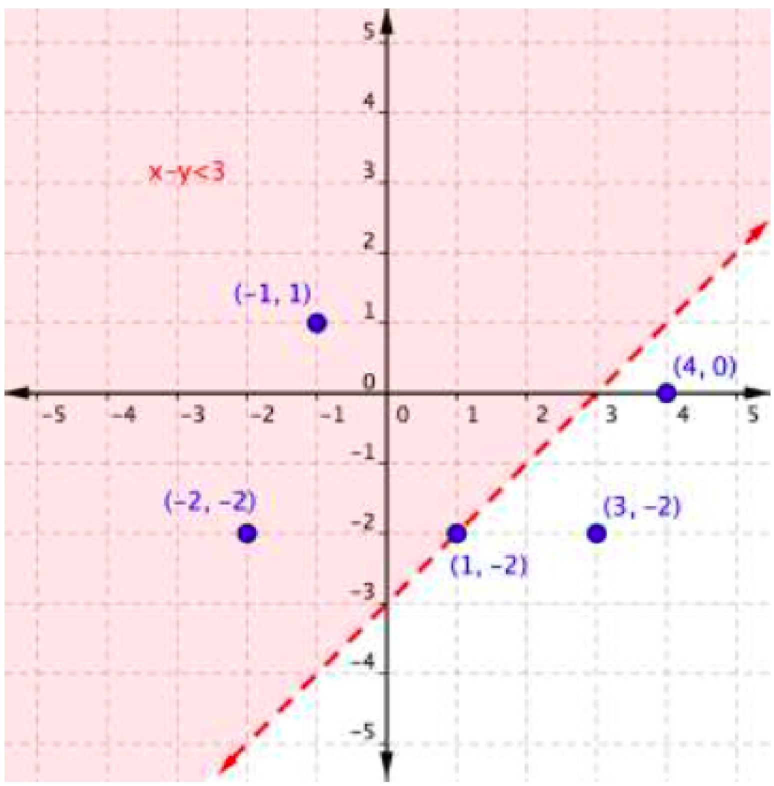
If substituting (x,y) into the inequality yields a true statement, then the ordered pair is a solution to the inequality, and the point will be plotted within the shaded region or the point will be part of a solid boundary line. A false statement means that the ordered pair is not a solution, and the point will graph outside the shaded region, or the point will be part of a dotted boundary line.

Determine if Ordered Pairs Satisfy a Linear Inequality

Watch this video online: https://youtu.be/-x-zt_yM0RM

Example

Use the graph to determine which ordered pairs plotted below are solutions of the inequality $x-y<3$.



Show Solution

Use a Graph Determine Ordered Pair Solutions of a Linear Inequality in Two Variable

Watch this video online: https://youtu.be/GQVdDRVq5_o

Example

Is $(2,-3)$ a solution of the inequality $y<-3x+1$?
Show Solution

Determine if Ordered Pairs Satisfy a Linear Inequality

Watch this video online: https://youtu.be/-x-zt_yM0RM

Graph an Inequality in Two Variables

So how do you get from the algebraic form of an inequality, like $y>3x+1$, to a graph of that inequality? Plotting inequalities is fairly straightforward if you follow a couple steps.

Graphing Inequalities

To graph an inequality:

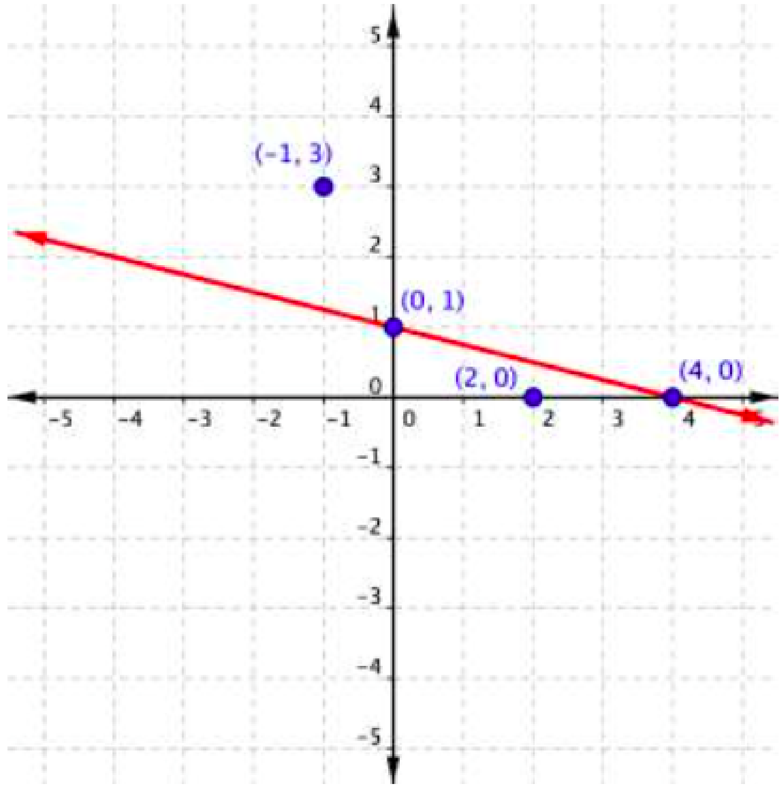
- Graph the related boundary line. Replace the <, >, ≤ or ≥ sign in the inequality with = to find the equation of the boundary line.
- Identify at least one ordered pair on either side of the boundary line and substitute those (x,y) values into the inequality. Shade the region that contains the ordered pairs that make the inequality a true statement.
- If points on the boundary line are solutions, then use a solid line for drawing the boundary line. This will happen for ≤ or ≥ inequalities.
- If points on the boundary line aren't solutions, then use a dotted line for the boundary line. This will happen for < or > inequalities.

Let's graph the inequality $x+4y\leq 4$.

To graph the boundary line, find at least two values that lie on the line $x+4y=4$. You can use the *x*– and *y*-intercepts for this equation by substituting 0 in for *x* first and finding the value of *y*; then substitute 0 in for *y* and find *x*.

x	*y*
0	1
4	0

Plot the points $(0,1)$ and $(4,0)$, and draw a line through these two points for the boundary line. The line is solid because ≤ means "less than or equal to," so all ordered pairs along the line are included in the solution set.



The next step is to find the region that contains the solutions. Is it above or below the boundary line? To identify the region where the inequality holds true, you can test a couple of ordered pairs, one on each side of the boundary line.

If you substitute $(-1,3)$ into $x+4y\leq 4$:

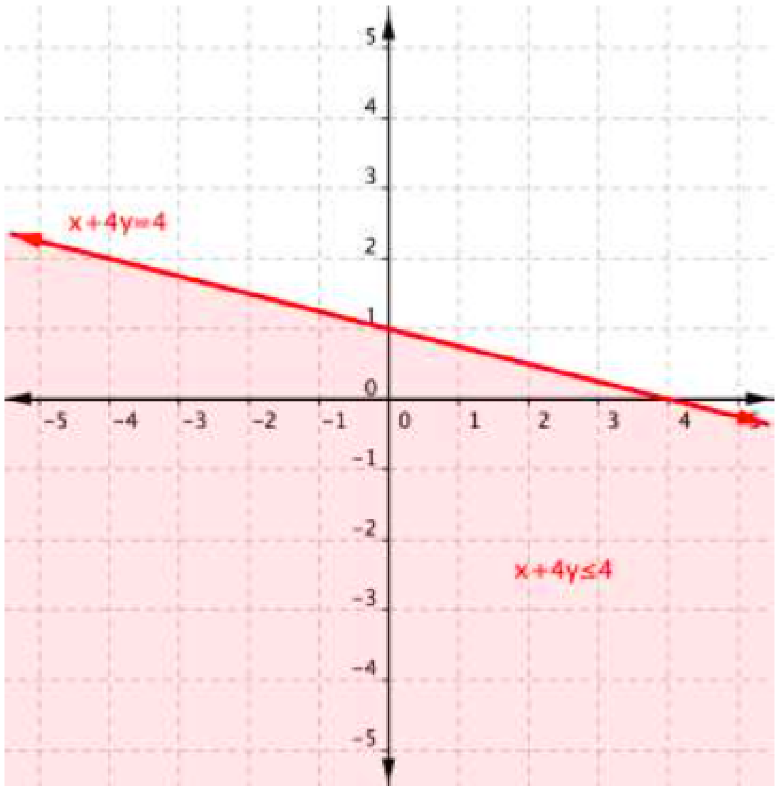
$$\begin{aligned} -1+4(3) &\leq 4 \\ -1+12 &\leq 4 \\ 11 &\leq 4 \end{aligned}$$

This is a false statement, since 11 is not less than or equal to 4.

On the other hand, if you substitute $(2,0)$ into $x+4y\leq 4$:

$$\begin{aligned} 2+4(0) &\leq 4 \\ 2+0 &\leq 4 \\ 2 &\leq 4 \end{aligned}$$

This is true! The region that includes $(2,0)$ should be shaded, as this is the region of solutions.



And there you have it—the graph of the set of solutions for $x + 4y \leq 4$.

Graphing Linear Inequalities in Two Variables

Watch this video online: https://youtu.be/2VgFg2ztspI

Example
Graph the inequality $2y > 4x - 6$. Show Solution

A quick note about the problem above—notice that you can use the points $(0, -3)$ and $(2, 1)$ to graph the boundary line, but that these points are not included in the region of solutions, since the region does not include the boundary line!

Graphing Linear Inequalities in Two Variables (Slope Intercept Form)

Watch this video online: https://youtu.be/Hzxc4HASygU

Summary

When inequalities are graphed on a coordinate plane, the solutions are located in a region of the coordinate plane, which is represented as a shaded area on the plane. The boundary line for the inequality is drawn as a solid line if the points on the line itself do satisfy the inequality, as in the cases of ≤ and ≥. It is drawn as a dashed line if the points on the line do not satisfy the inequality, as in the cases of < and >. You can tell which region to shade by testing some points in the inequality. Using a coordinate plane is especially helpful for visualizing the region of solutions for inequalities with two variables.

CONCLUSION

Suppose you are a semiprofessional blogger who writes about media (you'd really like to be a paid film critic, but no one has offered yet). Recently you posted a short blog piece complaining about the number of ads on TV these days, compared to when you were younger. You weren't very scientific about it, and a couple of your readers disputed your claim and tried to start an argument with you.

Well, that got your attention. It made you wonder whether your perception of there being "way more" TV ads was accurate. You get online and do a little research, and you find a website that reports some interesting data on the number of minutes of TV commercials per hour since 2009, shown below:

Years since 2009	Minutes
0	8.5
1	9.25
2	10
3	10.75
4	11.5
5	12.25
6	13

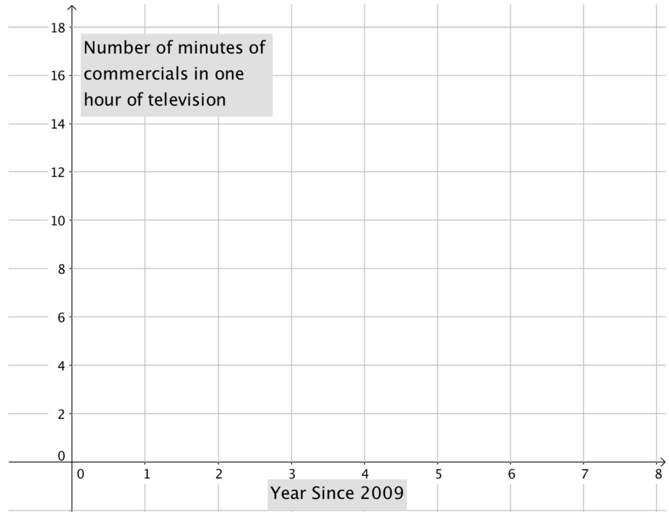
The data show that, yes, there are more commercials now than in 2009, but the table isn't very exciting, and you doubt that your readers will care. You may not be a famous film critic yet, but you realize you can use what you've learned in math class to present this information as a graph—the perfect thing to post on your blog and convince your skeptical readers!

You get to work.

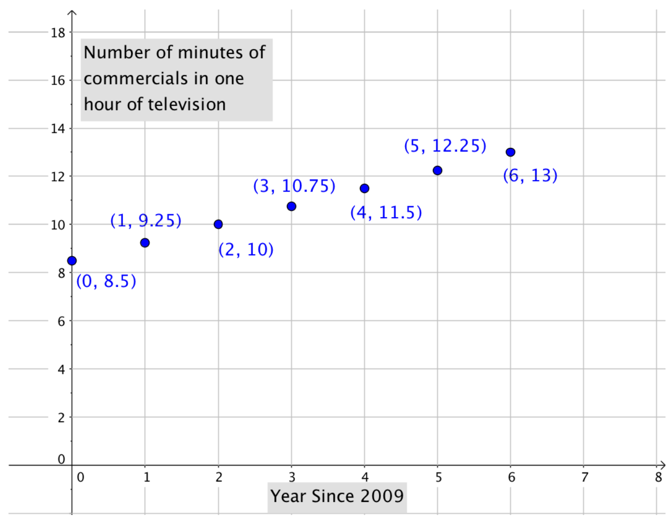
What information do you need to draw a graph of the line that represents the change in the number of minutes of commercials in one hour of TV since 2009?

The Cartesian Coordinate Plane

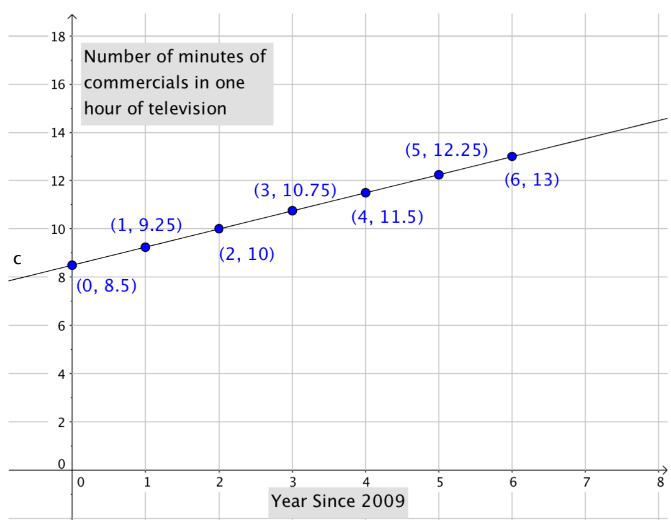
You remember that the coordinate plane gives your graph structure and meaning. A straight line on a page won't tell your readers much. You draw the axes and label the horizontal one "Years Since 2009," because that's the first data point you have. You label the vertical axis from 1 to 18 because your minute data range from 8.5 to 13 minutes, and that will give you room on either side.



Then you plot the ordered pairs from your table of values on your coordinate plane, as below.



The points give you a guide for drawing your line, which you do, as below.



Things are looking pretty good, so you post your graph to your blog to show people how much more time they are being exposed to commercials in one hour of TV watching since 2009.

Then, it happens . . .

One of your readers asks if you can guess how many minutes of commercials will be in one hour of television ten years from now (assuming the current trend continues). After thinking about the question for a while, you realize you don't have to guess! You have all the information you need to write the equation of the line you drew, and you recall that, with an equation, you can put in any value for the years since 2009.

Finding the Equation

You remember that knowing the slope and y-intercept of a line can help you write the equation of the line. You realize you have the y-intercept: $(0, 8.5)$. Now, you just need the slope.

You check your math notes to find the definition of slope, and use two of your data points to calculate it:

$\text{Slope} = \frac{\text{rise}}{\text{run}}$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{11.5 - 10.75}{4 - 3}$

$m = \frac{0.75}{1} = 0.75$

Now you have the two pieces of information you need to write the equation of the line that represents how many minutes of commercials will be in one hour of TV in *any* year before or after 2009.

First choose your variables: *x* = the year and *y* = the number of minutes. You can substitute the values for *m* and *b* into the slope-intercept form of a line:

$y = mx + b$
$y = 0.75x + 8.5$

Remembering that the whole point of this exercise was to answer your reader's question, next you figure out what 10 years from now would be in relation to 0 representing 2009 on your graph. If 10 years from now is 2026, then it's 17 years from 2009. The *x* value you need in order to answer the question is 17.

$y = 0.75(17) + 8.5$

$y = 12.75 + 8.5$

$y = 21.25$

You have a new data point: $(17, 21.25)$. This means that in 2026 there will be more than 20 minutes of commercials in one hour of TV. Yuck!

MODULE 3: SYSTEMS OF EQUATIONS AND INEQUALITIES

INTRODUCTION

When you play in a river you are surrounded by fluids, including water and air. At first it might seem strange to think of air as a fluid, but a fluid is defined as a substance that flows. Wind, therefore, is a great example of air that flows. Other examples of flows include traffic patterns and electrical currents. Flows can be turbulent like what you may experience in airplanes.

Early in the 19th Century, Claude-Louis Navier in France and George Gabriel Stokes in England both derived an equation that can explain and predict the flow of fluids. The Navier-Stokes equations are a system of equations used to describe the velocity of a fluid as it moves through three-dimensional space over a specific interval of time.

Sir George Gabriel Stokes

Interestingly, our understanding of solutions to the Navier-Stokes equations remains minimal. Surprisingly, given the equations' wide range of practical uses, it has not yet been proven that solutions *always* exist in three dimensions. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US $1,000,000 prize for a solution or a counter-example.

In this section, we will learn how to graph systems of equations in two dimensions and find whether solutions exist. We will also see how systems of equations can be used to solve problems where we have two unknown variables.

Learning Outcomes

- **Solutions for Systems of Equations and Inequalities**
  - Graph systems of equations
  - Evaluate ordered pairs as solutions to systems
  - Classify Solutions for Systems
- **Algebraic Methods for Solving Systems**
  - Use the substitution method
  - Use the elimination method without multiplication
  - Use the elimination method with multiplication
- **Problem Solving With Systems**
  - Solve mixture problems
  - Solve value problems
  - Solve cost and revenue problems

GRAPHS AND SOLUTIONS TO SYSTEMS OF LINEAR EQUATIONS

Learning Objectives

- Graph systems of equations
  - Graph a system of two linear equations
  - Graph a system of two linear inequalities
- Evaluate ordered pairs as solutions to systems
  - Determine whether an ordered pair is a solution to a system of linear equations
  - Determine whether an ordered pair is a solution to a system of linear inequalities
- Classify solutions to systems
  - Identify what type of solution a system will have based on its graph

The way a river flows depends on many variables including how big the river is, how much water it contains, what sorts of things are floating in the river, whether or not it is raining, and so forth. If you want to best describe its flow, you must take into account these other variables. A system of linear equations can help with that.

A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. You will find systems of equations in every application of mathematics. They are a useful tool for discovering and describing how behaviors or processes are interrelated. It is rare to find, for example, a pattern of traffic flow that that is only affected by weather. Accidents, time of day, and major sporting events are just a few of the other variables that can affect the flow of traffic in a city. In this section, we will explore some basic principles for graphing and describing the intersection of two lines that make up a system of equations.

Graph a system of linear equations

In this section, we will look at systems of linear equations and inequalities in two variables. First, we will practice graphing two equations on the same set of axes, and then we will explore the different considerations you need to make when graphing two linear inequalities on the same set of axes. The same techniques are used to graph a system of linear equations as you have used to graph single linear equations. We can use tables of values, slope and *y*-intercept, or *x*– and *y*-intercepts to graph both lines on the same set of axes.

For example, consider the following system of linear equations in two variables.

$$2x+y=-8$$
$$x-y=-1$$

Let’s graph these using slope-intercept form on the same set of axes. Remember that slope-intercept form looks like $y=mx+b,$ so we will want to solve both equations for y.

First, solve for y in $2x+y=-8$

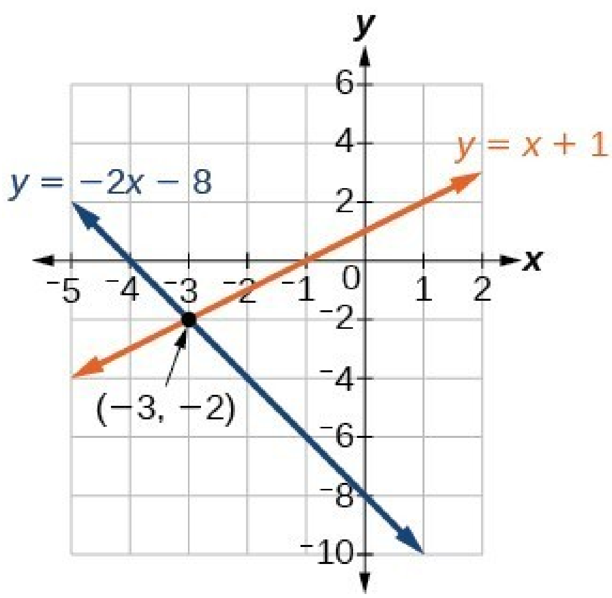
$2x+y=-8$
$y=-2x-8$
Second, solve for y in $x-y=-1$

$$x-y=-1$$
$$y=x+1$$

The system is now written as

$y=-2x-8$
$y=x+1$

Now you can graph both equations using their slopes and intercepts on the same set of axes, as seen in the figure below. Note how the graphs share one point in common. This is their point of intersection, a point that lies on both of the lines. In the next section we will verify that this point is a solution to the system.



In the following example, you will be given a system to graph that consists of two parallel lines.

Example

Graph the system $\begin{aligned} y=2x+1 \\ y=2x-3 \end{aligned}$ using the slopes and y-intercepts of the lines.

Show Solution

In the next example, you will be given a system whose equations look different, but after graphing, turn out to be the same line.

Example

Graph the system $\begin{array}{l} y = \frac{1}{2}x + 2 \\ 2y - x = 4 \end{array}$ using the x – and y-intercepts.

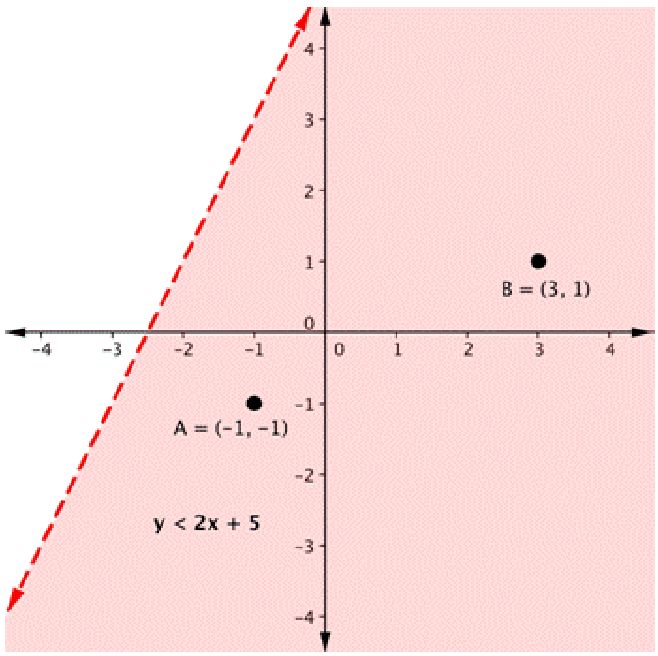
Show Solution

Watch this video online: https://youtu.be/BBmB3rFZLXU

Graphing a system of linear equations consists of choosing which graphing method you want to use and drawing the graphs of both equations on the same set of axes. When you graph a system of linear inequalities on the same set of axes, there are a few more things you will need to consider.

Graph a system of two inequalities

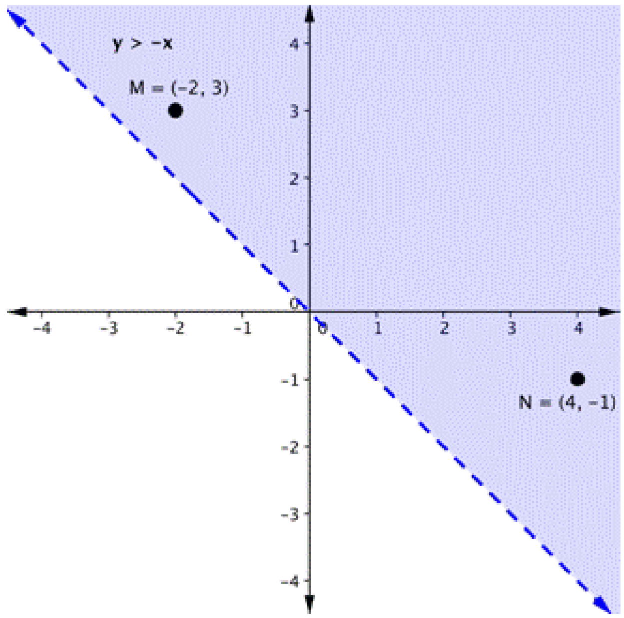
Remember from the module on graphing that the graph of a single linear inequality splits the **coordinate plane** into two regions. On one side lie all the solutions to the inequality. On the other side, there are no solutions. Consider the graph of the inequality $y < 2x + 5$.



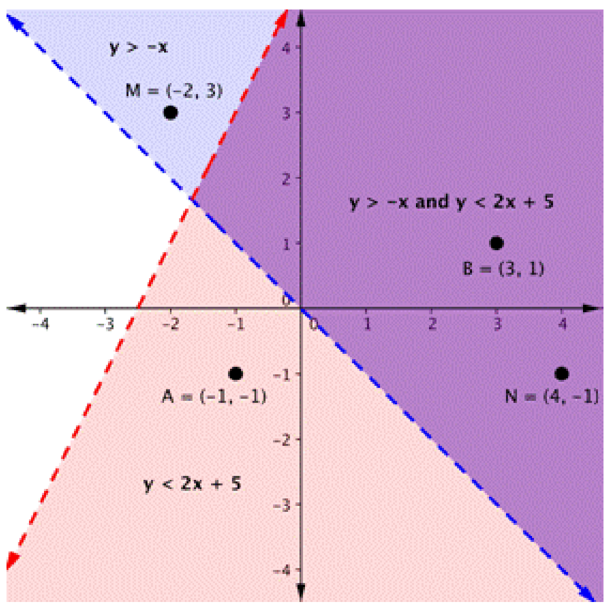
The dashed line is $y = 2x + 5$. Every ordered pair in the shaded area below the line is a solution to $y < 2x + 5$, as all of the points below the line will make the inequality true. If you doubt that, try substituting the *x* and *y* coordinates of Points A and B into the inequality—you'll see that they work. So, the shaded area shows all of the solutions for this inequality.

The boundary line divides the coordinate plane in half. In this case, it is shown as a dashed line as the points on the line don't satisfy the inequality. If the inequality had been $y \leq 2x + 5$, then the boundary line would have been solid.

Let's graph another inequality: $y > -x$. You can check a couple of points to determine which side of the boundary line to shade. Checking points M and N yield true statements. So, we shade the area above the line. The line is dashed as points on the line are not true.



To create a system of inequalities, you need to graph two or more inequalities together. Let's use $y < 2x + 5$ and $y > -x$ since we have already graphed each of them.



The purple area shows where the solutions of the two inequalities overlap. This area is the solution to the *system of inequalities*. Any point within this purple region will be true for both $y > -x$ and $y < 2x + 5$.

In the next example, you are given a system of two inequalities whose boundary lines are parallel to each other.

Examples

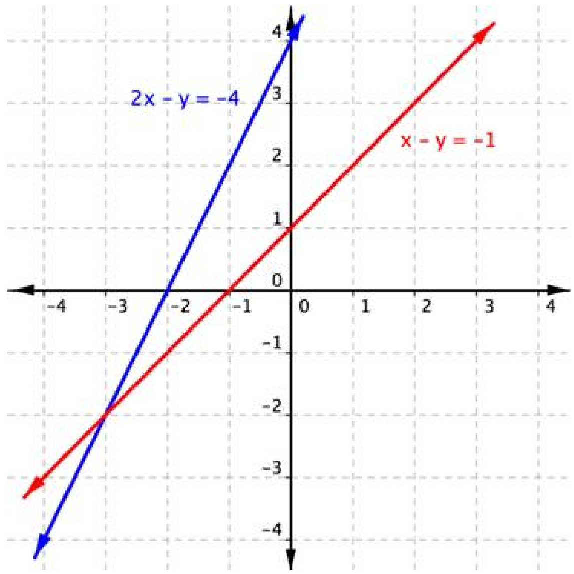
Graph the system $\begin{array}{l} y \geq 2x + 1 \\ y < 2x - 3 \end{array}$

Show Solution

Watch this video online: https://youtu.be/ACTxJv1h2_c

In the next section, we will see that points can be solutions to systems of equations and inequalities. We will verify algebraically whether a point is a solution to a linear equation or inequality.

Determine whether an ordered pair is a solution for a system of linear equations



The lines in the graph above are defined as

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}.$$

They cross at what appears to be $(-3, -2)$.

Using algebra, we can verify that this shared point is actually $(-3, -2)$ and not $(-2.999, -1.999)$. By substituting the *x*– and *y*-values of the ordered pair into the equation of each line, you can test whether the point is on both lines. If the substitution results in a true statement, then you have found a solution to the system of equations!

Since the solution of the system must be a solution to *all* the equations in the system, you will need to check the point in each equation. In the following example, we will substitute -3 for *x* and -2 for *y* in each equation to test whether it is actually the solution.

Example

Is $(-3, -2)$ a solution of the system

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Show Solution

Example

Is (3, 9) a solution of the system

$$\begin{aligned} y &= 3x \\ 2x - y &= 6 \end{aligned}$$

Show Solution

Think About It

Is $(-2, 4)$ a solution for the system

$$y = 2x$$
$$3x + 2y = 1$$

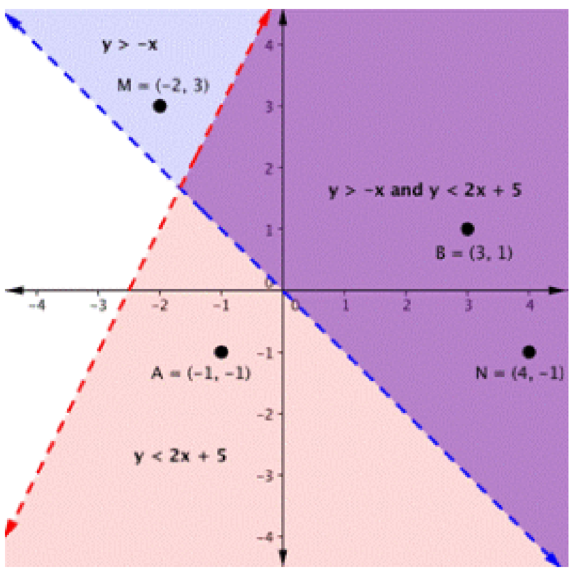
Before you do any calculations, look at the point given and the first equation in the system. Can you predict the answer to the question without doing any algebra?
Show Solution

Watch this video online: https://youtu.be/2IxgKgjX00k

Remember that in order to be a solution to the system of equations, the values of the point must be a solution for both equations. Once you find one equation for which the point is false, you have determined that it is not a solution for the system.

We can use the same method to determine whether a point is a solution to a system of linear inequalities.

Determine whether an ordered pair is a solution to a system of linear inequalities



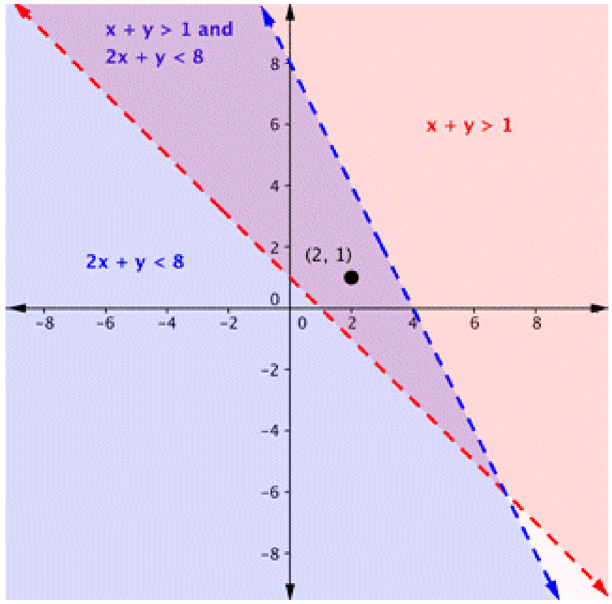
On the graph above, you can see that the points B and N are solutions for the system because their coordinates will make both inequalities true statements.

In contrast, points M and A both lie outside the solution region (purple). While point M is a solution for the inequality $y > -x$ and point A is a solution for the inequality $y < 2x + 5$, neither point is a solution for the *system*. The following example shows how to test a point to see whether it is a solution to a system of inequalities.

Example

Is the point (2, 1) a solution of the system $x + y > 1$ and $2x + y < 8$?
Show Solution

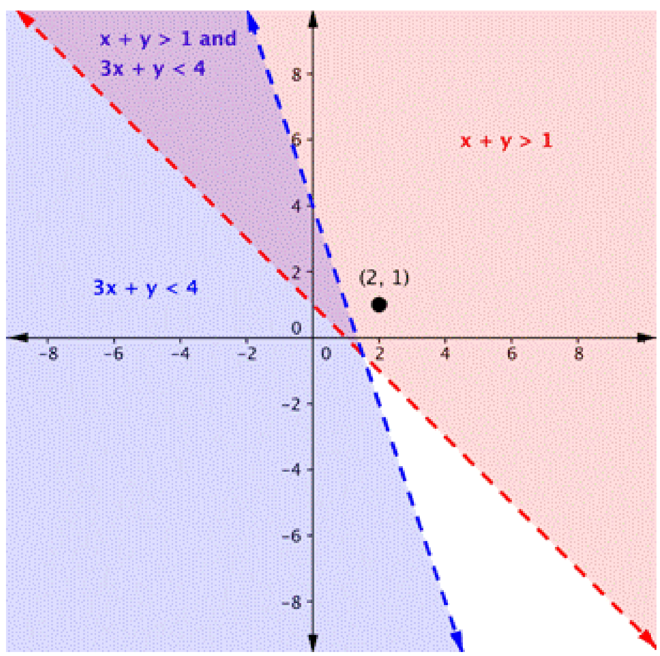
Here is a graph of the system in the example above. Notice that (2, 1) lies in the purple area, which is the overlapping area for the two inequalities.



Example

Is the point (2, 1) a solution of the system $x + y > 1$ and $3x + y < 4$?
Show Solution

Here is a graph of this system. Notice that (2, 1) is not in the purple area, which is the overlapping area; it is a solution for one inequality (the red region), but it is not a solution for the second inequality (the blue region).



As shown above, finding the solutions of a system of inequalities can be done by graphing each inequality and identifying the region they share. Below, you are given more examples that show the entire process of defining the region of solutions on a graph for a system of two linear inequalities. The general steps are outlined below:

- Graph each inequality as a line and determine whether it will be solid or dashed
- Determine which side of each boundary line represents solutions to the inequality by testing a point on each side
- Shade the region that represents solutions for both inequalities

Example

Shade the region of the graph that represents solutions for both inequalities. $x + y \geq 1$ and $y – x \geq 5$.
Show Solution

Watch this video online: https://youtu.be/o9hTFJEBcXs

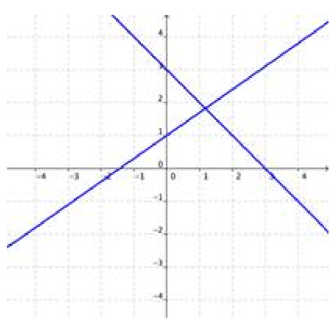
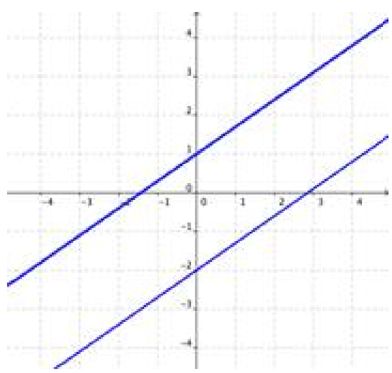
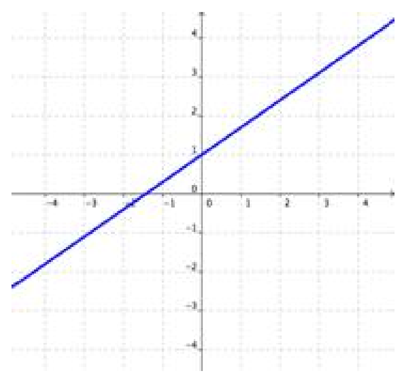
In this section we have seen that solutions to systems of linear equations and inequalities can be ordered pairs. In the next section, we will work with systems that have no solutions or infinitely many solutions.

Use a graph to classify solutions to systems

Recall that a linear equation graphs as a line, which indicates that all of the points on the line are solutions to that linear equation. There are an infinite number of solutions. As we saw in the last section, if you have a system of linear equations that intersect at one point, this point is a solution to the system. What happens if the lines never cross, as in the case of parallel lines? How would you describe the solutions to that kind of system? In this section, we will explore the three possible outcomes for solutions to a system of linear equations.

Three possible outcomes for solutions to systems of equations

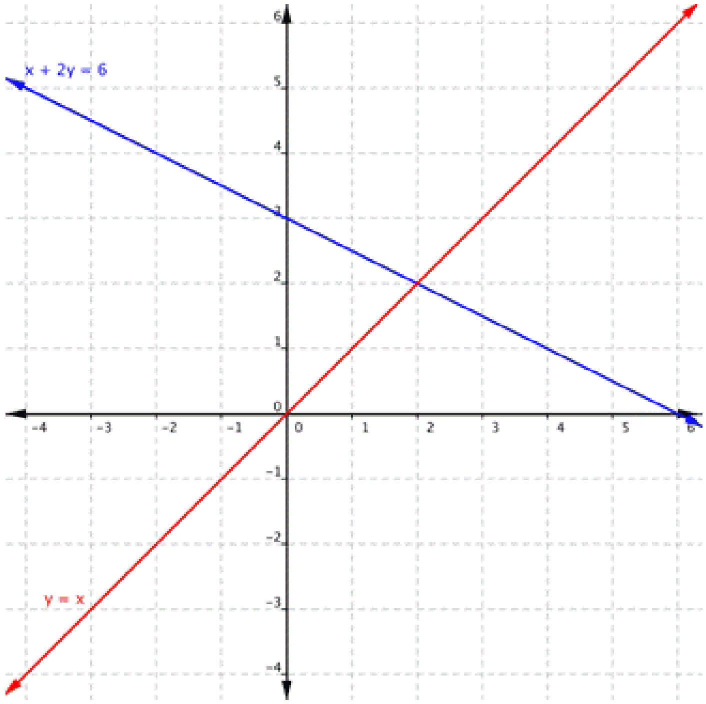
Recall that the solution for a system of equations is the value or values that are true for *all* equations in the system. There are three possible outcomes for solutions to systems of linear equations. The graphs of equations within a system can tell you how many solutions exist for that system. Look at the images below. Each shows two lines that make up a system of equations.

One Solution	**No Solutions**	**Infinite Solutions**
		
If the graphs of the equations intersect, then there is one solution that is true for both equations.	If the graphs of the equations do not intersect (for example, if they are parallel), then there are no solutions that are true for both equations.	If the graphs of the equations are the same, then there are an infinite number of solutions that are true for both equations.

- **One Solution:** When a system of equations intersects at an ordered pair, the system has one solution.
- **Infinite Solutions:** Sometimes the two equations will graph as the same line, in which case we have an infinite number of solutions.
- **No Solution:** When the lines that make up a system are parallel, there are no solutions because the two lines share no points in common.

Example

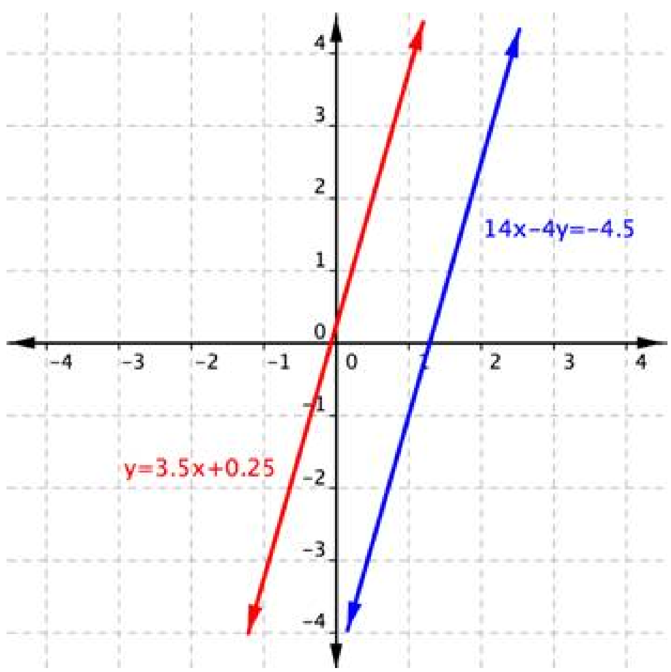
Using the graph of $\begin{array}{r} y = x \\ x + 2y = 6 \end{array}$, shown below, determine how many solutions the system has.



Show Solution

Example (Advanced)

Using the graph of $\begin{array}{c} y = 3.5x + 0.25 \\ 14x - 4y = -4.5 \end{array}$, shown below, determine how many solutions the system has.



Show Solution

Example

How many solutions does the system $\begin{array}{r} y = 2x + 1 \\ -4x + 2y = 2 \end{array}$ have?

Show Solution

Watch this video online: https://youtu.be/ZolxtOjcEQY

In the next section, we will learn some algebraic methods for finding solutions to systems of equations. Recall that linear equations in one variable can have one solution, no solution, or many solutions and we can verify this algebraically. We will use the same ideas to classify solutions to systems in two variables algebraically.

ALGEBRAIC METHODS FOR SOLVING SYSTEMS

Learning Objectives

- Use the substitution method
  - Solve a system of equations using the substitution method.
  - Recognize systems of equations that have no solution or an infinite number of solutions
- Use the elimination method without multiplication
  - Solve a system of equations when no multiplication is necessary to eliminate a variable
- Use the elimination method with multiplication
  - Use multiplication in combination with the elimination method to solve a system of linear equations
  - Recognize when the solution to a system of linear equations implies there are an infinite number of solutions

Solve a system of equations using the substitution method

In the last couple sections, we verified that ordered pairs were solutions to systems, and we used graphs to classify how many solutions a system of two linear equations had. What if we are not given a point of intersection, or it is not obvious from a graph? Can we still find a solution to the system? Of course you can, using algebra!

In this section we will learn the substitution method for finding a solution to a system of linear equations in two variables. We have used substitution in different ways throughout this course, for example when we were using the formulas for the area of a triangle and simple interest. We substituted values that we knew into the formula to solve for values that we did not know. The idea is similar when applied to solving systems, there are just a few different steps in the process. You will first solve for one variable, and then substitute that expression into the other equation. Let's start with an example to see what this means.

Example

Find the value of *x* for this system.

Equation A: $4x+3y=-14$

Equation B: $y=2$

Show Solution

You can substitute a value for a variable even if it is an expression. Here's an example.

Example

Solve for *x* and *y*.

Equation A: $y+x=3$

Equation B: $x=y+5$
Show Solution

Remember, a solution to a system of equations must be a solution to each of the equations within the system. The ordered pair $(4,-1)$ does work for both equations, so you know that it is a solution to the system as well.

Let's look at another example whose substitution involves the distributive property.

Example

Solve for *x* and *y*.

$y=3x+6$
$-2x+4y=4$
Show Solution

In the examples above, one of the equations was already given to us in terms of the variable *x* or *y*. This allowed us to quickly substitute that value into the other equation and solve for one of the unknowns.

Sometimes you may have to rewrite one of the equations in terms of one of the variables first before you can substitute. In the example below, you will first need to isolate one of the variables before you can substitute it into the other equation.

Example

Solve for *x* and *y*.

$2x+3y=22$
$3x+y=19$
Show Solution

In the following video, you will be given an example of solving a systems of two equations using the substitution method.

Watch this video online: https://youtu.be/MIXL35YRzRw

If you had chosen the other equation to start with in the previous example, you would still be able to find the same solution. It is really a matter of preference because sometimes solving for a variable will result in having to work with fractions. As you become more experienced with algebra, you will be able to anticipate what choices will lead to more desirable outcomes.

Recognize systems of equations that have no solution or an infinite number of solutions

When we learned methods for solving linear equations in one variable, we found that some equations didn't have any solutions, and others had an infinite number of solutions. We saw this behavior again when we started describing solutions to systems of equations in two variables.

Recall this example from Module 1 for solving linear equations in one variable:

Solve for *x*. $12+2x–8=7x+5–5x$

$$12+2x-8=7x+5-5x$$
$$2x+4=2x+5$$

$$\begin{array}{r} 2x+4=2x+5 \\ \underline{-2x \qquad -2x} \\ 4=\ 5 \end{array}$$

This false statement implies there are **no solutions** to this equation. In the same way, you may see an outcome like this when you use the substitution method to find a solution to a system of linear equations in two variables. In the next example, you will see an example of a system of two equations that does not have a solution.

Example

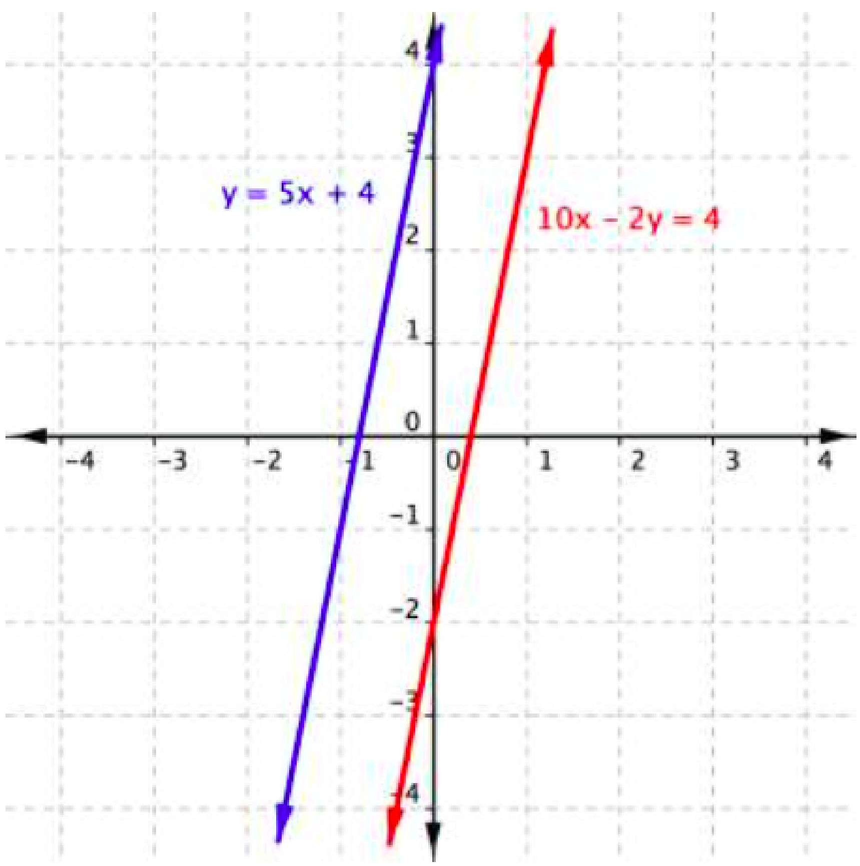
Solve for *x* and *y*.

$y=5x+4$

$10x-2y=4$

Show Solution

You get the false statement $-8=4$. What does this mean? The graph of this system sheds some light on what is happening.



The lines are parallel, they never intersect and there is no solution to this system of linear equations. Note that the result $-8=4$ is **not** a solution. It is simply a false statement and it indicates that there is **no** solution.

We have also seen linear equations in one variable and systems of equations in two variables that have an infinite number of solutions. In the next example, you will see what happens when you apply the substitution method to a system with an infinite number of solutions.

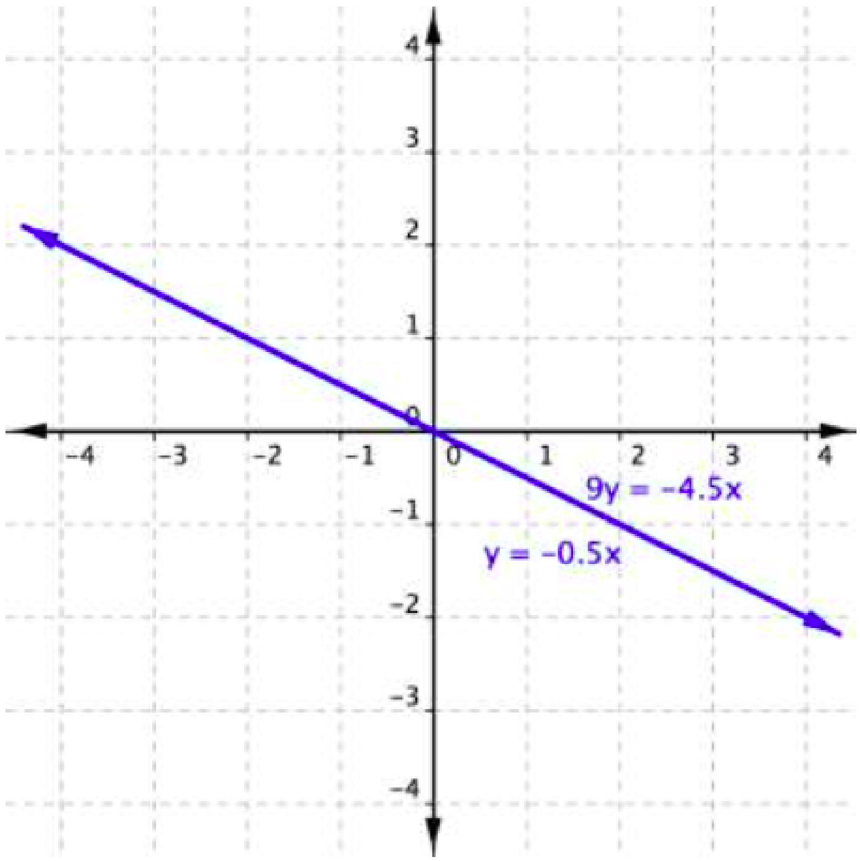
Example

Solve for x and y.

$$y = -0.5x$$
$$9y = -4.5x$$

Show Solution

This time you get a true statement: $-4.5x = -4.5x$. But what does this type of answer mean? Again, graphing can help you make sense of this system.



This system consists of two equations that both represent the same line; the two lines are collinear. Every point along the line will be a solution to the system, and that's why the substitution method yields a true statement. In this case, there are an infinite number of solutions.

In the following video you will see an example of solving a system that has an infinite number of solutions.

Watch this video online: https://youtu.be/Pcqb109yK5Q

In the following video you will see an example of solving a system of equations that has no solutions.

Watch this video online: https://youtu.be/kTtKfh5gFUc

Solve a system of equations using the elimination method

The **elimination method** for solving systems of linear equations uses the addition property of equality. You can add the same value to each side of an equation to eliminate one of the variable terms. In this method, you may or may not need to multiply the terms in one equation by a number first. We will first look at examples where no multiplication is necessary to use the elimination method. In the next section you will see examples using multiplication after you are familiar with the idea of the elimination method.

It is easier to show rather than tell with this method, so let's dive right into some examples.

If you add the two equations,

$x - y = -6$ and $x + y = 8$ together, watch what happens.

$$\begin{array}{r}
x - y = -6 \\
\underline{+x + y = \ \ 8} \\
2x + 0 = \ \ 2
\end{array}$$

You have eliminated the *y* term, and this equation can be solved using the methods for solving equations with one variable.

Let's see how this system is solved using the elimination method.

Example

Use elimination to solve the system.

$$\begin{aligned}
x - y &= -6 \\
x + y &= \ \ 8
\end{aligned}$$

Show Solution

Unfortunately not all systems work out this easily. How about a system like $2x + y = 12$ and $-3x + y = 2$. If you add these two equations together, no variables are eliminated.

$$\begin{array}{r}
2x + y = 12 \\
\underline{-3x + y = \ \ 2} \\
-x + 2y = 14
\end{array}$$

But you want to eliminate a variable. So let's add the opposite of one of the equations to the other equation. This means multiply every term in one of the equations by -1, so that the sign of every terms is opposite.

$$\begin{array}{l}
\ \ 2x + \ y \ = 12 \rightarrow 2x + y = 12 \rightarrow 2x + y = 12 \\
-3x + \ y \ = 2 \rightarrow -(-3x + y) = -(2) \rightarrow 3x - y = -2 \\
\ \ 5x + 0y = 10
\end{array}$$

You have eliminated the *y* variable, and the problem can now be solved.

The following video describe a similar problem where you can eliminate one variable by adding the two equations together.

Watch this video online: https://youtu.be/M4IEmwcqR3c

Caution! When you add the opposite of one entire equation to another, make sure to change the sign of EVERY term on both sides of the equation. This is a very common mistake to make.

Example

Use elimination to solve the system.

$$\begin{aligned}
2x + y &= 12 \\
-3x + y &= 2
\end{aligned}$$

Show Solution

The following are two more examples showing how to solve linear systems of equations using elimination.

Example

Use elimination to solve the system.

$$\begin{aligned} -2x+3y &= -1 \\ 2x+5y &= 25 \end{aligned}$$

Show Solution

Example

Use elimination to solve for *x* and *y.*

$4x+2y=14$

$5x+2y=16$

Show Solution

Go ahead and check this last example—substitute (2, 3) into both equations. You get two true statements: 14=14 and 16=16!

Notice that you could have used the opposite of the first equation rather than the second equation and gotten the same result.

Recognize systems that have no solution or an infinite number of solutions

Just as with the substitution method, the elimination method will sometimes eliminate *both* variables, and you end up with either a true statement or a false statement. Recall that a false statement means that there is no solution.

Let's look at an example.

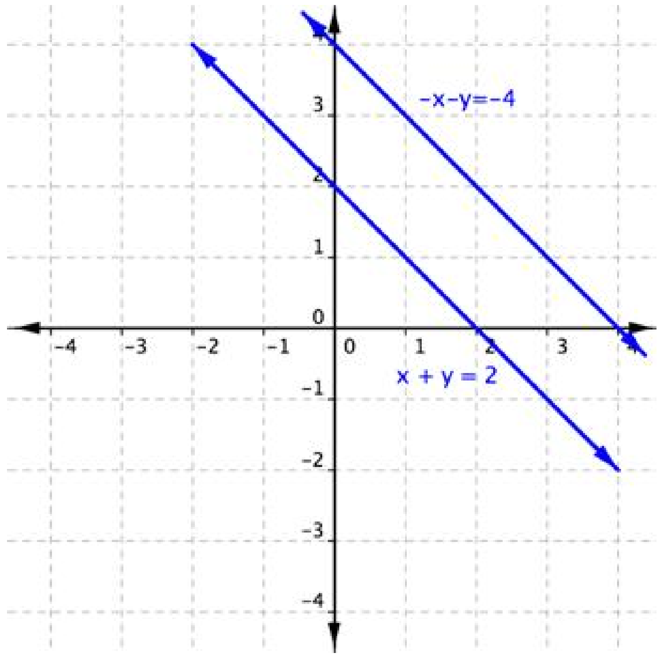
Example

Solve for *x* and *y.*

$$\begin{aligned} -x-y &= -4 \\ x+y &= 2 \end{aligned}$$

Show Solution

Graphing these lines shows that they are parallel lines and as such do not share any point in common, verifying that there is no solution.



If both variables are eliminated and you are left with a true statement, this indicates that there are an infinite number of ordered pairs that satisfy both of the equations. In fact, the equations are the same line.

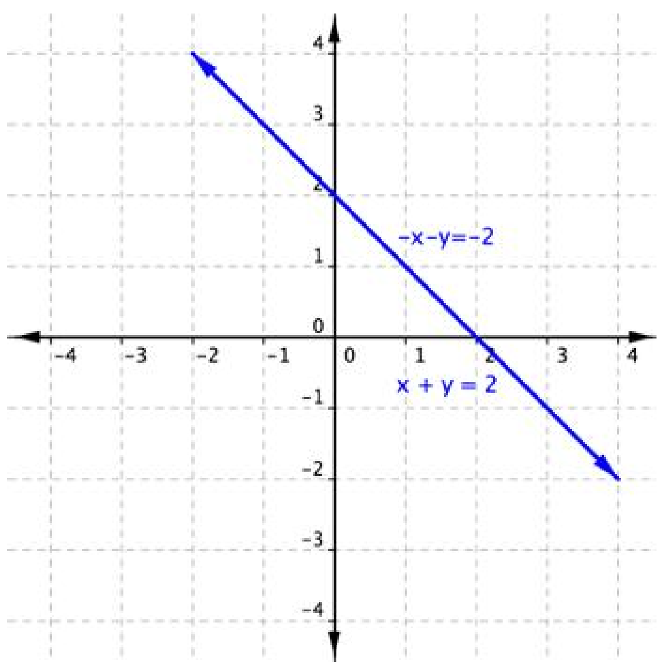
Example

Solve for *x* and *y*.

$$x+y=2$$
$$-x-y=-2$$

Show Solution

Graphing these two equations will help to illustrate what is happening.



In the following video, a system of equations which has no solutions is solved using the method of elimination.

Watch this video online: https://youtu.be/z5_ACYtzW98

Solve a system of equations when multiplication is necessary to eliminate a variable

Many times adding the equations or adding the opposite of one of the equations will not result in eliminating a variable. Look at the system below.

$$3x+4y=52$$
$$5x+y=30$$

If you add the equations above, or add the opposite of one of the equations, you will get an equation that still has two variables. So let's now use the multiplication property of equality first. You can multiply both sides of one of the equations by a number that will allow you to eliminate the same variable in the other equation.

We do this with multiplication. Notice that the first equation contains the term 4*y*, and the second equation contains the term *y*. If you multiply the second equation by –4, when you add both equations the *y* variables will add up to 0.

The following example takes you through all the steps to find a solution to this system.

Example

Solve for *x* and *y*.

Equation A: $3x+4y=52$

Equation B: $5x+y=30$
Show Solution

Caution! When you use multiplication to eliminate a variable, you must multiply EACH term in the equation by the number you choose. Forgetting to multiply every term is a common mistake.

There are other ways to solve this system. Instead of multiplying one equation in order to eliminate a variable when the equations were added, you could have multiplied *both* equations by different numbers.

Let's remove the variable *x* this time. Multiply Equation A by 5 and Equation B by -3.

Example

Solve for *x* and *y*.

$$3x+4y=52$$
$$5x+y=30$$
Show Solution

These equations were multiplied by 5 and -3 respectively, because that gave you terms that would add up to 0. Be sure to multiply all of the terms of the equation.

In the following video, you will see an example of using the elimination method for solving a system of equations.

Watch this video online: https://youtu.be/_liDhKops2w

It is possible to use the elimination method with multiplication and get a result that indicates no solutions or infinitely many solutions, just as with the other methods we have learned for finding solutions to systems. In the following example, you will see a system that has infinitely many solutions.

Example

Solve for *x* and *y*.

Equation A: $x - 3y = -2$

Equation B: $-2x + 6y = 4$
Show Solution

In the following video, the elimination method is used to solve a system of equations. Notice that one of the equations needs to be multiplied by a negative one first. Additionally, this system has an infinite number of solutions.

Watch this video online: https://youtu.be/NRxh9Q16Ulk

Summary

The substitution method is one way of solving systems of equations. To use the substitution method, use one equation to find an expression for one of the variables in terms of the other variable. Then substitute that expression in place of that variable in the second equation. You can then solve this equation as it will now have only one variable. Solving using the substitution method will yield one of three results: a single value for each variable within the system (indicating one solution), an untrue statement (indicating no solutions), or a true statement (indicating an infinite number of solutions).

Combining equations is a powerful tool for solving a system of equations. Adding or subtracting two equations in order to eliminate a common variable is called the elimination (or addition) method. Once one variable is eliminated, it becomes much easier to solve for the other one.

Multiplication can be used to set up matching terms in equations before they are combined to aid in finding a solution to a system. When using the multiplication method, it is important to multiply all the terms on both sides of the equation —not just the one term you are trying to eliminate.

APPLICATIONS OF SYSTEMS

Learning Objectives

- Solve mixture problems
  - Write a system of linear equations representing a mixture problem, solve the system and interpret the results
- Solve value problems
  - Write a system of linear equations representing a number problem
  - Determine and apply an appropriate method for solving the system
- Solve cost and revenue problems
  - Specify what the variables in a cost/ revenue system of linear equations represent
  - Determine and apply an appropriate method for solving the system
  - Write a system of inequalites that represents the profit region
  - Interpret the solutions to a system of cost/ revenue equations and inequalities

Write a system of linear equations representing a mixture problem, solve the system and interpret the results

One application of systems of equations are mixture problems. Mixture problems are ones where two different solutions are mixed together resulting in a new final solution. A solution is a mixture of two or more different substances like water and salt or vinegar and oil. Most biochemical reactions occur in liquid solutions, making them important for doctors, nurses, and researchers to understand. There are many other disciplines that use solutions as well.

The concentration or strength of a liquid solution is often described as a percentage. This number comes from the ratio of how much mass is in a specific volume of liquid. For example if you have 50 grams of salt in a 100mL of water you have a 50% salt solution based on the following ratio:

$$\frac{50 \text{ grams}}{100 \text{ mL}} = 0.50 \frac{\text{grams}}{\text{mL}} = 50\ \%$$

Solutions used for most purposes typically come in pre-made concentrations from manufacturers, so if you need a custom concentration, you would need to mix two different strengths. In this section, we will practice writing equations that represent the outcome from mixing two different concentrations of solutions.

We will use the following table to help us solve mixture problems:

	Amount	Concentration (%)	Total
Solution 1			
Solution 2			
Final Solution			

To demonstrate why the table is helpful in solving for unknown amounts or concentrations of a solution, consider two solutions that are mixed together, one is 120mL of a 9% solution, and the other is 75mL of a 23% solution. If we mix both of these solutions together we will have a new volume and a new mass of solute and with those we can find a new concentration.

First, find the total mass of solids for each solution by multiplying the volume by the concentration.

	Amount	Concentration (%)	Total Mass
Solution 1	120 mL	0.09 $\frac{\text{grams}}{\text{mL}}$	$\left(120\ \cancel{\text{mL}}\right)\left(0.09\frac{\text{grams}}{\cancel{\text{mL}}}\right) = 10.8 \text{ grams}$
Solution 2	75 mL	0.23 $\frac{\text{grams}}{\text{mL}}$	$\left(75\ \cancel{\text{mL}}\right)\left(0.23\frac{\text{grams}}{\cancel{\text{mL}}}\right) = 17.25 \text{ grams}$
Final Solution			

Next we add the new volumes and new masses.

	Amount	Concentration (%)	Total Mass
Solution 1	120 mL	0.09 $\frac{\text{grams}}{\text{mL}}$	$\left(120\ \cancel{\text{mL}}\right)\left(0.09\frac{\text{grams}}{\cancel{\text{mL}}}\right) = 10.8 \text{ grams}$
Solution 2	75 mL	0.23 $\frac{\text{grams}}{\text{mL}}$	$\left(75\ \cancel{\text{mL}}\right)\left(0.23\frac{\text{grams}}{\cancel{\text{mL}}}\right) = 17.25 \text{ grams}$

	Amount	Concentration (%)	Total Mass
Final Solution	195 mL	$\frac{28.05 \text{ grams}}{195 \text{ mL}} = 0.14 = 14\ \%$	$10.8 \text{ grams } + 17.25 \text{ grams } = 28.05 \text{ grams}$

Now we have used mathematical operations to describe the result of mixing two different solutions. We know the new volume, concentration and mass of solute in the new solution. In the following examples, you will see that we can use the table to find an unknown final volume or concentration. These problems can have either one or two variables. We will start with one variable problems, then move to two variable problems.

Example

A chemist has 70 mL of a 50% methane solution. How much of an 80% solution must she add so the final solution is 60% methane?
Show Solution

The above problem illustrates how we can use the mixture table to define an equation to solve for an unknown volume. In the next example we will start with two known concentrations and use a system of equations to find two starting volumes necessary to achieve a specified final concentration.

Example

A farmer has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 42 gallons of 20% butterfat?
Show Solution

In the following video you will be given an example of how to solve a mixture problem without using a table, and interpret the results.

Watch this video online: https://youtu.be/4s5MCqphpKo

Write a system of linear equations representing a value problem

Systems of equations are a very useful tool for modeling real-life situations and answering questions about them. If you can translate the application into two linear equations with two variables, then you have a system of equations that you can solve to find the solution. You can use any method to solve the system of equations.

One application of system of equations are known as value problems. Value problems are ones where each variable has a value attached to it. For example, the marketing team for an event venue wants to know how to focus their advertising based on who is attending specific events—children, or adults? They know the cost of a ticket to a basketball game is $25.00 for children and $50.00 for adults. Additionally, on a certain day, attendance at the game is 2,000 and the total gate revenue is $70,000. How can the marketing team use this information to find out whether to spend more money on advertising directed toward children or adults?

We will use a table to help us set up and solve this value problem. The basic structure of the table is shown below:

Number (usually what you are trying to find)	Value	Total
Item 1		
Item 2		
Total		

The first column in the table is used for the number of things we have. Quite often, this will be our variables. The second column is used for the value each item has. The third column is used for the total value which we calculate by multiplying the number by the value.

Example

Find the total number of child and adult tickets sold given that the cost of a ticket to a basketball game is $25.00 for children and $50.00 for adults. Additionally, on a certain day, attendance at the game is 2,000 and the total gate revenue is $70,000.
Show Solution

This example showed you how to find two unknown values given information that connected the two unknowns. With two equations, you are able to find a solution for two unknowns. If you were to have three unknowns, you would need three equations to find them, and so on.

In the following video, you are given an example of how to use a system of equations to find the number of children and adults admitted to an amusement park based on entrance fees and total revenue. This example shows how to write equations and solve the system without a table.

Watch this video online: https://youtu.be/uH4CgUhuDv0

In the next example, we will find the number of coins in a change jar given the total amount of money in the jar and the fact that the coins are either quarters or dimes.

Example

In a change jar there are 11 coins that have a value of S1.85. The coins are either quarters or dimes. How many of each kind of coin is in the jar?
Show Solution

In the following video, you will see an example similar to the previous one, except that the equations are written and solved without the use of a table.

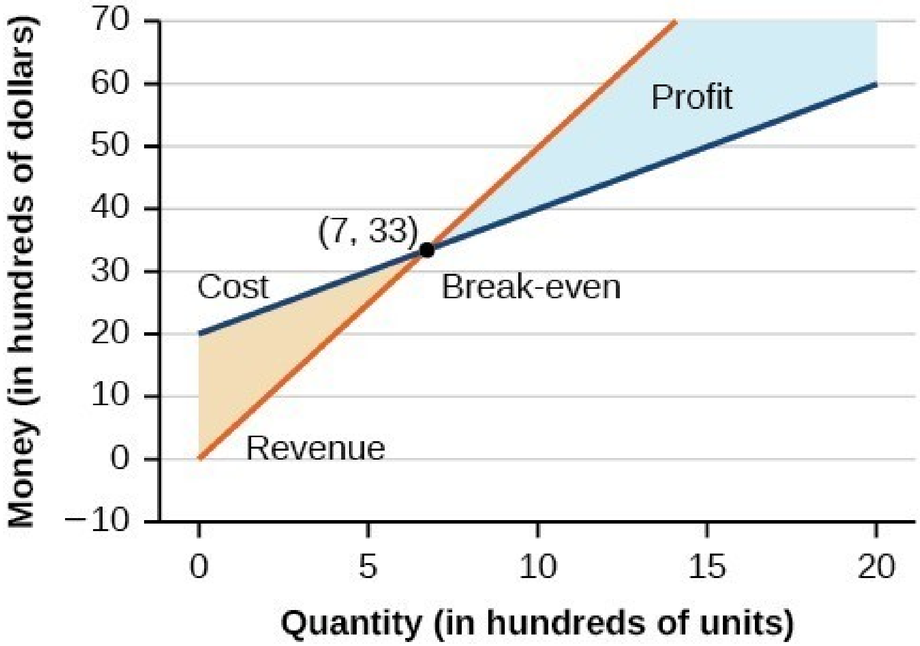
Watch this video online: https://youtu.be/GZYtSP-X_is

Cost and Revenue Problems

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible?

Using what we have learned about systems of equations, we can answer these questions. The skateboard manufacturer's revenue equation is the equation used to calculate the amount of money that comes into the business. It can be represented as $y = xp$, where $x =$ quantity and $p =$ price. The revenue equation is shown in orange in the graph below.

The cost equation is the equation used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost equation is shown in blue in the graph below. The x -axis represents quantity in hundreds of units. The *y*-axis represents both cost and revenue in hundreds of dollars. We won't learn how to write a cost equation in this example, they will be given to you. If you take any business or economics courses, you will learn more about how to write a cost equation.



The point at which the two lines intersect is called the break-even point, we learned that this is the solution to the system of linear equations that in this case comprise the cost and revenue equations.

Read the axes of the graph carefully, note that quantity is in hundreds, and money is in thousands. The solution to the graphed system is (7, 33). This means that if 700 units are produced, the cost to make them is \$3,300 and the revenue is also \$3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss.

Example

A business wants to manufacture bike frames. Before they start production, they need to make sure they can make a profit with the materials and labor force they have. Their accountant has given them a cost equation of $y = 0.85x + 35,000$ and a revenue equation of $y = 1.55x$:

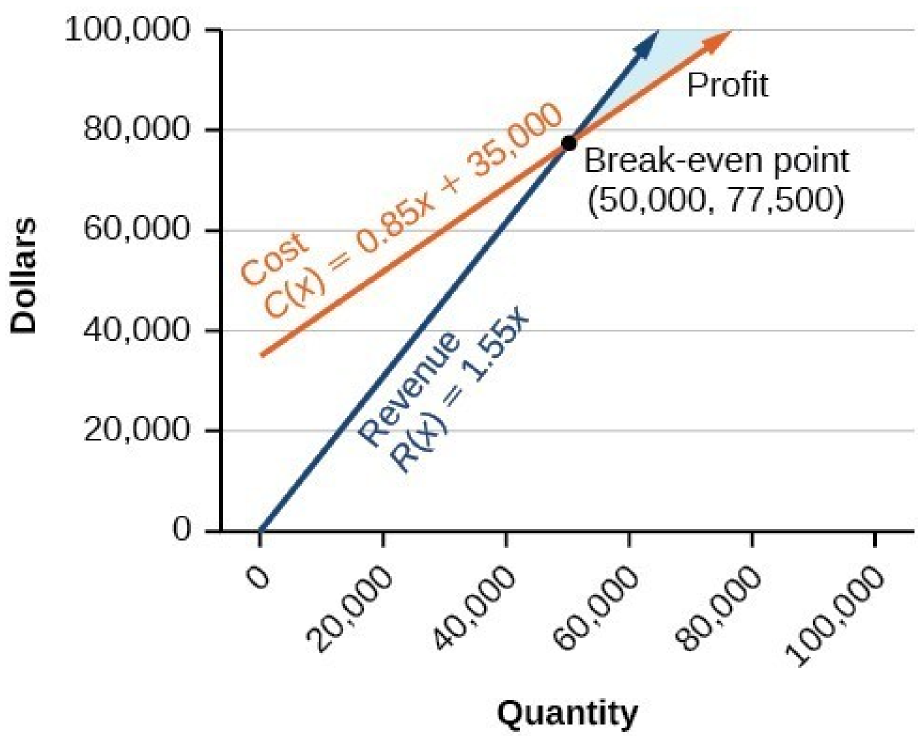
1. Interpret x and y for the cost equation
2. Interpret x and y for the revenue equation

Show Solution

Example

Given the same cost and revenue equations from the previous example, find the break-even point for the bike manufacturer. Interpret the solution with words.
Cost: $y = 0.85x + 35,000$
Revenue: $y = 1.55x$
Show Solution

In the next example, you will see how the information you learned about systems of linear inequalities can be applied to answering questions about cost and revenue. Below is a graph of the Cost/ Revenue system in the previous system:



Note how the blue shaded region between the Cost and Revenue equations is labeled Profit. This is the “sweet spot” that the company wants to achieve where they produce enough bike frames at a minimal enough cost to make money. They don’t want more money going out than coming in!

The following example shows how to write the system of linear equations as a system of linear inequalities whose solution set is the profit region for the system.

Example

Define the profit region for the bike manufacturing business using inequalities, given the system of linear equations:
Cost: $y = 0.85x + 35,000$
Revenue: $y = 1.55x$
Show Solution

In the following video you will see an example of how to find the break even point for a small sno-cone business.

Watch this video online: https://youtu.be/qey3FmE8saQ

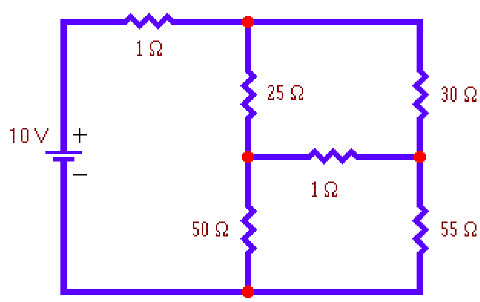
Summary

In this section, we saw two examples of writing a system of two linear equations to find two unknowns that were related to each other. In the first, the equations were related by the sum of the number of tickets bought and the sum of the total revenue brought in by the tickets sold. In the second problem, the relationships were similar. The two variables were related by the sum of the number of coins, and the total value of the coins.

We have seen that systems of linear equations and inequalities can help to define market behaviors that are very helpful to businesses. The intersection of cost and revenue equations gives the break even point, and also helps define the region for which a company will make a profit.

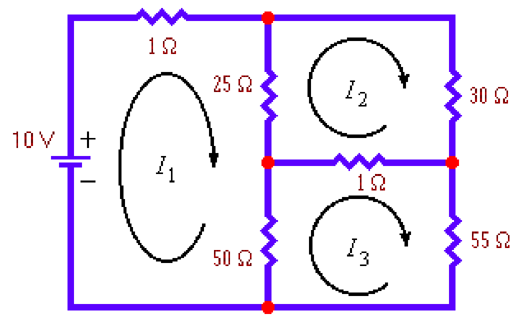
CONCLUSION

In this module, we came a few steps closer to solving the $1,000,000 question by learning some of the basic principles of systems of equations. We learned that using systems of linear equations with two variables allows us to describe relationships that are affected by a number of variables. We saw that graphing systems is the same as graphing single equations, and that the solution to a system of two linear equations is the point where the two lines intersect. The number of practical applications for systems of equations spans nearly all disciplines from geology to business to economics to engineering and beyond. For example, electrical engineers may be interested in defining the strength of a current flowing through a circuit as shown in the diagram below.



Electrical Circuit

A system of equations based on Kirchoff's Voltage Law can be constructed and solved to find the current in amperes that flows through each branch of the circuit. Kirchoff's Voltage Law is based on the principle that voltage changes in intensity as it moves around a circuit. Each equation in the system represents one branch of the circuit, so the number of equations needed to find the current depends on the number of branches in the circuit. For example, the figure below depicts the circuit from above with it's three loop currents identified:



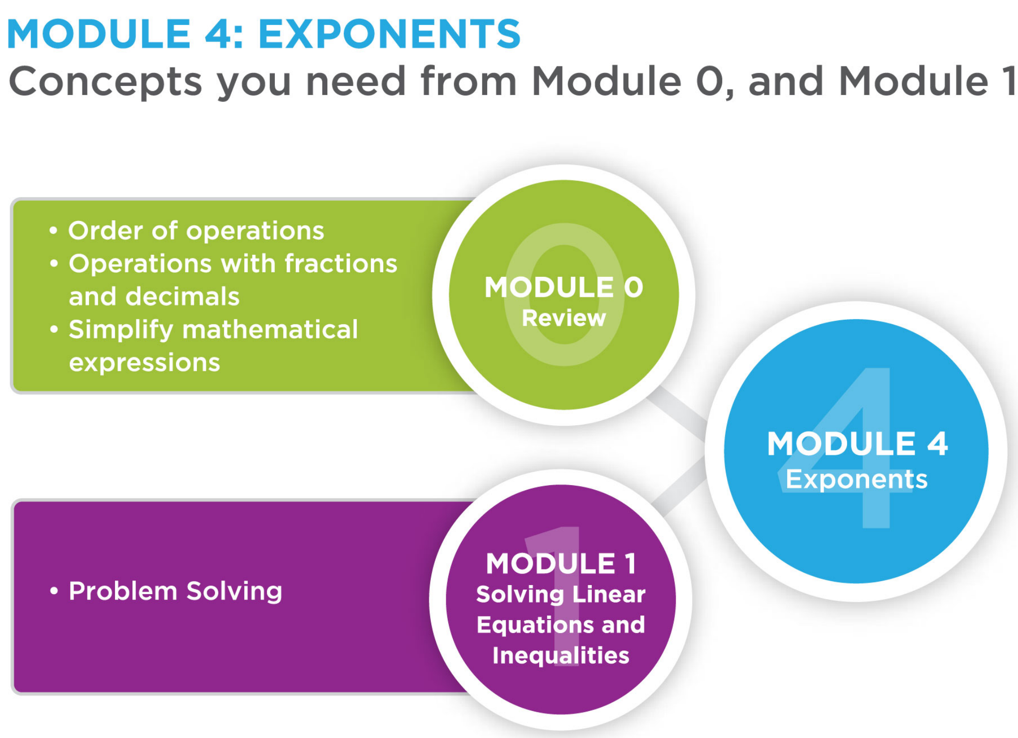
Electrical Circuit With 3 Loop Currents

Because there are three loop currents, we would need to write three equations with three variables based on Kirchoff's Law to find the three unknown currents that comprise the system. If you are curious to learn more about Kirchoff's Law and loop currents, visit this site.

If you continue to take more math courses, you may learn how to solve systems of three equations, and may even get to take an entire course in systems called Linear Algebra. And from there you can discover more about the Navier-Stokes equations and, perhaps, solve one of the seven most important math problems in the world!

MODULE 4: EXPONENTS

INTRODUCTION

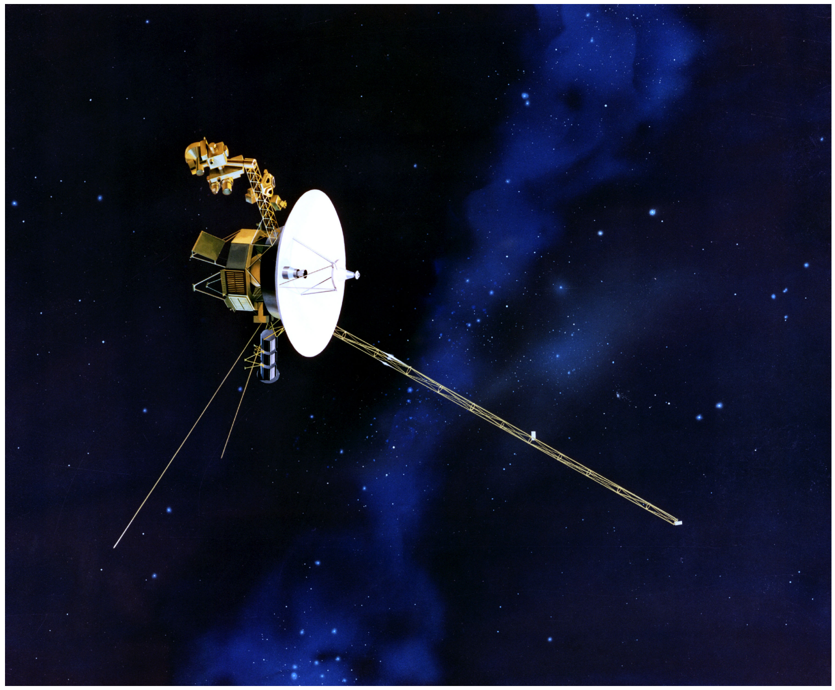


Concepts you need from module 0, and 1.

Why study exponents?

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. For example, Star Wars fans may remember Han Solo bragging about the Millennium Falcon's ability to make the Kessel Run in less than 12 parsecs in Episode IV. He was referring to a smuggler's route with sections that were flown in hyperspace, making length an important factor in how quickly a ship could make the run.

In reality, a parsec is a unit of length used to measure large distances to objects outside the solar system. A parsec is equal to about 31 trillion kilometers, or 19 trillion miles in length. Rather than writing all the zeros associated with the number 1 trillion (1,000,000,000,000) we commonly use the written words or scientific notation, which is also called exponential notation. Scientific notation uses exponents to represent the number of zeros that come before or after the important digits of a very small or large number. Using scientific notation, 19 trillion miles would be written 1.9×10^{13} miles. In this example the number 13 is the exponent and the number 10 is referred to as the base.



Artist's concept of Voyager in flight.

The most distant space probe, Voyager 1, was 0.0006 parsecs from Earth as of March 2015. It took Voyager 37 years to cover that distance. Voyager 1 was launched by NASA on September 5, 1977. As of 2013, the probe was moving with a relative velocity to the sun of about 17030 m/s. With the velocity the probe is currently maintaining, Voyager 1 is traveling about 325 million miles per year, or 520 million kilometers per year. Here are some more distances to well-known astronomical objects in parsecs:

- The distance to the open cluster Pleiades is 130 parsecs from Earth. That's 1.7×10^{15} miles.
- The center of the Milky Way is more than 8 kiloparsecs (a kiloparsec is 1000 parsecs) from Earth, and the Milky Way is roughly 34 kiloparsecs across.
- The nearest star to Earth (other than the sun), Proxima Centauri, is about 1.3 parsecs from the sun.
- Most of the stars visible to the unaided eye in the nighttime sky are within 500 parsecs of the sun.

In this section, you will learn the rules for algebraic operations in terms with exponents, then apply them to calculations involving very large or small numbers.

Learning Outcomes

Exponent Rules
- Evaluate terms and expressions with exponents
- Use the product and quotient rules
- Use the power rule for exponents
- Define and use the negative and zero exponent rules
- Simplify compound expressions

Scientific Notation
- Use scientific notation
- Multiply and divide numbers in scientific notation
- Solve problems with scientific notation

TERMS AND EXPRESSIONS WITH EXPONENTS

Learning Objectives

- Identify the components of a term containing integer exponents
- Evaluate expressions containing integer exponents

A lingua franca is a common language used to make communication possible between people who speak different languages. Math, as a general idea, is sometimes thought of as an example of a common language because formulas and equations don't rely on fluency in a specific language.

But even within mathematics a common language is needed in order to communicate mathematical ideas clearly and efficiently. **Exponential notation** (remember this can also called scientific notation) was developed to write repeated multiplication more efficiently. For example, growth occurs in living organisms by the division of cells. One type of cell divides 2 times in an hour. So in 12 hours, the cell will divide $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ times. This can be written more efficiently as 2^{12}. Expressing it this way is a much more efficient and clear way to express the ways cells divide.

In this section we will learn how to simplify and perform mathematical operations such as multiplication and division on terms that have exponents. We will also learn how to use scientific notation to represent very large or very small numbers, and perform mathematical operations on them.

Anatomy of exponential terms

We use exponential notation to write repeated multiplication. For example $10 \cdot 10 \cdot 10$ can be written more succinctly as 10^3. The 10 in 10^3 is called the **base**. The 3 in 10^3 is called the **exponent**. The expression 10^3 is called the exponential expression. Knowing the names for the parts of an exponential expression or term will help you learn how to perform mathematical operations on them.

$\text{base} \rightarrow 10^{3 \leftarrow \text{exponent}}$

10^3 is read as "10 to the third power" or "10 cubed." It means $10 \cdot 10 \cdot 10$, or 1,000.

8^2 is read as "8 to the second power" or "8 squared." It means $8 \cdot 8$, or 64.

5^4 is read as "5 to the fourth power." It means $5 \cdot 5 \cdot 5 \cdot 5$, or 625.

b^5 is read as "*b* to the fifth power." It means $b \cdot b \cdot b \cdot b \cdot b$. Its value will depend on the value of *b*.

The exponent applies only to the number that it is next to. Therefore, in the expression xy^4, only the *y* is affected by the 4. xy^4 means $x \cdot y \cdot y \cdot y \cdot y$. The *x* in this term is a **coefficient** of *y*.

If the exponential expression is negative, such as -3^4, it means $-(3 \cdot 3 \cdot 3 \cdot 3)$ or -81.

If -3 is to be the base, it must be written as $(-3)^4$, which means $-3 \cdot -3 \cdot -3 \cdot -3$, or 81.

Likewise, $(-x)^4 = (-x) \cdot (-x) \cdot (-x) \cdot (-x) = x^4$, while $-x^4 = -(x \cdot x \cdot x \cdot x)$.

You can see that there is quite a difference, so you have to be very careful! The following examples show how to identify the base and the exponent, as well as how to identify the expanded and exponential format of writing repeated multiplication.

Example

Identify the exponent and the base in the following terms, then simplify:

1. 7^2
2. $\left(\frac{1}{2}\right)^3$
3. $2x^3$
4. $(-5)^2$

Show Solution

In the following video you are provided more examples of applying exponents to various bases.

Watch this video online: https://youtu.be/ocedY91LHKU

Evaluate expressions

Evaluating expressions containing exponents is the same as evaluating the linear expressions from earlier in the course. You substitute the value of the variable into the expression and simplify.

You can use the order of operations to evaluate the expressions containing exponents. First, evaluate anything in Parentheses or grouping symbols. Next, look for Exponents, followed by Multiplication and Division (reading from left to right), and lastly, Addition and Subtraction (again, reading from left to right).

So, when you evaluate the expression $5x^3$ if $x=4$, first substitute the value 4 for the variable *x*. Then evaluate, using order of operations.

Example

Evaluate $5x^3$ if $x=4$.
Show Solution

In the example below, notice the how adding parentheses can change the outcome when you are simplifying terms with exponents.

Example

Evaluate $(5x)^3$ if $x=4$.
Show Solution

The addition of parentheses made quite a difference! Parentheses allow you to apply an exponent to variables or numbers that are multiplied, divided, added, or subtracted to each other.

Example

Evaluate x^3 if $x=-4$.
Show Solution

Caution! Whether to include a negative sign as part of a base or not often leads to confusion. To clarify whether a negative sign is applied before or after the exponent, here is an example.
What is the difference in the way you would evaluate these two terms?

1. -3^2
2. $(-3)^2$

To evaluate 1), you would apply the exponent to the three first, then apply the negative sign last, like this:

$$-\left(3^2\right)$$
$$=-(9)=-9$$

To evaluate 2), you would apply the exponent to the 3 and the negative sign:

$$(-3)^2$$
$$=(-3)\cdot(-3)$$
$$=9$$

The key to remembering this is to follow the order of operations. The first expression does not include parentheses so you would apply the exponent to the integer 3 first, then apply the negative sign. The second expression includes parentheses, so hopefully you will remember that the negative sign also gets squared.

In the next sections, you will learn how to simplify expressions that contain exponents. Come back to this page if you forget how to apply the order of operations to a term with exponents, or forget which is the base and which is the exponent!

In the following video you are provided with examples of evaluating exponential expressions for a given number.

Watch this video online: https://youtu.be/pQNz8IpVVg0

RULES FOR EXPONENTS

Learning Objectives

- Product and Quotient Rules
  - Use the product rule to multiply exponential expressions
  - Use the quotient rule to divide exponential expressions
- The Power Rule for Exponents
  - Use the power rule to simplify expressions involving products, quotients, and exponents
- Negative and Zero Exponents
  - Define and use the zero exponent rule
  - Define and use the negative exponent rule
- Simplify Expressions Using the Exponent Rules
  - Simplify expressions using a combination of the exponent rules
  - Simplify compound exponential expressions with negative exponents

Repeated Image

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3. $2x^3$
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So, when you evaluate the expression $5x^3$ if $x = 4$, first substitute the value 4 for the variable *x*. Then evaluate, using order of operations.

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Show Solution

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Example

Evaluate $(5x)^3$ if $x=4$.
Show Solution

The addition of parentheses made quite a difference! Parentheses allow you to apply an exponent to variables or numbers that are multiplied, divided, added, or subtracted to each other.

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What is the difference in the way you would evaluate these two terms?

1. -3^2
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Use the product rule to multiply exponential expressions

Exponential notation was developed to write repeated multiplication more efficiently. There are times when it is easier or faster to leave the expressions in exponential notation when multiplying or dividing. Let's look at rules that will allow you to do this.

For example, the notation 5^4 can be expanded and written as $5\cdot5\cdot5\cdot5$, or 625. And don't forget, the exponent only applies to the number immediately to its left, unless there are parentheses.

What happens if you multiply two numbers in exponential form with the same base? Consider the expression $2^3 2^4$. Expanding each exponent, this can be rewritten as $(2\cdot2\cdot2)(2\cdot2\cdot2\cdot2)$ or $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$. In exponential form, you would write the product as 2^7. Notice that 7 is the sum of the original two exponents, 3 and 4.

What about x^2x^6? This can be written as $(x \cdot x)(x \cdot x \cdot x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ or x^8. And, once again, 8 is the sum of the original two exponents. This concept can be generalized in the following way:

The Product Rule for Exponents

For any number *x* and any integers *a* and *b*, $(x^a)(x^b) = x^{a+b}$.
To multiply exponential terms with the same base, add the exponents.

Caution! When you are reading mathematical rules, it is important to pay attention to the conditions on the rule. For example, when using the product rule, you may only apply it when the terms being multiplied have the same base and the exponents are integers. Conditions on mathematical rules are often given before the rule is stated, as in this example it says "For any number *x*, and any integers *a* and *b*."

Example

Simplify.

$(a^3)(a^7)$
Show Solution

When multiplying more complicated terms, multiply the coefficients and then multiply the variables.

Example

Simplify.

$5a^4 \cdot 7a^6$
Show Solution

Caution! Do not try to apply this rule to sums.
Think about the expression $(2+3)^2$
Does $(2+3)^2$ equal $2^2 + 3^2$?
No, it does not because of the order of operations!
$(2+3)^2 = 5^2 = 25$
and
$2^2 + 3^2 = 4 + 9 = 13$
Therefore, you can only use this rule when the numbers inside the parentheses are being multiplied (or divided, as we will see next).

Watch this video online: https://youtu.be/hA9AT7QsXWo

Use the quotient rule to divide exponential expressions

Let's look at dividing terms containing exponential expressions. What happens if you divide two numbers in exponential form with the same base? Consider the following expression.

$$\frac{4^5}{4^2}$$

You can rewrite the expression as: $\frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4}$. Then you can cancel the common factors of 4 in the numerator and denominator:

Finally, this expression can be rewritten as 4^3 using exponential notation. Notice that the exponent, 3, is the difference between the two exponents in the original expression, 5 and 2.

So, $\frac{4^5}{4^2}=4^{5-2}=4^3$.

Be careful that you subtract the exponent in the denominator from the exponent in the numerator.

So, to divide two exponential terms with the same base, subtract the exponents.

The Quotient (Division) Rule for Exponents

For any non-zero number *x* and any integers *a* and *b*: $\frac{x^a}{x^b}=x^{a-b}$

Example

Evaluate. $\frac{4^9}{4^4}$

Show Solution

When dividing terms that also contain coefficients, divide the coefficients and then divide variable powers with the same base by subtracting the exponents.

Example

Simplify. $\frac{12x^4}{2x}$

Show Solution

In the following video we show another example of how to use the quotient rule to divide exponential expressions

Watch this video online: https://youtu.be/Jmf-CPhm3XM

Raise powers to powers

Another word for exponent is power. You have likely seen or heard an example such as 3^5 can be described as 3 raised to the 5th power. In this section we will further expand our capabilities with exponents. We will learn what to do when a term with a power is raised to another power, and what to do when two numbers or variables are multiplied and both are raised to an exponent. We will also learn what to do when numbers or variables that are divided are raised to a power. We will begin by raising powers to powers.

Let's simplify $\left(5^2\right)^4$. In this case, the base is 5^2 and the exponent is 4, so you multiply 5^2 four times:
$\left(5^2\right)^4=5^2\cdot5^2\cdot5^2\cdot5^2=5^8$ (using the Product Rule—add the exponents).

$\left(5^2\right)^4$ is a power of a power. It is the fourth power of 5 to the second power. And we saw above that the answer is 5^8. Notice that the new exponent is the same as the product of the original exponents: $2\cdot4=8$.

So, $\left(5^2\right)^4=5^{2\cdot4}=5^8$ (which equals 390,625, if you do the multiplication).

Likewise, $\left(x^4\right)^3=x^{4\cdot3}=x^{12}$

This leads to another rule for exponents—the **Power Rule for Exponents**. To simplify a power of a power, you multiply the exponents, keeping the base the same. For example, $\left(2^3\right)^5=2^{15}$.

The Power Rule for Exponents

For any positive number *x* and integers *a* and *b*: $\left(x^{a}\right)^{b}=x^{a\cdot b}$.
Take a moment to contrast how this is different from the product rule for exponents found on the previous page.

Example

Simplify $6\left(c^{4}\right)^{2}$.
Show Solution

Raise a product to a power

Simplify this expression.

$$(2a)^{4}=(2a)(2a)(2a)(2a)=(2\cdot 2\cdot 2\cdot 2)(a\cdot a\cdot a\cdot a\cdot a)=\left(2^{4}\right)\left(a^{4}\right)=16a^{4}$$

Notice that the exponent is applied to each factor of 2*a*. So, we can eliminate the middle steps.

$$(2a)^{4}=\left(2^{4}\right)\left(a^{4}\right)\text{, applying the 4 to each factor, 2 and }a$$

$$=16a^{4}$$

The product of two or more numbers raised to a power is equal to the product of each number raised to the same power.

A Product Raised to a Power

For any nonzero numbers *a* and *b* and any integer *x*, $(ab)^{x}=a^{x}\cdot b^{x}$.
How is this rule different from the power raised to a power rule? How is it different from the product rule for exponents on the previous page?

Example

Simplify. $(2yz)^{6}$
Show Solution

If the variable has an exponent with it, use the Power Rule: multiply the exponents.

Example

Simplify. $\left(-7a^{4}b\right)^{2}$
Show Solution

Watch this video online: https://youtu.be/Hgu9HKDHTUA

Raise a quotient to a power

Now let's look at what happens if you raise a quotient to a power. Remember that quotient means divide. Suppose you have $\frac{3}{4}$ and raise it to the 3rd power.

$$\left(\frac{3}{4}\right)^3=\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)=\frac{3\cdot3\cdot3}{4\cdot4\cdot4}=\frac{3^3}{4^3}$$

You can see that raising the quotient to the power of 3 can also be written as the numerator (3) to the power of 3, and the denominator (4) to the power of 3.

Similarly, if you are using variables, the quotient raised to a power is equal to the numerator raised to the power over the denominator raised to power.

$$\left(\frac{a}{b}\right)^4=\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)=\frac{a\cdot a\cdot a\cdot a}{b\cdot b\cdot b\cdot b}=\frac{a^4}{b^4}$$

When a quotient is raised to a power, you can apply the power to the numerator and denominator individually, as shown below.

$$\left(\frac{a}{b}\right)^4=\frac{a^4}{b^4}$$

A Quotient Raised to a Power

For any number *a*, any non-zero number *b*, and any integer *x*, $\left(\frac{a}{b}\right)^x=\frac{a^x}{b^x}$.

Example

Simplify. $\left(\frac{2x^2y}{x}\right)^3$

Show Solution

In the following video you will be shown examples of simplifying quotients that are raised to a power.

Watch this video online: https://youtu.be/ZbxgDRV35dE

Define and use the zero exponent rule

When we defined the quotient rule, we only worked with expressions like the following: $\frac{4^9}{4^4}$, where the exponent in the numerator (up) was greater than the one in the denominator (down), so the final exponent after simplifying was always a positive number, and greater than zero. In this section, we will explore what happens when we apply the quotient rule for exponents and get a negative or zero exponent.

What if the exponent is zero?

To see how this is defined, let us begin with an example. We will use the idea that dividing any number by itself gives a result of 1.

$$\frac{t^8}{t^8}=\frac{\cancel{t^8}}{\cancel{t^8}}=1$$

If we were to simplify the original expression using the quotient rule, we would have

$\frac{t^8}{t^8}=t^{8-8}=t^0$

If we equate the two answers, the result is $t^0=1$. This is true for any nonzero real number, or any variable representing a real number.

$$a^0=1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined, or DNE (Does Not Exist).

Exponents of 0 or 1

Any number or variable raised to a power of 1 is the number itself.
$n^1=n$
Any non-zero number or variable raised to a power of 0 is equal to 1
$n^0=1$
The quantity 0^0 is undefined.

As done previously, to evaluate expressions containing exponents of 0 or 1, substitute the value of the variable into the expression and simplify.

Example

Evaluate $2x^0$ if $x=9$
Show Solution

Example

Simplify $\frac{c^3}{c^3}$.
Show Solution

In the following video there is an example of evaluating an expression with an exponent of zero, as well as simplifying when you get a result of a zero exponent.

Watch this video online: https://youtu.be/jKihp_DVDa0

Define and use the negative exponent rule

We proposed another question at the beginning of this section. Given a quotient like $\frac{2^m}{2^n}$ what happens when *n* is larger than *m*? We will need to use the *negative rule of exponents* to simplify the expression so that it is easier to understand.

Let's look at an example to clarify this idea. Given the expression:

$\frac{h^3}{h^5}$

Expand the numerator and denominator, all the terms in the numerator will cancel to 1, leaving two *h*s multiplied in the denominator, and a numerator of 1.

$$\begin{aligned}\frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\ &= \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} \\ &= \frac{1}{h \cdot h} \\ &= \frac{1}{h^2}\end{aligned}$$

We could have also applied the quotient rule from the last section, to obtain the following result:

$$\begin{aligned}\frac{h^3}{h^5} &= h^{3-5} \\ &= h^{-2}\end{aligned}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true when *h*, or any variable, is a real number and is not zero.

The Negative Rule of Exponents

For any nonzero real number a and natural number n, the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n}$$

Let's looks at some examples of how this rule applies under different circumstances.

Example

Evaluate the expression 4^{-3}.

Show Solution

Example

Write $\frac{(t^3)}{(t^8)}$ with positive exponents.

Show Solution

Example

Simplify $\left(\frac{1}{3}\right)^{-2}$.

Show Solution

Example

Simplify. $\frac{1}{4^{-2}}$ Write your answer using positive exponents.

Show Solution

In the follwoing video you will see examples of simplifying expressions with negative exponents.

Watch this video online: https://youtu.be/WvFlHjlIITg

Simplify expressions using a combination of exponent rules

Once the rules of exponents are understood, you can begin simplifying more complicated expressions. There are many applications and formulas that make use of exponents, and sometimes expressions can get pretty cluttered. Simplifying an expression before evaluating can often make the computation easier, as you will see in the following example which makes use of the quotient rule to simplify before substituting 4 for x.

Example

Evaluate $\dfrac{24x^8}{2x^5}$ when $x=4$.
Show Solution

Example

Evaluate $\dfrac{24x^8y^2}{(2x^3y)^2}$ when $x=4$ and $y=-2$.
Show Solution

Notice that you could have worked this problem by substituting 4 for *x* and 2 for *y* in the original expression. You would still get the answer of 96, but the computation would be much more complex. Notice that you didn't even need to use the value of *y* to evaluate the above expression.

In the following video you are shown examples of evaluating an exponential expression for given numbers.

Watch this video online: https://youtu.be/mD06EyGja2w

Usually, it is easier to simplify the expression before substituting any values for your variables, but you will get the same answer either way. In the next examples, you will see how to simplify expressions using different combinations of the rules for exponents.

Example

Simplify. $a^2\left(a^5\right)^3$
Show Solution

The following examples require the use of all the exponent rules we have learned so far. Remember that the product, power, and quotient rules apply when your terms have the same base.

Example

Simplify. $\dfrac{a^2(a^5)^3}{8a^8}$
Show Solution

Simplify Expressions With Negative Exponents

Now we will add the last layer to our exponent simplifying skills and practice simplifying compound expressions that have negative exponents in them. It is standard convention to write exponents as positive because it is easier for the user to understand the value associated with positive exponents, rather than negative exponents.

Use the following summary of negative exponents to help you simplify expressions with negative exponents.

Rules for Negative Exponents

With *a*, *b*, *m*, and *n* not equal to zero, and *m* and *n* as integers, the following rules apply:

$a^{-m} = \frac{1}{a^m}$

$\frac{1}{a^{-m}} = a^m$

$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$

When you are simplifying expressions that have many layers of exponents, it is often hard to know where to start. It is common to start in one of two ways:

- Rewrite negative exponents as positive exponents
- Apply the product rule to eliminate any "outer" layer exponents such as in the following term: $\left(5y^3\right)^2$

We will explore this idea with the following example:

Simplify. $\left(4x^3\right)^5 \cdot \left(2x^2\right)^{-4}$

Write your answer with positive exponents. The table below shows how to simplify the same expression in two different ways, rewriting negative exponents as positive first, and applying the product rule for exponents first. You will see that there is a column for each method that describes the exponent rule or other steps taken to simplify the expression.

Rewrite with positive Exponents First	Description of Steps Taken	Apply the Product Rule for Exponents First	Description of Steps Taken
$\frac{\left(4x^3\right)^5}{\left(2x^2\right)^4}$	move the term $\left(2x^2\right)^{-4}$ to the denominator with a positive exponent	$\left(4^5x^{15}\right)\left(2^{-4}x^{-8}\right)$	Apply the exponent of 5 to each term in expression on the left, and the exponent of -4 to each term in the expression on the right.
$\frac{\left(4^5x^{15}\right)}{\left(2^4x^8\right)}$	Use the product rule to apply the outer exponents to the terms inside each set of parentheses.	$\left(4^5\right)\left(2^{-4}\right)\left(x^{15} \cdot x^{-8}\right)$	Regroup the numerical terms and the variables to make combining like terms easier
$\left(\frac{4^5}{2^4}\right)\left(\frac{x^{15}}{x^8}\right)$	Regroup the numerical terms and the variables to make combining like terms easier	$\left(4^5\right)\left(2^{-4}\right)\left(x^{15-8}\right)$	Use the rule for multiplying terms with exponents to simplify the x terms
$\left(\frac{4^5}{2^4}\right)\left(x^{15-8}\right)$	Use the quotient rule to simplify the x terms	$\left(\frac{4^5}{2^4}\right)\left(x^7\right)$	Rewrite all the negative exponents with positive exponents
$\left(\frac{1{,}024}{16}\right)\left(x^7\right)$	Expand the numerical terms	$\left(\frac{1{,}024}{16}\right)\left(x^7\right)$	Expand the numerical terms
$64x^7$	Divide the numerical terms	$64x^7$	Divide the numerical terms

If you compare the two columns that describe the steps that were taken to simplify the expression, you will see that they are all nearly the same, except the order is changed slightly. Neither way is better or more correct than the other, it truly is a matter of preference.

Example

Simplify $\frac{\left(t^3\right)^2}{\left(t^2\right)^{-8}}$

Write your answer with positive exponents.
Show Solution

Example

Simplify $\frac{(5x)^{-2}y}{x^3y^{-1}}$

Write your answer with positive exponents.
Show Solution

In the next section, you will learn how to write very large and very small numbers using exponents. This practice is widely used in science and engineering.

Summary

- Evaluating expressions containing exponents is the same as evaluating any expression. You substitute the value of the variable into the expression and simplify.
- The product rule for exponents: For any number *x* and any integers *a* and *b*, $\left(x^a\right)\left(x^b\right)=x^{a+b}$.
- The quotient rule for exponents: For any non-zero number *x* and any integers *a* and *b*: $\frac{x^a}{x^b}=x^{a-b}$
- The power rule for exponents:
  - For any nonzero numbers *a* and *b* and any integer *x*, $(ab)^x=a^x\cdot b^x$.
  - For any number *a*, any non-zero number *b*, and any integer *x*, $\left(\frac{a}{b}\right)^x=\frac{a^x}{b^x}$

SCIENTIFIC NOTATION

Learning Objectives

- Define decimal and scientific notation
- Convert between scientific and decimal notation
- Multiply and divide numbers expressed in scientific notation
- Solve application problems involving scientific notation

Convert between scientific and decimal notation

Before we can convert between scientific and decimal notation, we need to know the difference between the two. **Scientific notation** is used by scientists, mathematicians, and engineers when they are working with very large or very small numbers. Using exponential notation, large and small numbers can be written in a way that is easier to read.

When a number is written in scientific notation, the **exponent** tells you if the term is a large or a small number. A positive exponent indicates a large number and a negative exponent indicates a small number that is between 0 and 1. It is difficult to understand just how big a billion or a trillion is. Here is a way to help you think about it.

Word	How many thousands	Number	Scientific Notation
million	1000 x 1000 = a thousand thousands	1,000,000	10^6
billion	(1000 x 1000) x 1000 = a thousand millions	1,000,000,000	10^9
trillion	(1000 x 1000 x 1000) x 1000 = a thousand billions	1,000,000,000,000	10^{12}

1 billion can be written as 1,000,000,000 or represented as 10^9. How would 2 billion be represented? Since 2 billion is 2 times 1 billion, then 2 billion can be written as 2×10^9.

A light year is the number of miles light travels in one year, about 5,880,000,000,000. That's a lot of zeros, and it is easy to lose count when trying to figure out the place value of the number. Using scientific notation, the distance is 5.88×10^{12} miles. The exponent of 12 tells us how many places to count to the left of the decimal. Another example of how scientific notation can make numbers easier to read is the diameter of a hydrogen atom, which is about 0.00000005 mm, and in scientific notation is 5×10^{-8} mm. In this case the -8 tells us how many places to count to the right of the decimal.

Outlined in the box below are some important conventions of scientific notation format.

Scientific Notation

A positive number is written in scientific notation if it is written as $a \times 10^n$ where the coefficient *a* is $1 \leq a < 10$, and *n* is an integer.

Look at the numbers below. Which of the numbers is written in scientific notation?

Number	**Scientific Notation?**	**Explanation**
1.85×10^{-2}	yes	$1 \leq 1.85 < 10$ -2 is an integer
$1.083 \times 10^{\frac{1}{2}}$	no	$\frac{1}{2}$ is not an integer
0.82×10^{14}	no	0.82 is not ≥ 1
10×10^3	no	10 is not < 10

Now let's compare some numbers expressed in both scientific notation and standard decimal notation in order to understand how to convert from one form to the other. Take a look at the tables below. Pay close attention to the exponent in the scientific notation and the position of the decimal point in the decimal notation.

Large Numbers			**Small Numbers**	
Decimal Notation	**Scientific Notation**		**Decimal Notation**	**Scientific Notation**
500.0	5×10^2		0.05	5×10^{-2}
80,000.0	8×10^4		0.0008	8×10^{-4}

43,000,000.0	4.3×10^{7}		0.00000043	4.3×10^{-7}
62,500,000,000.0	6.25×10^{10}		0.000000000625	6.25×10^{-10}

Convert from decimal notation to scientific notation

To write a *large* number in scientific notation, move the decimal point to the left to obtain a number between 1 and 10. Since moving the decimal point changes the value, you have to multiply the decimal by a power of 10 so that the expression has the same value.

Let's look at an example.

$$\begin{aligned} 180{,}000. &= 18{,}000.0 \times 10^{1} \\ &\quad 1{,}800.00 \times 10^{2} \\ &\quad 180.000 \times 10^{3} \\ &\quad 18.0000 \times 10^{4} \\ &\quad 1.80000 \times 10^{5} \\ 180{,}000 &= 1.8 \times 10^{5} \end{aligned}$$

Notice that the decimal point was moved 5 places to the left, and the exponent is 5.

Example

Write the following numbers in scientific notation.

1. $920{,}000{,}000$
2. $10{,}200{,}000$
3. $100{,}000{,}000{,}000$

Show Solution

To write a small number (between 0 and 1) in scientific notation, you move the decimal to the *right* and the exponent will have to be *negative,* as in the following example.

$$\begin{aligned} 0.\underrightarrow{0000}4 &= 00.0004 \times 10^{-1} \\ &\quad 000.004 \times 10^{-2} \\ &\quad 0000.04 \times 10^{-3} \\ &\quad 00000.4 \times 10^{-4} \\ &\quad 000004. \times 10^{-5} \\ 0.00004 &= 4 \times 10^{-5} \end{aligned}$$

You may notice that the decimal point was moved five places to the *right* until you got to the number 4, which is between 1 and 10. The exponent is -5.

Example

Write the following numbers in scientific notation.

1. 0.0000000000035
2. 0.0000000102
3. 0.00000000000000793

Show Solution

In the following video you are provided with examples of how to convert both a large and a small number in decimal notation to scientific notation.

Watch this video online: https://youtu.be/fsNu3AdIgdk

Convert from scientific notation to decimal notation

You can also write scientific notation as decimal notation. Recall the number of miles that light travels in a year is 5.88×10^{12}, and a hydrogen atom has a diameter of 5×10^{-8} mm. To write each of these numbers in decimal notation, you move the decimal point the same number of places as the exponent. If the exponent is *positive*, move the decimal point to the *right.* If the exponent is *negative*, move the decimal point to the *left.*

$$5.88 \times 10^{12} = 5.880\underrightarrow{000}000000. = 5,880,000,000,000$$

$$5 \times 10^{-8} = 0.00\underleftarrow{000}005. = 0.00000005$$

For each power of 10, you move the decimal point one place. Be careful here and don't get carried away with the zeros—the number of zeros after the decimal point will always be 1 *less* than the exponent because it takes one power of 10 to shift that first number to the left of the decimal.

Example

Write the following in decimal notation.

1. $4.8 \times 10{-}4$
2. 3.08×10^{6}

Show Solution

Think About It

To help you get a sense of the relationship between the sign of the exponent and the relative size of a number written in scientific notation, answer the following questions. You can use the textbox to wirte your ideas before you reveal the solution.
1. You are writing a number that is greater than 1 in scientific notation. Will your exponent be positive or negative?

2. You are writing a number that is between 0 and 1 in scientific notation. Will your exponent be positive or negative?

3. What power do you need to put on 10 to get a result of 1?

Show Solution

In the next video you will see how to convert a number written in scientific notation into decimal notation.

Watch this video online: https://youtu.be/8BX0oKUMIjw

Multiplying and Dividing Numbers Expressed in Scientific Notation

Numbers that are written in scientific notation can be multiplied and divided rather simply by taking advantage of the properties of numbers and the rules of exponents that you may recall. To multiply numbers in scientific notation, first multiply the numbers that aren't powers of 10 (the *a* in $a \times 10^n$). Then multiply the powers of ten by adding the exponents.

This will produce a new number times a different power of 10. All you have to do is check to make sure this new value is in scientific notation. If it isn't, you convert it.

Let's look at some examples.

Example

$\left(3 \times 10^8\right)\left(6.8 \times 10^{-13}\right)$

Show Solution

Example

$\left(8.2 \times 10^6\right)\left(1.5 \times 10^{-3}\right)\left(1.9 \times 10^{-7}\right)$

Show Solution

In the following video you will see an example of how to multiply tow numbers that are written in scientific notation.

Watch this video online: https://youtu.be/5ZAY4OCkp7U

In order to divide numbers in scientific notation, you once again apply the properties of numbers and the rules of exponents. You begin by dividing the numbers that aren't powers of 10 (the *a* in $a \times 10^n$. Then you divide the powers of ten by subtracting the exponents.

This will produce a new number times a different power of 10. If it isn't already in scientific notation, you convert it, and then you're done.

Let's look at some examples.

Example

$\dfrac{2.829 \times 10^{-9}}{3.45 \times 10^{-3}}$

Show Solution

Example

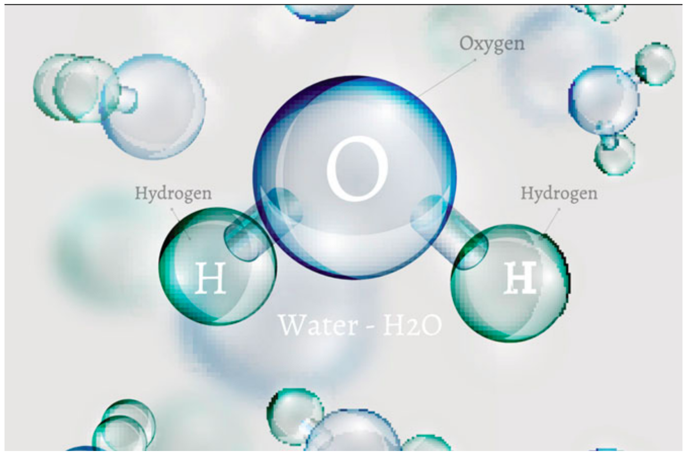
$\dfrac{\left(1.37 \times 10^4\right)\left(9.85 \times 10^6\right)}{5.0 \times 10^{12}}$

Show Solution

Notice that when you divide exponential terms, you subtract the exponent in the denominator from the exponent in the numerator. You will see another example of dividing numbers written in scientific notation in the following video.

Watch this video online: https://youtu.be/RlZck2W5pO4

Solve application problems



Water Molecule

Learning rules for exponents seems pointless without context, so let's explore some examples of using scientific notation that involve real problems. First, let's look at an example of how scientific notation can be used to describe real measurements.

Think About It

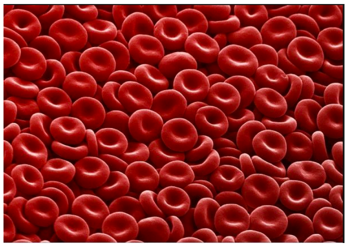
Match each length in the table with the appropriate number of meters described in scientific notation below.

The height of a desk	Diameter of water molecule	Diameter of Sun at its equator
Distance from Earth to Neptune	Diameter of Earth at the Equator	Height of Mt. Everest (rounded)
Diameter of average human cell	Diameter of a large grain of sand	Distance a bullet travels in one second

Power of 10, units in meters	Length from table above
10^{12}	
10^{9}	
10^{6}	
10^{4}	
10^{2}	
10^{0}	
10^{-3}	
10^{-5}	
10^{-10}	

Show Solution

One of the most important parts of solving a "real" problem is translating the words into appropriate mathematical terms, and recognizing when a well known formula may help. Here's an example that requires you to find the density of a cell, given its mass and volume. Cells aren't visible to the naked eye, so their measurements, as described with scientific notation, involve negative exponents.



Red Blood Cells

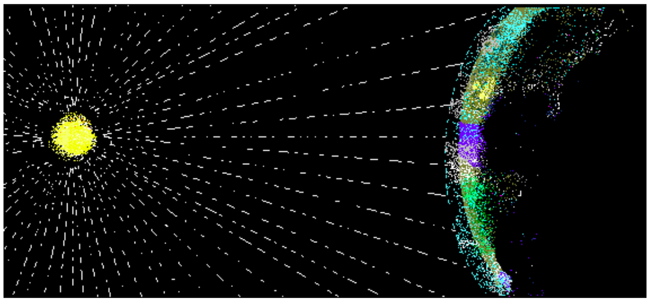
Example

Human cells come in a wide variety of shapes and sizes. The mass of an average human cell is about 2×10^{-11} grams ((Note: Orders of magnitude (mass). (n.d.). Retrieved May 26, 2016, from https://en.wikipedia.org/wiki/Orders_of_magnitude_(mass)))Red blood cells are one of the smallest types of cells ((Note: How Big is a Human Cell?)), clocking in at a volume of approximately 10^{-6} meters^3. ((Note: How big is a human cell? - Weizmann Institute of Science. (n.d.). Retrieved May 26, 2016, from http://www.weizmann.ac.il/plants/Milo/images/humanCellSize120116Clean.pdf)) Biologists have recently discovered how to use the density of some types of cells to indicate the presence of disorders such as sickle cell anemia or leukemia. ((Note: Grover, W. H., Bryan, A. K., Diez-Silva, M., Suresh, S., Higgins, J. M., & Manalis, S. R. (2011). Measuring single-cell density. Proceedings of the National Academy of Sciences, 108(27), 10992-10996. doi:10.1073/pnas.1104651108)) Density is calculated as the ratio of $\frac{\text{mass}}{\text{volume}}$. Calculate the density of an average human cell.
Show Solution

The following video provides an example of how to find the number of operations a computer can perform in a very short amount of time.

Watch this video online: https://youtu.be/Cbm6ejEbu-o

In the next example, you will use another well known formula, $d = r \cdot t$, to find how long it takes light to travel from the sun to the earth. Unlike the previous example, the distance between the earth and the sun is massive, so the numbers you will work with have positive exponents.



Light traveling from the sun to the earth.

Example

The speed of light is $3 \times 10^8 \frac{\text{meters}}{\text{second}}$. If the sun is 1.5×10^{11} meters from earth, how many seconds does it take for sunlight to reach the earth? Write your answer in scientific notation.
Show Solution

In the following video we calculate how many miles the participants of the New York marathon ran combined, and compare that to the circumference of the earth.

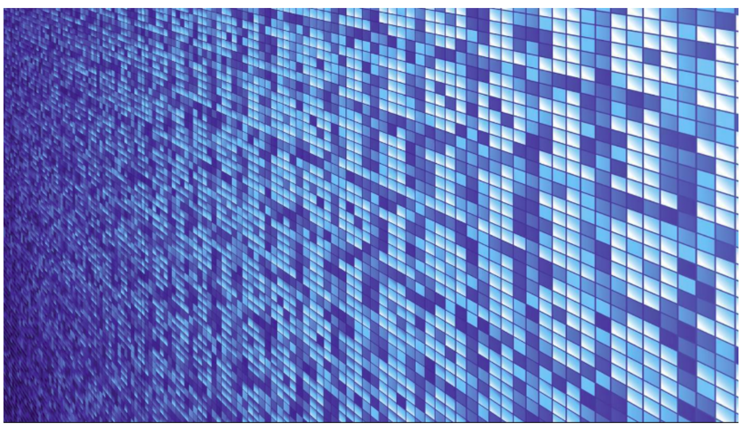
Visit this page in your course online to check your understanding.

Summary

Large and small numbers can be written in scientific notation to make them easier to understand. In the next section, you will see that performing mathematical operations such as multiplication and division on large and small numbers is made easier by scientific notation and the rules of exponents.

Scientific notation was developed to assist mathematicians, scientists, and others when expressing and working with very large and very small numbers. Scientific notation follows a very specific format in which a number is expressed as the product of a number greater than or equal to one and less than ten, and a power of 10. The format is written $a \times 10^{n}$, where $1 \leq a < 10$ and *n* is an integer. To multiply or divide numbers in scientific notation, you can use the commutative and associative properties to group the exponential terms together and apply the rules of exponents.

CONCLUSION



Bits and Bytes

Exponents allow us to perform repeated multiplication quickly and to write very large or small numbers in a way that is easy to understand. There are many and varied uses of exponents that you may be surprised to find are all around us. Have you ever heard a computer described as having a 32-bit processor with 64 megabytes of RAM and 2.1 gigabytes of hard disk space? What on earth does that even mean? This terminology is used to describe how fast a computer can process information, which results in how quickly it responds after you click on an app or open a file. It also describes how much memory the computer has to store the files and apps you want to use with it.

Computers use binary digits (0 and 1) instead of the decimal digits (0 to 9) we are used to working with. The descriptions 32-bit and 64 megabytes don't make much sense at first glance, but you can use the lingua franca of exponents to translate this information into an explanation of how memory and processing speed work.

Each binary digit is equivalent to one bit, and a byte contains 8 bits, often called a binary octet. Each character that you type on a computer needs one binary octet in order to exist. This is what a byte represents relative to decimal numbers:

Type of Number									Decimal Equivalent
Decimal Powers of 2	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	

Decimal Number	128	64	32	16	8	4	2	1	$128+64+32$ $+16+8+4$ $+2+1=255$
Binary Octet	0	0	0	0	0	0	0	1	1
	0	0	0	0	0	0	1	1	$2+1=3$
	0	0	0	0	1	1	1	0	$8+4+2=14$
	1	0	1	0	1	1	0	0	$128+32+8$ $+4=172$

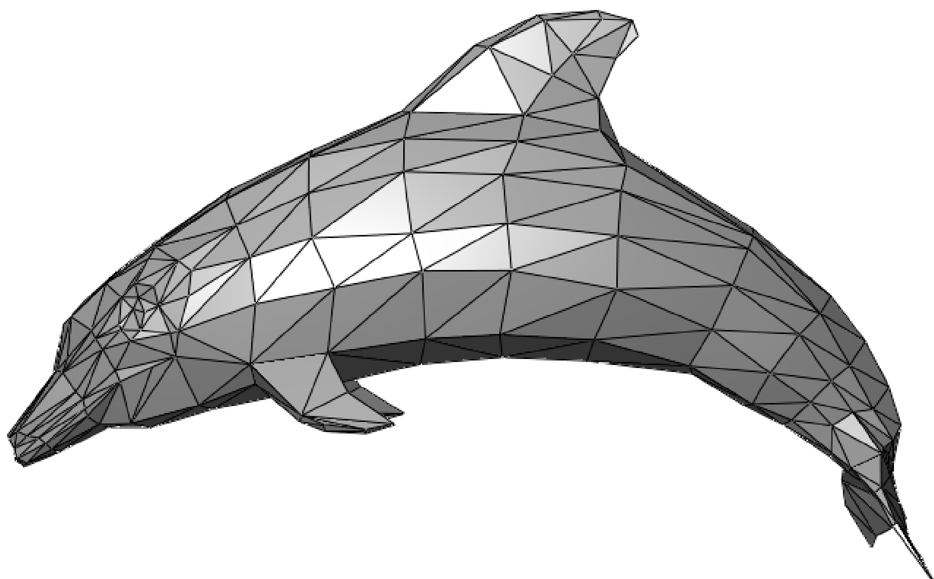
The 1s in the binary octet act as light switches within a computer. The computer uses the binary octets to represent characters such as numbers, letters, and other information. You can think of the number of bits assigned to a computer processor as the amount of “scratch pad” it has available to store calculations and information while it works. Having more “scratch pad” available means the computer can do more without needing to access memory which takes time. There are many more factors that influence the speed and agility of a computer, but the basics can be described with exponents. Now try to imagine how many bytes the Millennium Falcon’s computer must have needed to make the Kessel Run in less than 12 parsecs!

MODULE 5: POLYNOMIALS

INTRODUCTION

Why learn about polynomials?

If you have ever watched a Pixar movie, you have seen computer generated images. A very common method for generating graphics with a computer is to use what is called a wire mesh. You can think of a wire mesh as a grid – like the ones we have used to graph lines – that has been bent and stretched to make a shape we want, as in the image of a dolphin below.



The dolphin in the image was created by plotting points in space to create connected triangles. This method of rendering graphics works well and is in wide use, but it takes a lot of computer memory. Recently, researchers have been investigating the use of polynomials for rendering graphics in part because it demands less memory in the process. ((Note: Loop, Charles, and Jim Blinn. Real-time GPU Rendering of Piecewise Algebraic Surfaces. *ACM SIGGRAPH 2006 Papers on - SIGGRAPH '06* (2006): n. pag. Web.)). In this process, the surfaces that are rendered are made from solutions to algebraic polynomials. The image below shows some images of smooth-surfaced objects that were rendered using polynomials by researchers Charles Loop and Jim Blinn from Microsoft.

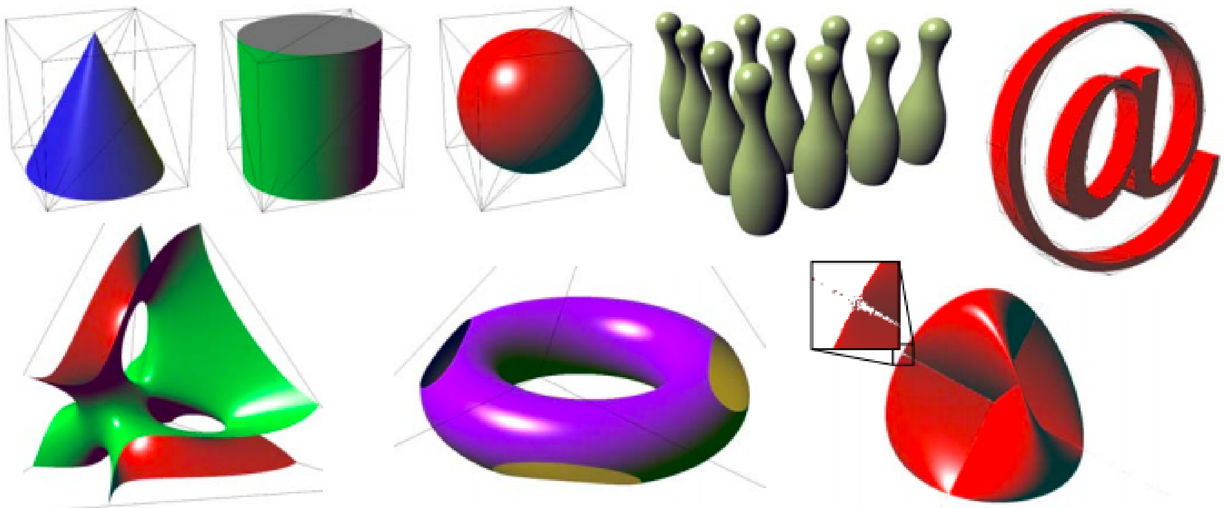


Figure 5: Some sample piecewise algebraic surfaces composed of Bézier tetrahedra and rendered using our technique.

Surfaces Rendered Using Polynomials.

In this module, you will learn how to identify a polynomial and how to perform algebraic operations on them. Like the linear equations and inequalities you learned about earlier, polynomials are useful in many applications of mathematics as well as in other disciplines like biology, economics, and even cryptology. Gaining a basic understanding of their qualities and how the rules of algebra we have learned so far apply to them will help you learn how to use polynomials both inside and out of your math class.

In your next math class you will likely learn how to solve certain kinds of polynomials and how to graph them as well.

Learning Outcomes

Single Variable Polynomials
- Define and evaluate polynomials

Operations on Polynomials
- Add polynomials
- Multiply polynomials
- Subtract polynomials
- Multiply binomials
- Divide by a polynomial

Applications of Polynomials
- Divide by a monomial
- Write polynomials involving perimeter, area, and volume
- Write polynomials involving cost, revenue, and profit

IDENTIFY AND EVALUATE POLYNOMIALS

Learning Objectives

- Identify the terms, the coefficients, and the exponents of a polynomial
- Evaluate a polynomial for given values of the variable
- Simplify polynomials by collecting like terms

Identify the terms, the coefficients, and the exponents of a polynomial

Polynomials are algebraic expressions that are created by combining numbers and variables using arithmetic operations such as addition, subtraction, multiplication, division, and exponentiation. You can create a **polynomial** by adding or subtracting terms. Polynomials are very useful in applications from science and engineering to business. You may see a resemblance between expressions, which we have been studying in this course, and polynomials. Polynomials are a special sub-group of mathematical expressions and equations.

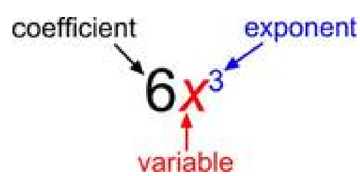
The following table is intended to help you tell the difference between what is a polynomial and what is not.

IS a Polynomial	Is NOT a Polynomial	Because
$2x^2 - \frac{1}{2}x - 9$	$\frac{2}{x^2} + x$	Polynomials only have variables in the numerator
$\frac{y}{4} - y^3$	$\frac{2}{y} + 4$	Polynomials only have variables in the numerator
$\sqrt{12}\left(a\right) + 9$	$\sqrt{a} + 7$	Variables under a root are not allowed in polynomials

The basic building block of a polynomial is a **monomial**. A monomial is one term and can be a number, a variable, or the product of a number and variables with an exponent. The number part of the term is called the **coefficient**.

Examples of monomials:

- number: 2
- variable: x
- product of number and variable: $2x$
- product of number and variable with an exponent: $2x^3$



The coefficient can be any real number, including 0. The exponent of the variable must be a whole number—0, 1, 2, 3, and so on. A monomial cannot have a variable in the denominator or a negative exponent.

The value of the exponent is the **degree** of the monomial. Remember that a variable that appears to have no exponent really has an exponent of 1. And a monomial with no variable has a degree of 0. (Since x^0 has the value of 1 if $x \neq 0$, a number such as 3 could also be written $3x^0$, if $x \neq 0$ as $3x^0 = 3 \cdot 1 = 3$.)

Example

Identify the coefficient, variable, and degree of the variable for the following monomial terms:
1) 9
2) *x*
3) $\dfrac{3}{5}k^8$
Show Solution

A polynomial is a monomial or the sum or difference of two or more polynomials. Each monomial is called a **term** of the polynomial.

Some polynomials have specific names indicated by their prefix.

- **monomial**—is a polynomial with exactly one term (“mono”—means one)
- **binomial**—is a polynomial with exactly two terms (“bi”—means two)
- **trinomial**—is a polynomial with exactly three terms (“tri”—means three)

The word “polynomial” has the prefix, “poly,” which means many. However, the word polynomial can be used for all numbers of terms, including only one term.

Because the exponent of the variable must be a whole number, monomials and polynomials cannot have a variable in the denominator.

Polynomials can be classified by the degree of the polynomial. The degree of a polynomial is the degree of its highest degree term. So the degree of $2x^3+3x^2+8x+5$ is 3.

A polynomial is said to be written in standard form when the terms are arranged from the highest degree to the lowest degree. When it is written in standard form it is easy to determine the degree of the polynomial.

The table below illustrates some examples of monomials, binomials, trinomials, and other polynomials. They are all written in standard form.

Monomials	**Binomials**	**Trinomials**	**Other Polynomials**
15	$3y+13$	x^3-x^2+1	$5x^4+3x^3-6x^2+2x$
$\frac{1}{2}x$	$4p-7$	$3x^2+2x-9$	$\frac{1}{3}x^5-2x^4+\frac{2}{9}x^3-x^2+4x-\frac{5}{6}$
$-4y^3$	$3x^2+\frac{5}{8}x$	$3y^3+y^2-2$	$3t^3-3t^2-3t-3$
$16n^4$	$14y^3+3y$	$a^7+2a^5-3a^3$	$q^7+2q^5-3q^3+q$

When the coefficient of a polynomial term is 0, you usually do not write the term at all (because 0 times anything is 0, and adding 0 doesn’t change the value). The last binomial above could be written as a trinomial, $14y^3+0y^2+3y$.

A term without a variable is called a **constant** term, and the degree of that term is 0. For example 13 is the constant term in $3y+13$. You would usually say that $14y^3+3y$ has no constant term or that the constant term is 0.

Example

For the following expressions, determine whether they are a polynomial. If so, categorize them as a monomial, binomial, or trinomial.

1. $\frac{x-3}{1-x}+x^2$
2. t^2+2t-3
3. $x^3+\frac{x}{8}$
4. $\frac{\sqrt{y}}{2}-y-1$

Show Solution

In the following video, you will be shown more examples of how to identify and categorize polynomials.

Watch this video online: https://youtu.be/nPAqfuoSbPI

Evaluate a polynomial for given values of the variable

You can evaluate polynomials just as you have been evaluating expressions all along. To evaluate an expression for a value of the variable, you substitute the value for the variable *every time* it appears. Then use the order of operations to find the resulting value for the expression.

Example

Evaluate $3x^2 - 2x + 1$ for $x = -1$.
Show Solution

Example

Evaluate $-\frac{2}{3}p^4 + 2p^3 - p$ for $p = 3$.
Show Solution

The following video presents more examples of evaluating a polynomial for a given value.

Watch this video online: https://youtu.be/2EeFrgQP1hM

Simplify polynomials by collecting like terms

Apple and Orange

A polynomial may need to be simplified. One way to simplify a polynomial is to combine the **like terms** if there are any. Two or more terms in a polynomial are like terms if they have the same variable (or variables) with the same exponent. For example, $3x^2$ and $-5x^2$ are like terms: They both have *x* as the variable, and the exponent is 2 for each. However, $3x^2$ and $3x$ are not like terms, because their exponents are different.

Here are some examples of terms that are alike and some that are unlike.

Term	Like Terms	UNLike Terms
a	$3a,\ -2a,\ \frac{1}{2}a$	$a^2,\ \frac{1}{a},\ \sqrt{a}$
a^2	$-5a^2,\ \frac{1}{4}a^2,\ 0.56a^2$	$\frac{1}{a^2},\ \sqrt{a^2},\ a^3$
ab	$7ab,\ 0.23ab,\ \frac{2}{3}ab,\ -ab$	$a^2b,\ \frac{1}{ab},\ \sqrt{ab}$
ab^2	$4ab^2,\ \frac{ab^2}{7},\ 0.4ab^2,\ -a^2b$	$a^2b,\ ab,\ \sqrt{ab^2},\ \frac{1}{ab^2}$

Example

Which of these terms are like terms?

$7x^3 \, 7x \, 7y - 8x^3 \, 9y - 3x^2 \, 8y^2$
Show Solution

You can use the distributive property to simplify the sum of like terms. Recall that the distributive property of addition states that the product of a number and a sum (or difference) is equal to the sum (or difference) of the products.

$2\left(3+6\right)=2\left(3\right)+2\left(6\right)$

Both expressions equal 18. So you can write the expression in whichever form is the most useful.

Let's see how we can use this property to combine like terms.

Example

Simplify $3x^2-5x^2$.
Show Solution

You may have noticed that combining like terms involves combining the coefficients to find the new coefficient of the like term. You can use this as a shortcut.

Example

Simplify $6a^4+4a^4$.
Show Solution

When you have a polynomial with more terms, you have to be careful that you combine *only* like terms. If two terms are not like terms, you can't combine them.

Example

Simplify $3x^2+3x+x+1+5x$
Show Solution

Watch this video online: https://youtu.be/1epjbVO_qU4

Summary

Polynomials are algebraic expressions that contain any number of terms combined by using addition or subtraction. A term is a number, a variable, or a product of a number and one or more variables with exponents. Like terms (same variable or variables raised to the same power) can be combined to simplify a polynomial. The polynomials can be evaluated by substituting a given value of the variable into *each* instance of the variable, then using order of operations to complete the calculations.

OPERATIONS ON POLYNOMIALS

Learning Objectives

- Add polynomials
  - Use horizontal and vertical organization to add polynomials
- Find the opposite of a polynomial
  - Subtract polynomials using both horizontal and vertical organization
- Multiply Polynomials
-
  - Find the product of monomials
  - Find the product of polynomials and monomials
  - Find the product of two binomials
- Multiply Binomials
-
  - Apply the FOIL method to multiply two binomials
  - Use a table to multiply two binomials
  - Simplify the product of two binomials given a wide variety of variables, constants, signs, and arrangement of terms in the binomial
- Divide Polynomials
-
  - Divide a binomial by a monomial
  - Divide a trinomial by a monomial
  - Apply polynomial long division to divide by a binomial

Adding and subtracting **polynomials** may sound complicated, but it's really not much different from the addition and subtraction that you do every day. The main thing to remember is to look for and combine **like terms**.

You can add two (or more) polynomials as you have added algebraic expressions. You can remove the parentheses and combine like terms.

Example

Add. $(3b+5)+(2b+4)$
Show Solution

When you are adding polynomials that have subtraction, it is important to remember to keep the sign on each term as you are collecting like terms. The next example will show you how to regroup terms that are subtracted when you are collecting like terms.

Example

Add. $\left(-5x^2-10x+2\right)+\left(3x^2+7x-4\right)$
Show Solution

The above examples show addition of polynomials horizontally, by reading from left to right along the same line. Some people like to organize their work vertically instead, because they find it easier to be sure that they are combining like terms. The example below shows this "vertical" method of adding polynomials:

Example

Add. $\left(3x^2+2x-7\right)+\left(7x^2-4x+8\right)$
Show Solution

Sometimes in a vertical arrangement, you can line up every term beneath a like term, as in the example above. But sometimes it isn't so tidy. When there isn't a matching like term for every term, there will be empty places in the vertical arrangement.

Example

Add. $\left(4x^3+5x^2-6x+2\right)+\left(-4x^2+10\right)$

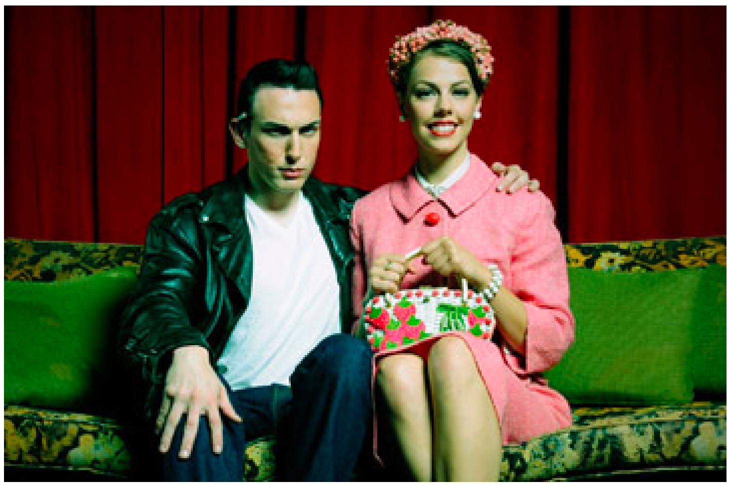
Show Solution

You may be thinking, how is this different than combining like terms, which we did in the last section? The answer is, it's not really. We just added a layer to combining like terms by adding more terms to combine. □ Polynomials are a useful tool for describing the behavior of anything that isn't linear, and sometimes you may need to add them.

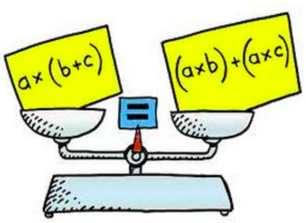
In the following video, you will see more examples of combining like terms by adding polynomials.

Watch this video online: https://youtu.be/KYZR7g7QcF4

Find the opposite of a polynomial



Opposites



When you are solving equations, it may come up that you need to subtract polynomials. This means subtracting each term of a polynomial, which requires changing the sign of each term in a polynomial. Recall that changing the sign of 3 gives -3, and changing the sign of -3 gives 3. Just as changing the sign of a number is found by multiplying the number by -1, we can change the sign of a polynomial by multiplying it by -1. Think of this in the same way as you would the distributive property. You are distributing -1 to each term in the polynomial. Changing the sign of a polynomial is also called finding the opposite.

Example

Find the opposite of $9x^2+10x+5$.
Show Solution

Be careful when there are negative terms or subtractions in the polynomial already. Just remember that you are changing the sign, so if it is negative, it will become positive.

Example

Find the opposite of $3p^2-5p+7$.
Show Solution

Notice that in finding the opposite of a polynomial, you change the sign of *each term* in the polynomial, then rewrite the polynomial with the new signs on each term.

Subtract polynomials

When you subtract one polynomial from another, you will first find the opposite of the polynomial being subtracted, then combine like terms. The easiest mistake to make when subtracting one polynomial from another is to forget to change the sign of EVERY term in the polynomial being subtracted.

Example

Subtract. $\left(15x^2+12x+20\right)-\left(9x^2+10x+5\right)$
Show Solution

When polynomials include a lot of terms, it can be easy to lose track of the signs. Be careful to transfer them correctly, especially when subtracting a negative term.

Example

Subtract. $\left(14x^3+3x^2-5x+14\right)-\left(7x^3+5x^2-8x+10\right)$
Show Solution

When you have many terms, like in the example above, try the vertical approach from the previous page to keep your terms organized. However you choose to combine polynomials is up to you—the key point is to identify like terms, keep track of their signs, and be able to organize them accurately.

Example

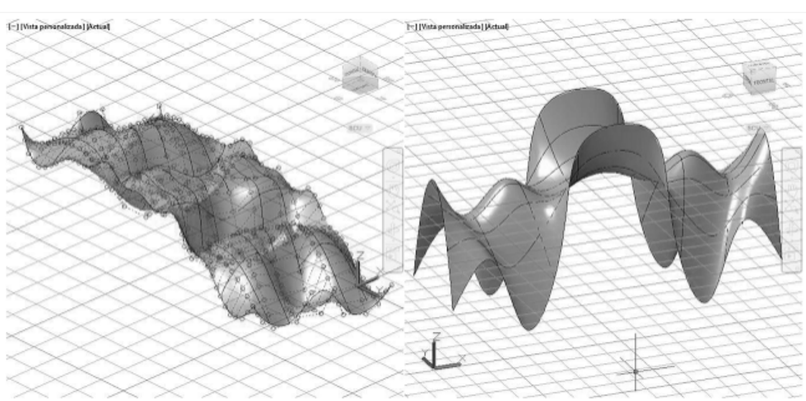
Subtract. $\left(14x^3+3x^2-5x+14\right)-\left(7x^3+5x^2-8x+10\right)$
Show Solution

In the following video, you will see more examples of subtracting polynomials.

Watch this video online: https://youtu.be/xq-zVm25VC0

Find the product of monomials

Multiplying **polynomials** involves applying the rules of exponents and the distributive property to simplify the product. Polynomial multiplication can be useful in modeling real world situations. Understanding polynomial products is an important step in learning to solve algebraic equations involving polynomials. There are many, varied uses for polynomials including the generation of 3D graphics for entertainment and industry, as in the image below.



NURBS surfaces.

Surfaces made from polynomials with AutoCAD

In the exponents section, we practiced multiplying monomials together, like we did with this expression: $24x^8 2x^5$. The only thing different between that section and this one is that we called it simplifying, and now we are calling it polynomial multiplication. Remember that simplifying a mathematical expression means performing as many operations as we can until there are no more to perform, including multiplication. In this section we will show examples of how to multiply more than just monomials. We will multiply monomials with binomials and trinomials. We will also learn some techniques for multiplying two binomials together.

Example

Multiply. $-9x^3 \cdot 3x^2$
Show Solution

That's it! When multiplying monomials, multiply the coefficients together, and then multiply the variables together. Remember, if two variables have the same base, follow the rules of exponents, like this:

$$5a^4 \cdot 7a^6 = 35a^{10}$$

The following video provides more examples of multiplying monomials with different exponents.

Watch this video online: https://youtu.be/30x8hY32B0o

Find the product of polynomials and monomials

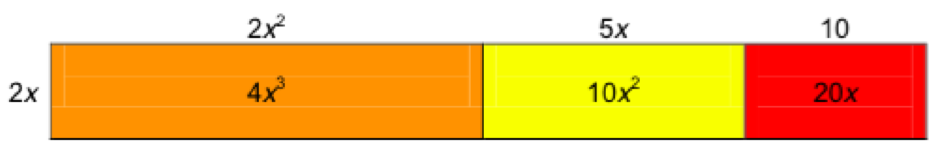
The distributive property can be used to multiply a monomial and a binomial. Just remember that the monomial must be multiplied by each term in the binomial. In the next example, you will see how to multiply a second degree monomial with a binomial. Note the use of exponent rules.

Example

Simplify. $5x^2\left(4x^2+3x\right)$
Show Solution

Now let's add another layer by multiplying a monomial by a trinomial. Consider the expression $2x\left(2x^2+5x+10\right)$.

This expression can be modeled with a sketch like the one below.



The only difference between this example and the previous one is there is one more term to distribute the monomial to.

$$\begin{aligned}2x\left(2x^2+5x+10\right) &= 2x\left(2x^2\right)+2x\left(5x\right)=2x\left(10\right)\\ &= 4x^3+10x^2+20x\end{aligned}$$

You will always need to pay attention to negative signs when you are multiplying. Watch what happens to the sign on the terms in the trinomial when it is multiplied by a negative monomial in the next example.

Example

Simplify. $-7x\left(2x^2-5x+1\right)$
Show Solution

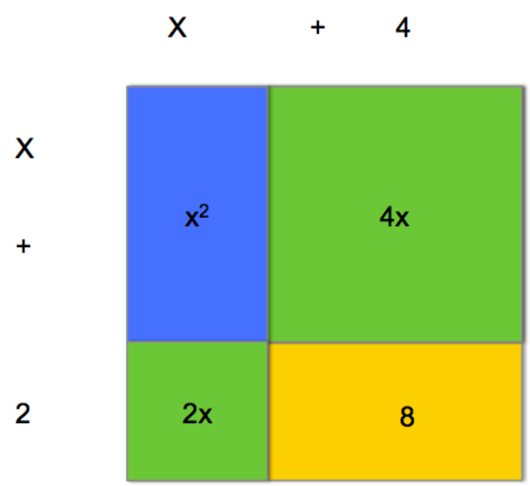
The following video provides more examples of multiplying a monomial and a polynomial.

Watch this video online: https://youtu.be/bwTmApTV_8o

Find the product of two binomials

Now let's explore multiplying two binomials. For those of you that use pictures to learn, you can draw an area model to help make sense of the process. You'll use each binomial as one of the dimensions of a rectangle, and their product as the area.

The model below shows $(x+4)(x+2)$:



Visual representation of multiplying two binomials.

Each binomial is expanded into variable terms and constants, $x+4$, along the top of the model and $x+2$ along the left side. The product of each pair of terms is a colored rectangle. The total area is the sum of all of these small rectangles, $x^2+2x+4x+8$, If you combine all the like terms, you can write the product, or area, as x^2+6x+8.

You can use the distributive property to determine the product of two binomials.

Example

Simplify. $(x+4)(x+2)$
Show Solution

Look back at the model above to see where each piece of $x^2+2x+4x+8$ comes from. Can you see where you multiply x by $x+2$, and where you get x^2 from $x(x)$?

Another way to look at multiplying binomials is to see that each term in one binomial is multiplied by each term in the other binomial. Look at the example above: the x in $x+4$ gets multiplied by both the x and the 2 from $x+2$, and the 4 gets multiplied by both the x and the 2.

The following video provides an example of multiplying two binomials using an area model as well as repeated distribution.

Watch this video online: https://youtu.be/u4HgI0BrUlo

In the next section we will explore other methods for multiplying two binomials, and become aware of the different forms that binomials can have.

FOIL

Foil Crane

In the last section we finished with an example of multiplying two binomials, $(x+4)(x+2)$. In this section we will provide examples of how to use two different methods to multiply to binomials. Keep in mind as you read through the page that simplify and multiply are used interchangeably.

Some people use the FOIL method to keep track of which pairs of terms have been multiplied when you are multiplying two binomials. This is not the same thing you use to wrap up leftovers, but an acronym for **First, Outer, Inner, Last.** Let's go back to the example from the previous page, where we were asked to multiply the two binomials: $(x+4)(x+2)$. The following steps show you how to apply this method to multiplying two binomials.

$$\begin{array}{lll} \text{First term in each binomial :} & (x+4)(x+2) & x(x)=x^2 \\ \text{Outer terms :} & (x+4)(x+2) & x(2)=2x \\ \text{Inner terms :} & (x+4)(x+2) & 4(x)=4x \\ \text{Last terms in each binomial :} & (x+4)(x+2) & 4(2)=8 \end{array}$$

When you add the four results, you get the same answer, $x^2+2x+4x+8=x^2+6x+8$.
The last step in multiplying polynomials is to combine like terms. Remember that a polynomial is simplified only when there are no like terms remaining.

> Caution! Note that the FOIL method only works for multiplying two binomials together. It sill not work for multiplying a binomial and a trinomial, or two trinomials.

One of the neat things about multiplication is that terms can be multiplied in either order. The expression $(x+2)(x+4)$ has the same product as $(x+4)(x+2)$, x^2+6x+8. (Work it out and see.) The order in which you multiply binomials does not matter. What matters is that you multiply each term in one binomial by each term in the other binomial.

Polynomials can take many forms. So far we have seen examples of binomials with variable terms on the left and constant terms on the right, such as this binomial $(2r-3)$. Variables may also be on the right of the constant term, as in this binomial $(5+r)$. In the next example, we will show that multiplying binomials in this form requires one extra step at the end.

Order Doesn't Matter When You Multiply

> Example
>
> Find the product. $(3 - s)(1 - s)$
> Show Solution

In the next example, you will see that sometimes there are constants in front of the variable. They will get multiplied together just as we have done before.

> Example
>
> Simplify $(4x - 10)(2x + 3)$ using the FOIL acronym.
> Show Solution

The video that follows gives another example of multiplying two binomials using the FOIL acronym. Remember this method only works when you are multiplying two binomials.

Watch this video online: https://youtu.be/_MrdEFnXNGA

The Table Method

You may see a binomial multiplied by itself written as $(x+3)^2$ instead of $(x+3)(x+3)$. To find this product, let's use another method. We will place the terms of each binomial along the top row and first column of a table, like this:

	x	$+3$
x		
$+3$		

Now multiply the term in each column by the term in each row to get the terms of the resulting polynomial. Note how we keep the signs on the terms, even when they are positive, this will help us write the new polynomial.

	x	$+3$
x	$x \cdot x = x^2$	$3 \cdot x = +3x$
$+3$	$x \cdot 3 = +3x$	$3 \cdot 3 = +9$

Now we can write the terms of the polynomial from the entries in the table:

$(x+3)^2$

$= x^2 + 3x + 3x + 9$

= x^2 + $6x$ + 9.

Pretty cool, huh?

So far, we have shown two methods for multiplying two binomials together. Why are we focusing so much on binomials? They are one of the most well studied and widely used polynomials, so there is a lot of information out there about them. In the previous example, we saw the result of squaring a binomial that was a sum of two terms. In the next example we will find the product of squaring a binomial that is the difference of two terms.

Example

Square the binomial difference $(x–7)$
Show solution

Caution! It is VERY important to remember the caution from the exponents section about squaring a binomial:
You can't move the exponent into a grouped sum because of the order of operations!!!!!
INCORRECT: $(2+x)^2 \neq 2^2+x^2$
CORRECT: $(2+x)^2 = (2+x)(2+x)$

In the video that follows, you will see another examples of using a table to multiply two binomials.

Watch this video online: https://youtu.be/tWsLJ_pn5mQ

Further Examples

The next couple of examples show you some different forms binomials can take. In the first, we will square a binomial that has a coefficient in front of the variable, like the product in the first example on this page. In the second we will find the product of two binomials that have the variable on the right instead of the left. We will use both the FOIL method and the table method to simplify.

Example

Find the product. $(2x+6)^2$
Show Solution

In the last example, we want to show you another common form a binomial can take, each of the terms in the two binomials is the same, but the signs are different. You will see that in this case, the middle term will disappear.

Example

Multiply the binomials. $(x+8)(x–8)$
Show Solution

Think About It

There are predictable outcomes when you square a binomial sum or difference. In general terms, for a binomial difference,
$(a-b)^2 = (a-b)(a-b)$,
the resulting product, after being simplified, will look like this:
$a^2-2ab+b^2$.
The product of a binomial sum will have the following predictable outcome:
$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$.

Note that a and b in these generalizations could be integers, fractions, or variables with any kind of constant. You will learn more about predictable patterns from products of binomials in later math classes.

In this section we showed two ways to find the product of two binomials, the FOIL method, and by using a table. Some of the forms a product of two binomials can take are listed here:

- $\left(x+5\right)\left(2x-3\right)$
- $\left(x+7\right)^{2}$
- $\left(x-1\right)^{2}$
- $\left(2-y\right)\left(5+y\right)$
- $\left(x+9\right)\left(x-9\right)$
- $\left(2x-4\right)\left(x+3\right)$

And this is just a small list, the possible combinations are endless. For each of the products in the list, using one of the two methods presented here will work to simplify.

Divide a polynomial by a monomial

The fourth arithmetic operation is division, the inverse of multiplication. Division of polynomials isn't much different from division of numbers. In the exponential section, you were asked to simplify expressions such as: $\frac{a^{2}\left(a^{5}\right)^{3}}{8a^{8}}$. This expression is the division of two monomials. To simplify it, we divided the coefficients and then divided the variables. In this section we will add another layer to this idea by dividing polynomials by monomials, and by binomials.

The distributive property states that you can distribute a factor that is being multiplied by a sum or difference, and likewise you can distribute a *divisor* that is being divided into a sum or difference. In this example, you can add all the terms in the numerator, then divide by 2.

$\frac{\text{dividend}\rightarrow}{\text{divisor}\rightarrow}\quad\frac{8+4+10}{2}=\frac{22}{2}=11$

Or you can first divide each term by 2, then simplify the result.

$\frac{8}{2}+\frac{4}{2}+\frac{10}{2}=4+2+5=11$

Either way gives you the same result. The second way is helpful when you can't combine like terms in the numerator. Let's try something similar with a binomial.

Example

Divide. $\frac{9a^{3}+6a}{3a^{2}}$

Show Solution

In the next example, you will see that the same ideas apply when you are dividing a trinomial by a monomial. You can distribute the divisor to each term in the trinomial and simplify using the rules for exponents. As we have throughout the course, simplifying with exponents includes rewriting negative exponents as positive. Pay attention to the signs of the terms in the next example, we will divide by a negative monomial.

Example

Divide. $\frac{27y^{4}+6y^{2}-18}{-6y}$

Show Solution

Now, we ask you to think about what would happen if you were given a quotient like this to simplify: $\frac{27y^{4}+6y^{2}-18}{-6y+3}$. You may be tempted to divide each term of $27y^{4}+6y^{2}-18$ individually by $-6y$, then 3. This would go against the order of operations because the division sign is a grouping symbol, and the addition in the denominator cannot be simplified anymore. The result is that no further operations can be performed with the tools we know. We can, however, call into use a tool that you may have learned in gradeschool: long division.

Polynomial Long Division

Recall how you can use long division to divide two whole numbers, say 900 divided by 37.

$$37\overline{)900}$$

First, you would think about how many 37s are in 90, as 9 is too small. (*Note:* you could also think, how many 40s are there in 90.)

$$\begin{array}{r} 2 \\ 37\overline{)900} \\ 74 \end{array}$$

There are two 37s in 90, so write 2 above the last digit of 90. Two 37s is 74; write that product below the 90.

$$\begin{array}{r} 2 \\ 37\overline{)900} \\ \underline{-74} \\ 16 \end{array}$$

Subtract: $90-74$ is 16. (If the result is larger than the divisor, 37, then you need to use a larger number for the quotient.)

$$\begin{array}{r} 2 \\ 37\overline{)900} \\ \underline{-74} \\ 160 \end{array}$$

Bring down the next digit (0) and consider how many 37s are in 160.

$$\begin{array}{r} 24 \\ 37\overline{)900} \\ \underline{-74} \\ 160 \\ \underline{-148} \end{array}$$

There are four 37s in 160, so write the 4 next to the two in the quotient. Four 37s is 148; write that product below the 160.

$$\begin{array}{r} 24 \\ 37\overline{)900} \\ \underline{-74} \\ 160 \\ \underline{-148} \\ 12 \end{array}$$

Subtract: $160–148$ is 12. This is less than 37 so the 4 is correct. Since there are no more digits in the dividend to bring down, you're done.

The final answer is 24 R12, or $24\frac{12}{37}$. You can check this by multiplying the quotient (without the remainder) by the divisor, and then adding in the remainder. The result should be the dividend:

$24\cdot37+12=888+12=900$

To divide polynomials, use the same process. This example shows how to do this when dividing by a **binomial**.

Example

Divide: $\frac{(x^2–4x–12)}{(x+2)}$
Show Solution

Check this by multiplying:

$(x-6)(x+2)=x^2+2x-6x-12=x^2-4x-12$

Polynomial long division involves many steps. Hopefully this video will help you determine what step to do next when you are using polynomial long division.

Watch this video online: https://youtu.be/KUPFg__Djzw

Let's try another example. In this example, a term is "missing" from the dividend.

Example

Divide: $\frac{(x^3–6x–10)}{(x–3)}$
Show Solution

Check the result:

$$\begin{array}{rcl} (x–3)\left(x^2+3x+3\right) & = & x\left(x^2+3x+3\right)–3\left(x^2+3x+3\right) \\ & = & x^3+3x^2+3x–3x^2–9x–9 \\ & = & x^3–6x–9 \\ x^3–6x–9+(-1) & = & x^3–6x–10 \end{array}$$

The video that follows shows another example of dividing a third degree trinomial by a first degree binomial.

Watch this video online: https://youtu.be/Rxds7Q_UTeo

The last video example shows how to divide a third degree trinomial by a second degree binomial.

Watch this video online: https://youtu.be/P6OTbUf8f60

Summary

To divide a monomial by a monomial, divide the coefficients (or simplify them as you would a fraction) and divide the variables with like bases by subtracting their exponents. To divide a polynomial by a monomial, divide each term of the polynomial by the monomial. Be sure to watch the signs! Final answers should be written without any negative exponents. Dividing polynomials by polynomials of more than one term can be done using a process very much like long division of whole numbers. You must be careful to subtract entire expressions, not just the first term. Stop when the degree of the remainder is less than the degree of the divisor. The remainder can be written using R notation, or as a fraction added to the quotient with the remainder in the numerator and the divisor in the denominator.

Summary

We have seen that subtracting a polynomial means changing the sign of each term in the polynomial and then reorganizing all the terms to make it easier to combine those that are alike. How you organize this process is up to you, but we have shown two ways here. One method is to place the terms next to each other horizontally, putting like terms next to each other to make combining them easier. The other method was to place the polynomial being subtracted underneath the other after changing the signs of each term. In this method it is important to align like terms and use a blank space when there is no like term.

Multiplication of binomials and polynomials requires an understanding of the distributive property, rules for exponents, and a keen eye for collecting like terms. Whether the polynomials are monomials, binomials, or trinomials, carefully multiply each term in one polynomial by each term in the other polynomial. Be careful to watch the addition and subtraction signs and negative coefficients. A product is written in simplified form if all of its like terms have been combined.

APPLICATIONS OF POLYNOMIALS

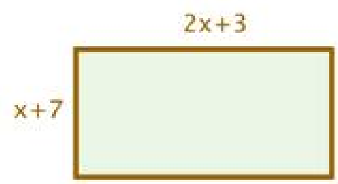
Learning Objectives

- Geometric Applications
  - Write a polynomial representing the perimeter of a shape
  - Write a polynomial representing the area of a surface
  - Write a polynomial representing the volume of a solid
- Cost, Revenue, and Profit Polynomials
-
  - Write a profit polynomial given revenue and cost polynomials
  - Find profit for given quantities produced

In this section we will explore ways that polynomials are used in applications of perimeter, area, and volume. First, we will see how a polynomial can be used to describe the perimeter of a rectangle.

Example

A rectangular garden has one side with a length of $x+7$ and another with a length $2x+3$. Find the perimeter of the garden.



Show Solution

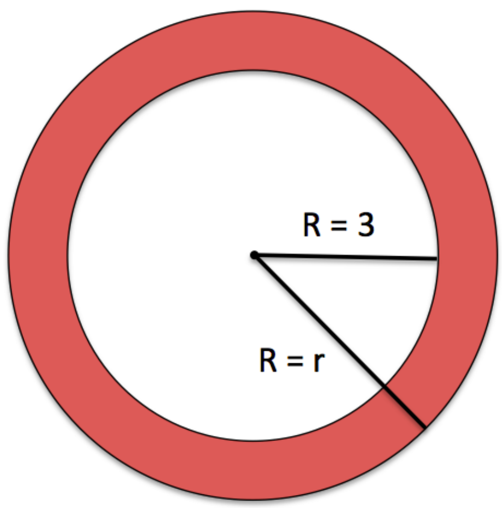
In the following video you are shown how to find the perimeter of a triangle whose sides are defined as polynomials.

Watch this video online: https://youtu.be/BhRpZv0_0jE

The area of a circle can be found using the radius of the circle and the constant pi in the formula $A=\pi r^2$. In the next example we will use this formula to find a polynomial that describes the area of an irregular shape.

Example

Find a polynomial for the shaded region of the figure.

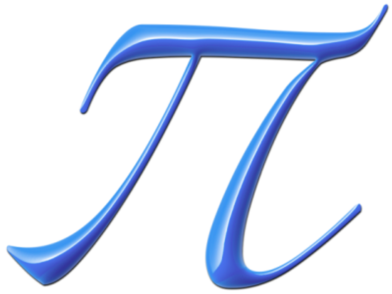


Show Solution

In the video that follows, you will be shown an example of determining the area of a rectangle whose sides are defined as polynomials.

Visit this page in your course online to check your understanding.

A note about pi.



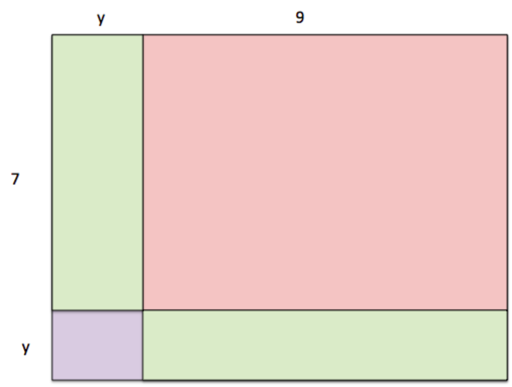
Pi

It is easy to confuse pi as a variable because we use a greek letter to represent it. We use a greek letter instead of a number because nobody has been able to find an end to the number of digits of pi. To be precise and thorough, we use the greek letter as a way to say: “we are including all the digits of pi without having to write them”. The expression for the area of the shaded region in the example above included both the variable r, which represented an unknown radius and the number pi. If we needed to use this expression to build a physical object or instruct a machine to cut specific dimensions, we would round pi to an appropriate number of decimal places.

In the next example, we will write the area for a rectangle in two different ways, one as the product of two binomials and the other as the sum of four rectangles. Because we are describing the same shape two different ways, we should end up with the same expression no matter what way we define the area.

Example

Write two different polynomials that describe the area of of the figure. For one expression, think of the rectangle as one large figure, and for the other expression, think of the rectangle as the sum of 4 different rectangles.

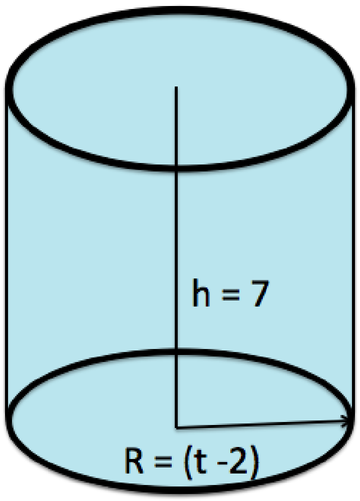


Show Solution

The last example we will provide in this section is one for volume. The volume of regular solids such as spheres, cylinders, cones and rectangular prisms are known. We will find an expression for the volume of a cylinder, which is defined as $V = \pi r^2 h$.

Example

Define a polynomial that describes the volume of the cylinder shown in the figure:



Show Solution

In this last video, we present another example of finding the volume of a cylinder whose dimensions include polynomials.

Watch this video online: https://youtu.be/g-g_nSsfGs4

Cost, Revenue, and Profit Polynomials

In the systems of linear equations section, we discussed how a company's cost and revenue can be modeled with two linear equations. We found that the profit region for a company was the area between the two lines where the company would make money based on how much was produced. In this section, we will see that sometimes polynomials are used to describe cost and revenue.

Profit is typically defined in business as the difference between the amount of money earned (revenue) by producing a certain number of items and the amount of money it takes to produce that number of items. When you are in business, you definitely want to see profit, so it is important to know what your cost and revenue is.

Cell Phones

For example, let's say that the cost to a manufacturer to produce a certain number of things is C and the revenue generated by selling those things is R. The profit, P, can then be defined as

P = R-C

The example we will work with is a hypothetical cell phone manufacturer whose cost to manufacture x number of phones is $C=2000x+750,000$, and the Revenue generated from manufacturing x number of cell phones is $R=-0.09x^{2}+7000x$.

Example

Define a Profit polynomial for the hypothetical cell phone manufacturer.
Show Solution

Mathematical models are great when you use them to learn important information. The cell phone manufacturing company can use the profit equation to find out how much profit they will make given x number of phones are manufactured. In the next example, we will explore some profit values for this company.

Example

Given the following numbers of cell phones manufactured, find the profit for the cell phone manufacturer:

1. x = 100 phones
2. x = 25,000 phones
3. x=60,000 phones

Interpret your results.
Show Solution

In the video that follows, we present another example of finding a polynomial profit equation.

Watch this video online: https://youtu.be/-TWjDC4g9dU

Summary

We have shown that profit can be modeled with a polynomial, and that the profit a company can make based on a business model like this has it's bounds.

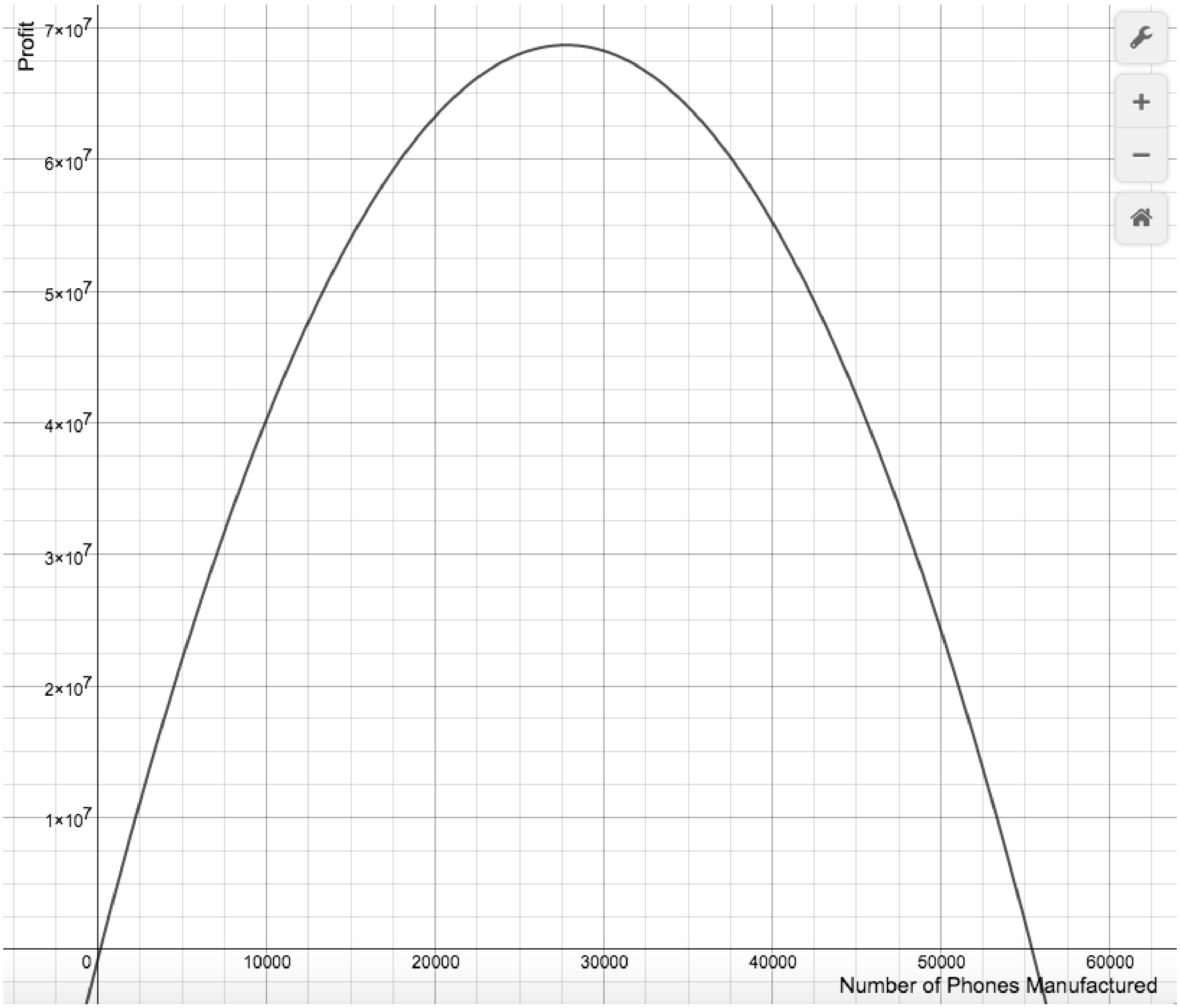
In this section we defined polynomials that represent perimeter, area and volume of well-known shapes. We also introduced some convention about how to use and write π when it is combined with other constants and variables. The next application will introduce you to cost and revenue polynomials. We explored cost and revenue equations in the module on Systems of Linear Equations, now we will see that they can be more than just linear equations, they can be polynomials.

CONCLUSION

You have just learned how polynomials are defined and how algebraic principles can be applied to polynomials. In future math courses, you will likely learn how to graph polynomials, as we did with linear equations. For example, in the last section on cost, revenue, and profit equations we wrote a polynomial that represented the profit for a cell phone company.

$$P=-0.09x^2+5000x-750,000$$

Using an online graphing calculator, we generated this graph of the profit equation.



The x-axis represents the number of phones manufactured, and the y-axis represents the profit earned by the company. Because the company must invest money into machinery and workforce to manufacture phones, when no phones have been made the profit for the company is negative. In other words, the point (0,-750,000) represents no phones made and $750,000 invested.

The top of the curve represents the “sweet spot” that many business would like to achieve between making and spending money. Using some tools from later math classes, we can find that the maximum amount of profit can be gained from producing 27,777 phones.

Polynomials have many practical uses and have been studied widely by mathematicians for many years. Our vast knowledge of how they behave mixed with high speed computing allows us to continue to find even more uses for them.

MODULE 6: FACTORING

INTRODUCTION

Why learn how to factor?

The way you organize your belongings when you move can have a huge impact on the success of your move. For example, you wouldn't want to try and move all your things using only shoeboxes. Not only would it take forever, you probably own things that are larger than a shoebox. Storing and sorting the nuts and bolts in your toolshed is very different from storing and sorting all of your possessions when you move.

Moving Madness

In this module, we will present some factoring techniques for polynomials that will help you solve polynomial equations. Factoring is a complementary operation to the distributive property, it is a way to "unpack" the multiplication done by applying the distributive property. Reorganizing polynomials by factoring allows us to find solutions for certain types of polynomials.

You may be thinking, we already know how to solve linear equations, why don't we just use the same techniques for solving polynomial equations? Let's investigate the example below to see why we need different techniques for solving polynomial equations than for solving linear equations with an example.

Solve the following second degree polynomial equation:
$x^2 + 6x - 7 = 2x - 4$

We use complementary operations to isolate the variable when we solve linear equations, so we will try to do the same with this polynomial equation.

$$\begin{aligned}
&x^2 + 6x - 7 = 2x - 4 \\
&\quad \underline{-2x} \qquad \underline{-2x} \\
&x^2 + 4x - 7 = -4 \\
&\qquad\quad \underline{+7} \quad \underline{+7} \\
&x^2 + 4x = 3
\end{aligned}$$

Ok, now what? We could try dividing all the terms by 4 to isolate x, but we would still have x^2. We will need different techniques than we use for solving linear equations to solve this equation. There are many methods for solving different kinds of polynomials using algebraic principles, but the truth is that most polynomials cannot be solved with the algebra we have used here.

At the end of this module, we will share some of the practical uses for solving polynomials that occur in our everyday lives. Polynomials are everywhere! They appear in electrical circuitry, mechanical systems, population ecology, roller coaster design, classrooms around the US, and even in the way Google's search engine ranks pages.

Learning Outcomes

Simple Polynomial Equations

- Use the zero product principle
- Identify the greatest common factor of a polynomial
- Solve simple polynomial equations

Factoring Methods

- Factor by grouping
- Factor trinomials

Applications of Trinomials

- Solve quadratic equations
- Solve quadratic equations involving projectile motion
- Use the pythagorean theorem

INTRODUCTION TO FACTORING

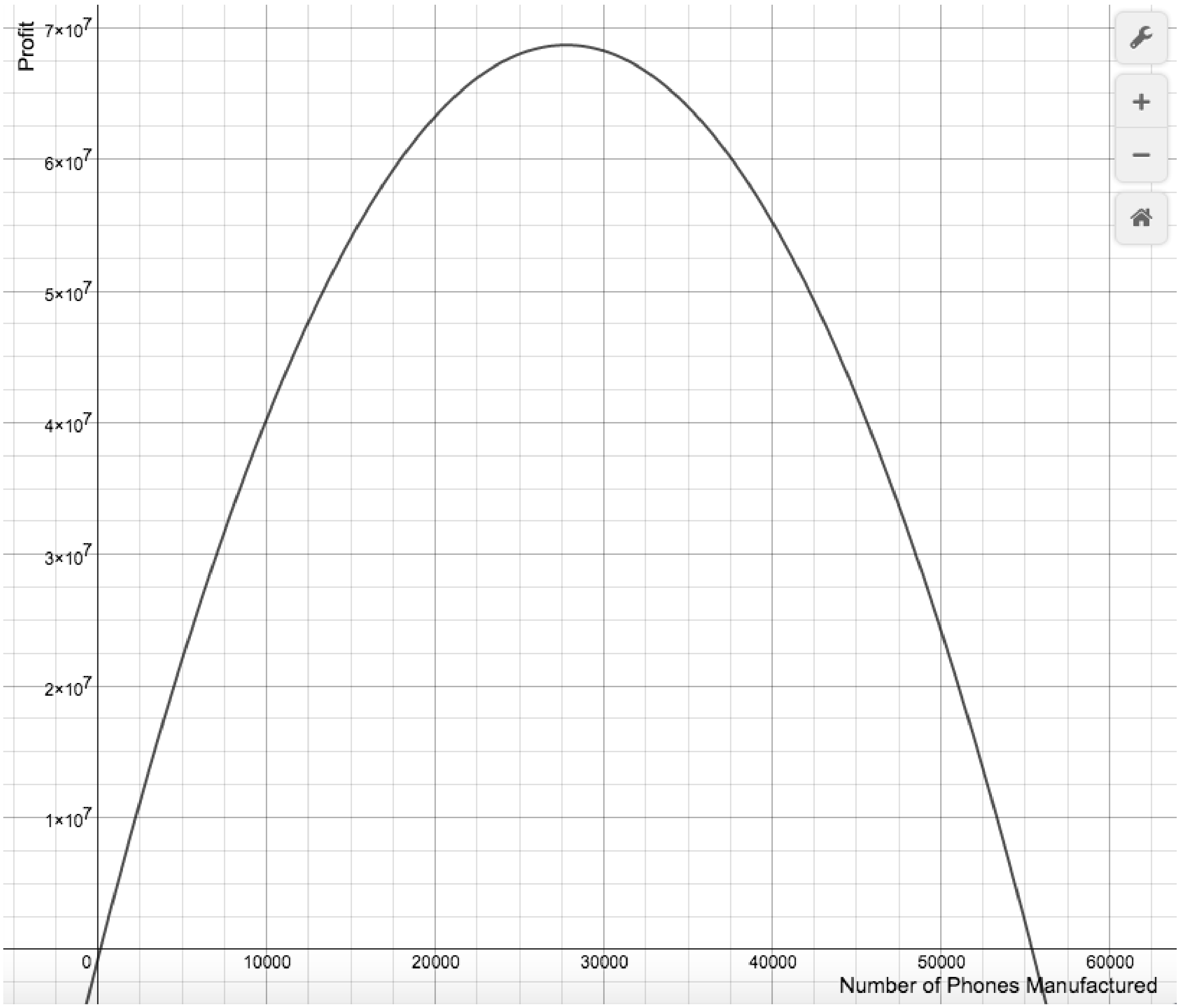
Learning Objectives

- Use the principle of zero products to solve equations
- Determine for what kind of equations the principle of zero products can be used
- Explain why some techniques for solving linear equations don't work for solving polynomial equations

Consider the profit equation for a cell phone manufacturer:

$P = -0.09x^2 + 5000x - 750,000$

Using an online graphing calculator, we generated this graph of the profit equation.



A manager may want to know for what amount of phones manufactured and sold will a profit be made. By substituting 100 in for x, we discovered that when 100 phones were manufactured and sold, the company did not make a profit.

Substitute x = 100

$$\begin{aligned} P &= -0.09x^2 + 5000x - 750,000 \\ &= -0.09(100)^2 + 5000(100) - 750,000 \\ &= -900 + 500,000 - 750,000 \\ &= -250,900 \end{aligned}$$

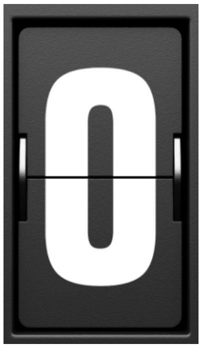
Finding the points where profit is greater than or equal to zero will guide management decisions, and help the company plan to have enough labor and materials on hand. Knowing where profit equals zero gives the company a baseline from which to plan. On the graph, profit is zero when the parabola crosses the x-axis. In algebraic terms, this means finding where the profit equation is equal to zero:

$$\begin{aligned} P &= -0.09x^2 + 5000x - 750,000 \\ 0 &= -0.09x^2 + 5000x - 750,000 \\ 0 &< -0.09x^2 + 5000x - 750,000 \end{aligned}$$

We know how to solve linear equations like this: $x - 4 = 0$, and linear inequalities like this: $x - 4 > 0$. But, how do you solve a polynomial equation like this: $0 = -0.09x^2 + 5000x - 750,000$, or an inequality like this: $0 < -0.09x^2 + 5000x - 750,000$?

In this section we will learn some very useful tools for solving certain kinds of polynomial equations, and in later math courses you will likely learn how to extend these ideas to solving polynomial inequalities. The first concept we will explore is that of the zero product property, then we will discuss how that can be used to solve polynomial equations.

The Principle of Zero Products



Zero

What if we told you that we multiplied two numbers together and got an answer of zero? What could you say about the two numbers? Could they be 2 and 5? Could they be 9 and 1? No! When the result (answer) from multiplying two numbers is zero, that means that one of them *had* to be zero. This idea is called the zero product principle, and it is useful for solving certain kinds of equations, like we described in the cell phone manufacturer example.

Principle of Zero Products

The Principle of Zero Products states that if the product of two numbers is 0, then at least one of the factors is 0. If $ab=0$, then either $a=0$ or $b=0$, or both *a* and *b* are 0.

Example

Use the principle of zero products to solve. $5y=0$
Show Solution

We can extend this idea to products of more than just two numbers. In the next example we will show that we can use the principle of zero products to solve an equation containing the product of a number and a binomial.

Example

Use the principle of zero products to solve:
$7\left(y-2\right)=0$
Show Solution

We could have used the distributive property and the addition and multiplication properties of equality to solve the equation in the previous example. It would look something like this:

Solve $7\left(y-2\right)=0$ using the distributive property.

$$
\begin{aligned}
7(y-2) &= 0\\
7y-14 &= 0\\
\underline{+14}\quad & \underline{+14}\\
7y &= 14\\
\frac{7y}{7} &= \frac{14}{7}\\
y &= 2
\end{aligned}
$$

We have the same answer that we verified in the example, but we used different algebraic principles to find it.

In the next example we add another layer to the idea that we can use the principle of zero products to solve equations. We will solve an equation that contains the product of a variable and a binomial.

Example

Use the principle of zero products to solve:
$t(5-t)=0$
Show Solution

Why don't we just use the distributive property to solve these kind of equations? Let's try using the distributive property on the previous example to explain why this will not always work.

$t(5-t)=0$

$$
\begin{aligned}
5t-t^2 &= 0\\
\underline{-5t}\quad & \underline{-5t}\\
t^2 &= -5t\\
t\cdot t &= -5t\\
\frac{t\cdot t}{t} &= \frac{-5t}{t}\\
t &= -5
\end{aligned}
$$

Wait, in the example, our solution was $t=0 \text{ OR } t=5$. Let's check this new answer to see if it is correct.

Substitute $t=-5$

$$
\begin{aligned}
-5(5-(-5)) &= 0\\
-5(5+5) &= 0\\
-5(10) &= 0\\
-50 &\neq 0
\end{aligned}
$$

This isn't even the right answer!

When we are solving polynomial equations, we need to use some different methods than we used to solve linear equations to make sure we get *all* of the *correct* answers. The principle of zero products is one tool that allows us to do this.

Caution! It is important to remember that the principle of zero products **only** works when we have an equation with zero on one side, and only a product on the other side. The table below gives examples of equations for which you can and cannot apply the principle of zero products.

YES Zero Products Works to Solve	NO Zero Products Does Not Work to Solve	WHY NOT?
$\frac{1}{2}(x-2)=0$	$\frac{1}{2}(x-2)=28$	There is a product on the left, but it is not equal to zero.
$s(9+s)=0$	$s^2+9s=0$	There is a sum equal to zero but no product equal to zero.

Let's look at one more example of how the principle of zero products can be used to solve equations involving products that are binomials.

Example

Use the principle of zero products to solve:
$(s+1)(s-5)=0$
Show Solution

The last two examples we showed were both polynomial equations that were degree two. Remember that degree means the largest exponent in the polynomial. Degree two polynomials are often called quadratic. Using the distributive property to multiply the products in the last example will help you see that it is a degree two polynomial.

Use a table:

	s	$+1$
s	s^2	s
-5	$-5s$	-5

Combine terms and simplify:

$$s^2+s-5s-5$$
$$=s^2-4s-5$$

When the polynomial is simplified, you can tell that it is a degree two polynomial, or a quadratic polynomial. At the start of this page, we proposed that a business would be interested in where the quadratic polynomial that represented profit was equal to or greater than zero. We have presented one way to find where a quadratic polynomial is equal to zero, as long as it is in the form of a product of two binomials, and hasn't been multiplied out.

In the following video we present more examples of how to use the zero product principle to solve polynomial equations that are in factored form.

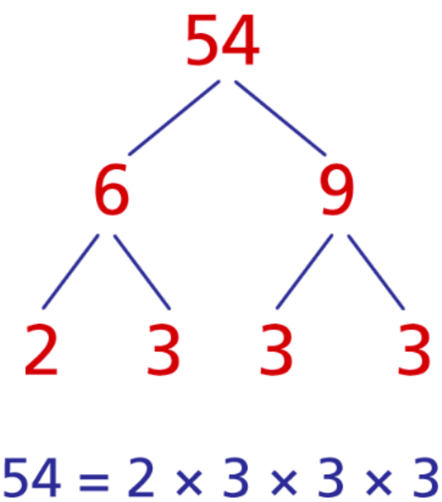
Watch this video online: https://youtu.be/yCcMCPHFrVc

What would you do if you were asked to solve a quadratic equation such as $y^2+2y=0$? We have already shown that we can't use the same techniques we used to solve linear equations. In the next section, we will show you how to re-write $y^2+2y=0$ as the product of a monomial and a binomial so you can use the zero product principle to solve it.

SOLVE SIMPLE POLYNOMIAL EQUATIONS

Learning Objectives

- Greatest Common Factor
  - Review the concept of greatest common factor
  - Factor a Polynomial
- Simple Polynomial Equations
  - Factor the greatest common monomial out of a polynomial
  - Solve a polynomial in factored form by setting it equal to zero



Factor Tree for 54

In the section on the zero product principle, we showed that using the techniques for solving equations that we learned for linear equations did not work to solve

$t\left(5-t\right)=0$

But because the equation was written as the product of two terms, we could use the zero product principle. What if we are given a polynomial equation that is not written as a product of two terms, such as this one $2y^2+4y=0$? We can use a technique called factoring, where we try to find factors that can be divided into each term of the polynomial so it can be rewritten as a product.

In this section we will explore how to find common factors from the terms of a polynomial, and rewrite it as a product. This technique will help us solve polynomial equations in the next section.

Greatest Common Factor

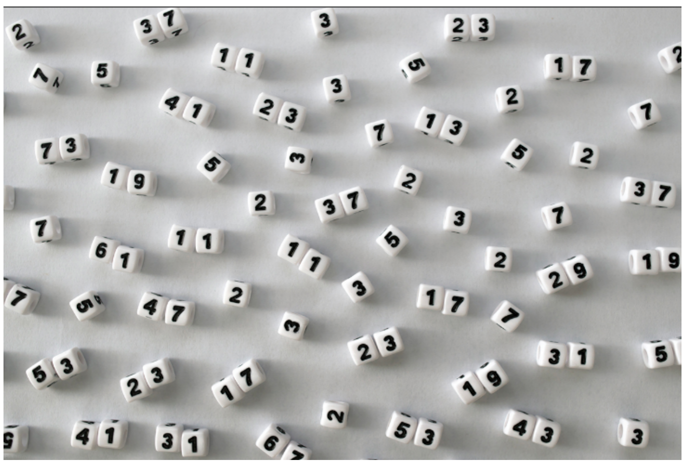
When we studied fractions, we learned that the **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20.The GCF of polynomials works the same way: 4x is the GCF of 16x and $20x^2$ because it is the largest polynomial that divides evenly into both 16x and $20x^2$.

A General Note: Greatest Common Factor

The **greatest common factor** (GCF) of a group of given polynomials is the largest polynomial that divides evenly into the polynomials.

Factors are the building blocks of multiplication. They are the numbers that you can multiply together to produce another number: 2 and 10 are factors of 20, as are 4 and 5 and 1 and 20. To factor a number is to rewrite it as a product. $20 = 4 \cdot 5$. In algebra, we use the word factor as both a noun – something being multiplied – and as a verb – rewriting a sum or difference as a product.

To factor a **polynomial**, you rewrite it as a product. Any integer can be written as the product of factors, and we can apply this technique to **monomials** or polynomials. **Factoring** is very helpful in simplifying and solving equations using polynomials.



Prime Numbers

A **prime factor** is similar to a **prime number**—it has only itself and 1 as factors. The process of breaking a number down into its prime factors is called **prime factorization**.

To get acquainted with the idea of factoring, let's first find the **greatest common factor (GCF)** of two whole numbers. The GCF of two numbers is the greatest number that is a factor of *both* of the numbers. Take the numbers 50 and 30.

$$50 = 10 \cdot 5$$
$$30 = 10 \cdot 3$$

Their greatest common factor is 10, since 10 is the greatest factor that both numbers have in common.

To find the GCF of greater numbers, you can factor each number to find their prime factors, identify the prime factors they have in common, and then multiply those together.

Example

Find the greatest common factor of 210 and 168.
Show Solution

Because the GCF is the product of the prime factors that these numbers have in common, you know that it is a factor of both numbers. (If you want to test this, go ahead and divide both 210 and 168 by 42—they are both evenly divisible by this number!)

The video that follows show another example of finding the greatest common factor of two whole numbers.

Watch this video online: https://youtu.be/KbBJcdDY_VE

Finding the greatest common factor in a set of monomials is not very different from finding the GCF of two whole numbers. The method remains the same: factor each monomial independently, look for common factors, and then multiply them to get the GCF.

Example

Find the greatest common factor of $25b^3$ and $10b^2$.

Show Solution

The monomials have the factors 5, *b*, and *b* in common, which means their greatest common factor is $5 \cdot b \cdot b$, or simply $5b^2$.

The video that follows gives another example of finding the greatest common factor of two monomials with only one variable.

Watch this video online: https://youtu.be/EhkVBXRBC2s

Example

Find the greatest common factor of $81c^3d$ and $45c^2d^2$.
Show Solution

The video that follows shows another example of finding the greatest common factor of two monomials with more than one variable.

Watch this video online: https://youtu.be/GfJvoIO3gKQ

Factor a Polynomial

One of these things is not like the others.

Before we solve polynomial equations, we will practice finding the greatest common factor of a polynomial. If you can find common factors for each term of a polynomial, then you can factor it, and solving will be easier.

To help you practice finding common factors, identify factors that the terms of the polynomial have in common in the table below.

Polynomial	Terms	Common Factors
$6x+9$	6*x* and 9	3 is a factor of 6*x* and 9
a^2-2a	a^2 and $-2a$	*a* is a factor of a^2 and $-2a$
$4c^3+4c$	$4c^3$ and $4c$	4 and *c* are factors of $4c^3$ and 4*c*

To factor a polynomial, first identify the greatest common factor of the terms. You can then use the distributive property to rewrite the polynomial in a factored form. Recall that the **distributive property of multiplication over addition** states that a product of a number and a sum is the same as the sum of the products.

Distributive Property Forward and Backward

Forward: Product of a number and a sum: $a\left(b+c\right)=a\cdot b+a\cdot c$. You can say that "$a$ is being distributed over $b+c$."
Backward: Sum of the products: $a\cdot b+a\cdot c=a\left(b+c\right)$. Here you can say that "*a* is being factored out."

We first learned that we could distribute a factor over a sum or difference, now we are learning that we can "undo" the distributive property with factoring.

Example

Factor $25b^3+10b^2$.
Show Solution

The factored form of the polynomial $25b^3+10b^2$ is $5b^2\left(5b+2\right)$. You can check this by doing the multiplication. $5b^2\left(5b+2\right)=25b^3+10b^2$.

Note that if you do not factor the greatest common factor at first, you can continue factoring, rather than start all over.

For example:

$$\begin{aligned}25b^3+10b^2&=5\left(5b^3+2b^2\right) &&\text{Factor out 5.}\\&=5b^2\left(5b+2\right)&&\text{Factor out } b^2.\end{aligned}$$

Notice that you arrive at the same simplified form whether you factor out the GCF immediately or if you pull out factors individually.

Example

Factor $81c^3d+45c^2d^2$.
Show Solution

The following video provides two more examples of finding the greatest common factor of a binomial

Watch this video online: https://youtu.be/25_f_mVab_4

Simple Polynomial Equations

In this section we will apply factoring a monomial from a polynomial to solving polynomial equations. Recall that not all of the techniques we use for solving linear equations will apply to solving polynomial equations, so we will be using the zero product principle to solve for a variable.

We will begin with an example where the polynomial is already equal to zero.

Example

Solve:
$-t^2+t=0$
Show Solution

Notice how we were careful with signs in the last example. Even though one of the terms was negative, we factored out the positive common term of t. In the next example we will see what to do when the polynomial you are working with is not set equal to zero. Int eh followign video, we present more examples of solving quadratic equations by factoring.

Watch this video online: https://youtu.be/Hpb8DVYBDzA

Example

Solve: $6t = 3t^2 - 12t$
Show Solution

The video that follows provides another example of solving a polynomial equation using the zero product principle and factoring.

Watch this video online: https://youtu.be/oYytjgbd6Q0

We will work through one more example that is similar to the ones above, except this example has fractions and the greatest common monomial is negative.

Example

Solve $\frac{1}{2}y = -4y - \frac{1}{2}y^2$
Show Solution

Wow! In the last example, we used *many* skills to solve one equation. Let's summarize them:

- We needed a common denominator to combine the like terms $-4y$ and $-\frac{1}{2}y$, after we moved all the terms to one side of the equation
- We found the GCF of the terms $-\frac{9}{2}y$ and $-\frac{1}{2}y^2$
- We used the GCF to factor the polynomial $-\frac{9}{2}y - \frac{1}{2}y^2$
- We used the zero product principle to solve the polynomial equation $0 = -\frac{1}{2}y\,(9 + y)$

Sometimes solving an equation requires the combination of many algebraic principles and techniques. The last facet of solving the polynomial equation $\frac{1}{2}y = -4y - \frac{1}{2}y^2$ that we should talk about is negative signs.

We found that the GCF $-\frac{1}{2}y$ contained a negative coefficient. This meant that when we factored it out of all the terms in the polynomial, we were left with two positive factors, 9 and y. This explains why we were left with $(9 + y)$ as one of the factors of our final product.

In the following video we present another example of solving a quadratic polynomial with fractional coefficients using factoring and the zero product principle.

Watch this video online: https://youtu.be/wm6DJ1bnaJs

If the GCF of a polynomial is negative, pay attention to the signs that are left when you factor it from the terms of a polynomial.

In the next unit, we will learn more factoring techniques that will allow you to be able to solve a wider variety of polynomial equations such as $3x^2 - x = 2$.

Summary

A whole number, monomial, or polynomial can be expressed as a product of factors. You can use some of the same logic that you apply to factoring integers to factoring polynomials. To factor a polynomial, first identify the greatest common factor of the terms, and then apply the distributive property to rewrite the expression. Once a polynomial in $a \cdot b + a \cdot c$ form has been rewritten as $a\,(b + c)$, where *a* is the GCF, the polynomial is in factored form.

In this section we practiced using the zero product principle as a method for solving polynomial equations. We found that the techniques we used to solve linear equations did not give us the correct answer when used to solve a polynomial equation. We also found that a polynomial can be rewritten as a product by factoring out the greatest common factor. We used both factoring and the zero product principle to solve second degree polynomials.

MORE FACTORING METHODS

Learning Objectives

- Factor by Grouping
  - Identify patterns that result from multiplying two binomials and how they affect factoring by grouping
  - Factor a four term polynomial by grouping terms
- Methods for Factoring Trinomials
  - Apply an algorithm to rewrite a trinomial as a four term polynomial
  - Use factoring by grouping to factor a trinomial
  - Use a shortcut to factor trinomials of the form x^2+bx+c
  - Factor trinomials of the form ax^2+bx+c
  - Recognize where to place negative signs when factoring a trinomial
  - Recognize when a polynomial is a difference of squares, and how it would factor as the product of two binomials

When we learned to multiply two binomials, we found that the result, before combining like terms, was a four term polynomial, as in this example: $\left(x+4\right)\left(x+2\right)=x^2+2x+4x+8$.

We can apply what we have learned about factoring out a common monomial to return a four term polynomial to the product of two binomials. Why would we even want to do this?

Why Should I Care?

Because it is an important step in learning techniques for factoring trinomials, such as the one you get when you simplify the product of the two binomials from above:

$$\left(x+4\right)\left(x+2\right)$$
$$=x^2+2x+4x+8$$
$$=x^2+6x+8$$

Additionally, factoring by grouping is a technique that allows us to factor a polynomial whose terms don't all share a GCF. In the following example, we will introduce you to the technique. Remember, one of the main reasons to factor is because it will help solve polynomial equations.

Example

Factor $a^2 + 3a + 5a + 15$
Show Solution

Notice that when you factor a two term polynomial, the result is a monomial times a polynomial. But the factored form of a four-term polynomial is the product of two binomials. As we noted before, this is an important middle step in learning how to factor a three term polynomial.

This process is called the *grouping technique*. Broken down into individual steps, here's how to do it (you can also follow this process in the example below).

- Group the terms of the polynomial into pairs that share a GCF.
- Find the greatest common factor and then use the distributive property to pull out the GCF
- Look for the common binomial between the factored terms
- Factor the common binomial out of the groups, the other factors will make the other binomial

Let's try factoring a few more four-term polynomials. Note how there is a now a constant in front of the x^2 term. We will just consider this another factor when we are finding the GCF.

Example

Factor $2x^2 + 4x + 5x + 10$.
Show Solution

Another example follows that contains subtraction. Note how we choose a positive GCF from each group of terms, and the negative signs stay.

Example

Factor $2x^2 – 3x + 8x – 12$.
Show Solution

The video that follows provides another example of factoring by grouping.

Watch this video online: https://youtu.be/RR5nj7RFSiU

In the next example, we will have a GCF that is negative. It is important to pay attention to what happens to the resulting binomial when the GCF is negative.

Example

Factor $3x^2 + 3x – 2x – 2$.
Show Solution

In the following video we present another example of factoring by grouping when one of the GCF is negative.

Watch this video online: https://youtu.be/0dvGmDGVC5U

Sometimes, you will encounter polynomials that, despite your best efforts, cannot be factored into the product of two binomials.

Example

Factor $7x^2 – 21x + 5x – 5$.
Show Solution

In the example above, each pair can be factored, but then there is no common factor between the pairs!

Factor Trinomials Part I

In the last section we introduced the technique of factoring by grouping as a means to be able to factor a trinomial. Now we will actually get to the work of starting with a three term polynomial, and rewriting it as a four term polynomial so it can be factored.

We will start with factoring trinomials of the form $x^2 + bx + c$ that don't have a coefficient in front of the x^2 term.

Remember that when $(x + 2)$ and $(x + 5)$, are multiplied, the result is a four term polynomial and then it is simplified into a trinomial:

$$(x + 2)(x + 5) = x^2 + 5x + 2x + 10 = x^2 + 7x + 10$$

Factoring is the reverse of multiplying, so let's go in reverse and factor the trinomial $x^2 + 7x + 10$. The individual terms x^2, $7x$, and 10 share no common factors. If we rewrite the middle term as the sum of the two terms $7x = 5x + 2x$ then we can use the grouping technique:

$$(x^2 + 5x) + (2x + 10)$$

Factor each pair: $x\underbrace{(x+5)}_{\text{common binomial factor}} + 2\underbrace{(x+5)}$

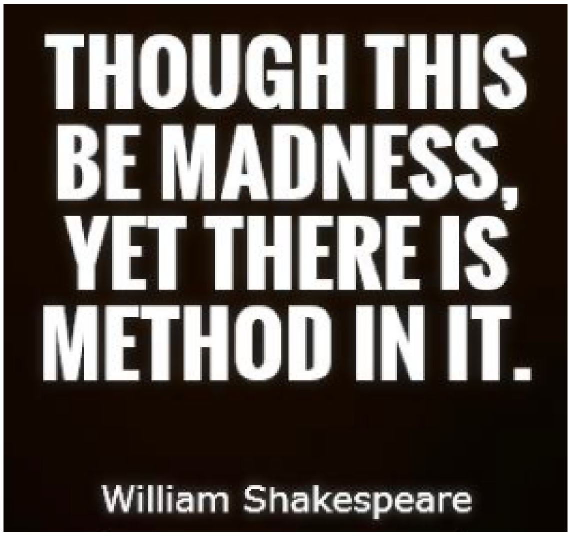
Then pull out the common binomial factor: $(x + 5)(x + 2)$

What would have happened if we had rewritten $7x$ as $6x + x$?

$$(x^2 + 6x) + (x + 10)$$

Factor each pair: $x(x + 6) + 1(x + 10)$

Then we don't have a common factor of $(x + 5)$ like we did before. There is a method to the madness of choosing how to rewrite the middle terms so that you will end up with a common binomial factor.



Method to the Madness

The following is a summary of the method, then we will show some examples of how to use it.

Factoring Trinomials in the form $x^2 + bx + c$

To factor a trinomial in the form $x^2 + bx + c$, find two integers, *r* and *s*, whose product is *c* and whose sum is *b*.

$r \cdot s = c$

and

$r + s = b$

Rewrite the trinomial as $x^2 + rx + sx + c$ and then use grouping and the distributive property to factor the polynomial. The resulting factors will be $(x + r)$ and $(x + s)$.

For example, to factor $x^2 + 7x + 10$, you are looking for two numbers whose sum is 7 (the coefficient of the middle term) and whose product is 10 (the last term).

Look at factor pairs of 10:1 and 10, 2, and 5. Do either of these pairs have a sum of 7? Yes, 2 and 5. So you can rewrite $7x$ as $2x + 5x$, and continue factoring as in the example above. Note that you can also rewrite $7x$ as $5x + 2x$. Both will work.

Let's factor the trinomial $x^2 + 5x + 6$. In this polynomial, the *b* part of the middle term is 5 and the *c* term is 6. A chart will help us organize possibilities. On the left, list all possible factors of the *c* term, 6; on the right you'll find the sums.

Factors whose product is 6	Sum of the factors
$1 \cdot 6 = 6$	$1 + 6 = 7$
$2 \cdot 3 = 6$	$2 + 3 = 5$

There are only two possible factor combinations, 1 and 6, and 2 and 3. You can see that $2 + 3 = 5$. So $2x + 3x = 5x$, giving us the correct middle term.

Example

Factor $x^2 + 5x + 6$.
Show Solution

Note that if you wrote $x^2 + 5x + 6$ as $x^2 + 3x + 2x + 6$ and grouped the pairs as $\left(x^2 + 3x\right) + (2x + 6)$; then factored, $x\,(x + 3) + 2\,(x + 3)$, and factored out $x + 3$, the answer would be $(x + 3)\,(x + 2)$. Since multiplication is commutative, the order of the factors does not matter. So this answer is correct as well; they are equivalent answers.

In the following video, we present another example of how to use grouping to factor a quadratic polynomial.

Watch this video online: https://youtu.be/_Rtp7nSxf6c

Finally, let's take a look at the trinomial $x^2 + x - 12$. In this trinomial, the *c* term is -12. So look at all of the combinations of factors whose product is -12. Then see which of these combinations will give you the correct middle term, where *b* is 1.

Factors whose product is -12	Sum of the factors
$1 \cdot -12 = -12$	$1 + -12 = -11$
$2 \cdot -6 = -12$	$2 + -6 = -4$

Factors whose product is	Sum of the factors
$3\cdot -4=-12$	$3+-4=-1$
$4\cdot -3=-12$	$4+-3=1$
$6\cdot -2=-12$	$6+-2=4$
$12\cdot -1=-12$	$12+-1=11$

There is only one combination where the product is -12 and the sum is 1, and that is when $r=4$, and $s=-3$. Let's use these to factor our original trinomial.

Example

Factor x^2+x-12.
Show Solution

In the above example, you could also rewrite x^2+x-12 as $x^2-3x+4x-12$ first. Then factor $x\left(x-3\right)+4\left(x-3\right)$, and factor out $\left(x-3\right)$ getting $\left(x-3\right)\left(x+4\right)$. Since multiplication is commutative, this is the same answer.

Factoring Tips

Factoring trinomials is a matter of practice and patience. Sometimes, the appropriate number combinations will just pop out and seem so obvious! Other times, despite trying many possibilities, the correct combinations are hard to find. And, there are times when the trinomial cannot be factored.

While there is no foolproof way to find the right combination on the first guess, there are some tips that can ease the way.

Tips for Finding Values that Work

When factoring a trinomial in the form x^2+bx+c, consider the following tips.
Look at the *c* term first.

- If the *c* term is a positive number, then the factors of *c* will both be positive or both be negative. In other words, *r* and *s* will have the same sign.
- If the *c* term is a negative number, then one factor of *c* will be positive, and one factor of *c* will be negative. Either *r* or *s* will be negative, but not both.

Look at the *b* term second.

- If the *c* term is positive and the *b* term is positive, then both *r* and *s* are positive.
- If the *c* term is positive and the *b* term is negative, then both *r* and *s* are negative.
- If the *c* term is negative and the *b* term is positive, then the factor that is positive will have the greater absolute value. That is, if $|r|>|s|$, then *r* is positive and *s* is negative.
- If the *c* term is negative and the *b* term is negative, then the factor that is negative will have the greater absolute value. That is, if $|r|>|s|$, then *r* is negative and *s* is positive.

After you have factored a number of trinomials in the form x^2+bx+c, you may notice that the numbers you identify for *r* and *s* end up being included in the factored form of the trinomial. Have a look at the following chart, which reviews the three problems you have seen so far.

Trinomial	$x^2+7x+10$	x^2+5x+6	x^2+x-12

***r* and *s* values**	$r = +5, s = +2$	$r = +2, s = +3$	$r = +4, s = -3$
Factored form	$(x+5)(x+2)$	$(x+2)(x+3)$	$(x+4)(x-3)$

The Shortcut

Notice that in each of the examples above, the *r* and *s* values are repeated in the factored form of the trinomial. So what does this mean? It means that in trinomials of the form $x^2 + bx + c$ (where the coefficient in front of x^2 is 1), if you can identify the correct *r* and *s* values, you can effectively skip the grouping steps and go right to the factored form. For those of you that like shortcuts, let's look at some examples where we use this idea.

Shortcut This Way

In the next two examples, we will show how you can skip the step of factoring by grouping and move directly to the factored form of a product of two binomials with the r and s values that you find. The idea is that you can build factors for a trinomial in this form: $x^2 + bx + c$ by finding r and s, then placing them in two binomial factors like this:

$(x+r)(x+s)$ OR $(x+s)(x+r)$

Example

Factor: $y^2 + 6y - 27$
Show Solution

We will show one more example so you can gain more experience.

Example

Factor: $-m^2 + 16m - 48$

Show Solution

In the following video, we present two more examples of factoring a trinomial using the shortcut presented here.

Watch this video online: https://youtu.be/-SVBVVYVNTM

Factor Trinomials Part II

The next goal is for you to be comfortable with recognizing where to place negative signs, and whether a trinomial can even be factored. Additionally, we will explore one special case to look out for at the end of this page.

Not all trinomials look like x^2+5x+6, where the coefficient in front of the x^2 term is 1. In these cases, your first step should be to look for common factors for the three terms.

Trinomial	**Factor out Common Factor**	**Factored**
$2x^2+10x+12$	$2(x^2+5x+6)$	$2\left(x+2\right)\left(x+3\right)$
$-5a^2-15a-10$	$-5(a^2+3a+2)$	$-5\left(a+2\right)\left(a+1\right)$
c^3-8c^2+15c	$c\left(c^2-8c+15\right)$	$c\left(c-5\right)\left(c-3\right)$
$y^4-9y^3-10y^2$	$y^2\left(y^2-9y-10\right)$	$y^2\left(y-10\right)\left(y+1\right)$

Notice that once you have identified and pulled out the common factor, you can factor the remaining trinomial as usual. This process is shown below.

Example

Factor $3x^3-3x^2-90x$.
Show Solution

The following video contains two more examples of factoring a quadratic trinomial where the first step is to factor out a GCF. We use the shortcut method instead of factoring by grouping.

Watch this video online: https://youtu.be/pgH77rAtsbs

The general form of trinomials with a leading coefficient of *a* is ax^2+bx+c. Sometimes the factor of *a* can be factored as you saw above; this happens when *a* can be factored out of all three terms. The remaining trinomial that still needs factoring will then be simpler, with the leading term only being an x^2 term, instead of an ax^2 term.

However, if the coefficients of all three terms of a trinomial don't have a common factor, then you will need to factor the trinomial with a coefficient of something other than 1.

Factoring Trinomials in the form ax^2+bx+c

To factor a trinomial in the form ax^2+bx+c, find two integers, *r* and *s*, whose sum is *b* and whose product is *ac.*
$r\cdot s=a\cdot c$
$r+s=b$
Rewrite the trinomial as $ax^2+rx+sx+c$ and then use grouping and the distributive property to factor the polynomial.

This is almost the same as factoring trinomials in the form x^2+bx+c, as in this form $a=1$. Now you are looking for two factors whose product is $a\cdot c$, and whose sum is *b*.

Let's see how this strategy works by factoring $6z^2 + 11z + 4$.

In this trinomial, $a = 6$, $b = 11$, and $c = 4$. According to the strategy, you need to find two factors, *r* and *s*, whose sum is $b = 11$ and whose product is $a \cdot c = 6 \cdot 4 = 24$. You can make a chart to organize the possible factor combinations. (Notice that this chart only has positive numbers. Since *ac* is positive and *b* is positive, you can be certain that the two factors you're looking for are also positive numbers.)

Factors whose product is 24	**Sum of the factors**
$1 \cdot 24 = 24$	$1 + 24 = 25$
$2 \cdot 12 = 24$	$2 + 12 = 14$
$3 \cdot 8 = 24$	$3 + 8 = 11$
$4 \cdot 6 = 24$	$4 + 6 = 10$

There is only one combination where the product is 24 and the sum is 11, and that is when $r = 3$, and $s = 8$. Let's use these values to factor the original trinomial.

Example

Factor $6z^2 + 11z + 4$.
Show Solution

In the following video, we present another example of factoring a trinomial using grouping. In this example, the middle term, b, is negative. Note how having a negative middle term and a positive c term influence the options for r and s when factoring.

Watch this video online: https://youtu.be/agDaQ_cZnNc

Before going any further, it is worth mentioning that not all trinomials can be factored using integer pairs. Take the trinomial $2z^2 + 35z + 7$, for instance. Can you think of two integers whose sum is $b = 35$ and whose product is $a \cdot c = 2 \cdot 7 = 14$? There are none! This type of trinomial, which cannot be factored using integers, is called a prime trinomial.

In some situations, *a* is negative, as in $-4h^2 + 11h + 3$. It often makes sense to factor out -1 as the first step in factoring, as doing so will change the sign of ax^2 from negative to positive, making the remaining trinomial easier to factor.

Example

Factor $-4h^2 + 11h + 3$
Show Solution

Note that the answer above can also be written as $(-h + 3)(4h + 1)$ or $(h - 3)(-4h - 1)$ if you multiply -1 times one of the other factors.

In the following video we present another example of factoring a trinomial in the form $-ax^2 + bx + c$ using the grouping technique.

Watch this video online: https://youtu.be/zDAMjdBfkDs

Difference of Squares

We would be remiss if we failed to introduce one more type of polynomial that can be factored. This polynomial can be factored into two binomials but has only two terms. Let's start from the product of two binomials to see the pattern.

Given the product of two binomials: $(x-2)(x+2)$, if we multiply them together, we lose the middle term that we are used to seeing as a result.

Multiply:

$(x-2)(x+2)$

$=x^2-2x+2x-2^2$

$=x^2-2^2$

$=x^2-4$

The polynomial x^2-4 is called a difference of squares because teach term can be written as something squared. A difference of squares will always factor in the following way:

Factor a Difference of Squares

Given a^2-b^2, it's factored form will be $(a+b)(a-b)$

Let's factor $x^2{-}4$ by writing it as a trinomial, $x^2+0x{-}4$. This is similar in format to the trinomials we have been factoring so far, so let's use the same method.

Find the factors of $a\cdot c$ whose sum is *b,* in this case, 0*:*

Factors of -4	**Sum of the factors**
$1\cdot-4=-4$	$1-4=-3$
$2\cdot-2=-4$	$2-2=0$
$-1\cdot4=-4$	$-1+4=3$

2, and -2 have a sum of 0. You can use these to factor $x^2{-}4$.

Example

Factor $x^2{-}4$.
Show Solution

Since order doesn't matter with multiplication, the answer can also be written as $(x+2)(x{-}2)$.

You can check the answer by multiplying $(x{-}2)(x+2)=x^2+2x{-}2x{-}4=x^2{-}4$.

The following video show two more examples of factoring a difference of squares.

Watch this video online: https://youtu.be/Li9IBp5HrFA

Summary

When a trinomial is in the form of ax^2+bx+c, where *a* is a coefficient other than 1, look first for common factors for all three terms. Factor out the common factor first, then factor the remaining simpler trinomial. If the remaining trinomial is still of the form ax^2+bx+c, find two integers, *r* and *s*, whose sum is *b* and whose product is *ac.* Then rewrite the trinomial as $ax^2+rx+sx+c$ and use grouping and the distributive property to factor the polynomial.

When ax^2 is negative, you can factor -1 out of the whole trinomial before continuing.

A difference of squares a^2-b^2 will factor in this way: $\left(a+b\right)\left(a-b\right)$.

Summary

Trinomials in the form x^2+bx+c can be factored by finding two integers, *r* and *s*, whose sum is *b* and whose product is *c.* Rewrite the trinomial as $x^2+rx+sx+c$ and then use grouping and the distributive property to factor the polynomial.

QUADRATIC EQUATIONS

Learning Objectives

- Quadratic Equations
  - Recognize a quadratic equation
  - Use the zero product principle to solve a quadratic equation that can be factored
  - Determine when solutions to quadratic equations can be discarded
- Pythagorean Theorem
  - Recognize a right triangle from other types of triangles
  - Use the Pythagorean theorem to find the lengths of a right triangle
- Projectiles
  - Define projectile motion
  - Solve a quadratic equation that represents projectile motion
  - Interpret the solution to a quadratic equation that represents projectile motion

When a polynomial is set equal to a value (whether an integer or another polynomial), the result is an equation. An equation that can be written in the form $ax^2+bx+c=0$ is called a **quadratic equation**. You can solve a quadratic equation using the rules of algebra, applying factoring techniques where necessary, and by using the **Principle of Zero Products**.

There are many applications for quadratic equations. When you use the Principle of Zero Products to solve a quadratic equation, you need to make sure that the equation is equal to zero. For example, $12^2+11x+2=7$ must first be changed to $12x^2+11x+-5=0$ by subtracting 7 from both sides.

Example

The area of a rectangular garden is 30 square feet. If the length is 7 feet longer than the width, find the dimensions.
Show Solution

In the example in the following video, we present another area application of factoring trinomials.

Watch this video online: https://youtu.be/PvXsWZp588o

The example below shows another quadratic equation where neither side is originally equal to zero. (Note that the factoring sequence has been shortened.)

Example

Solve $5b^2+4=-12b$ for *b.*
Show Solution

The following video contains another example of solving a quadratic equation using factoring with grouping.

Watch this video online: https://youtu.be/04zEXaOiO4U

If you factor out a constant, the constant will never equal 0. So it can essentially be ignored when solving. See the following example.

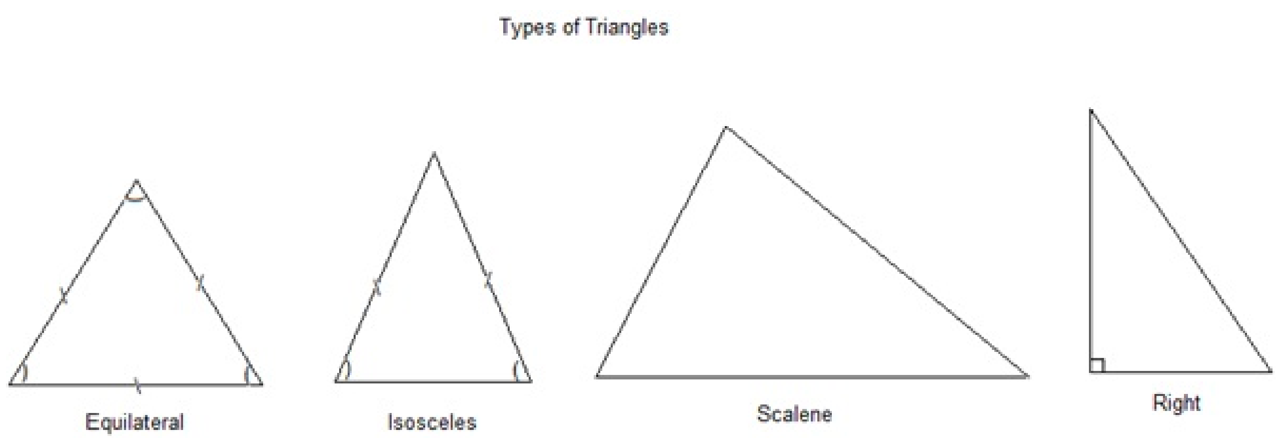
Example

Solve for k: $-2k^2+90=-8k$
Show Solution

In this last video example, we solve a quadratic equation with a leading coefficient of -1 using the shortcut method of factoring and the zero product principle.

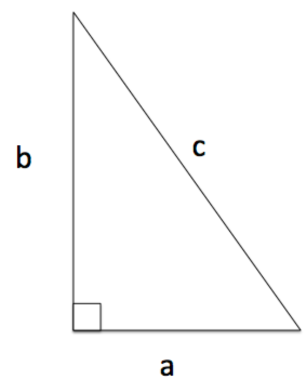
Watch this video online: https://youtu.be/nZYfgHygXis

Pythagorean Theorem



Triangles

The **Pythagorean theorem** or **Pythagoras's theorem** is a statement about the sides of a right triangle. One of the angles of a right triangle is always equal to 90 degrees. This angle is the right angle. The two sides next to the right angle are called the legs and the other side is called the hypotenuse. The hypotenuse is the side opposite to the right angle, and it is always the longest side. The image above shows four common kinds of triangle, including a right triangle.



Right Triangle Labeled

The Pythagorean theorem is often used to find unknown lengths of the sides of right triangles. If the longest leg of a right triangle is labeled c, and the other two a, and b as in the image on teh left, The Pythagorean Theorem states that

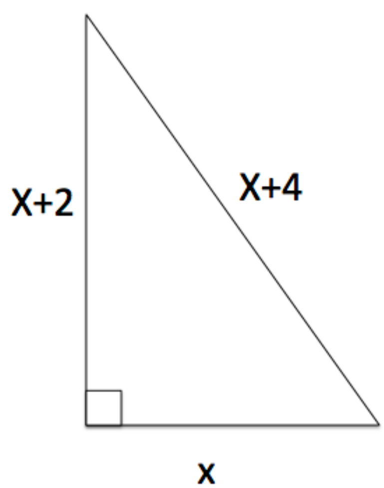
$a^2 + b^2 = c^2$

Given enough information, we can solve for an unknown length. This relationship has been used for many, many years for things such as celestial navigation and early civil engineering projects. We now have digital GPS and survey equipment that have been programmed to do the calculations for us.

In the next example we will combine the power of the Pythagorean theorem and what we know about solving quadratic equations to find unknown lengths of right triangles.

Example

A right triangle has one leg with length x, another whose length is greater by two, and the length of the hypotenuse is greater by four. Find the lengths of the sides of the triangle. Use the image below.



Show Solution

This video example shows another way a quadratic equation can be used to find and unknown length of a right triangle.

Watch this video online: https://youtu.be/xeP5pRBqsNs

If you are interested in celestial navigation and the mathematics behind it, watch this video for fun.

Watch this video online: https://youtu.be/XWLZKmPU17M

Projectile Motion

Projectile motion happens when you throw a ball into the air and it comes back down because of gravity. A projectile will follow a curved path that behaves in a predictable way. This predictable motion has been studied for centuries, and in simple cases it’s height from the ground at a given time, t, can be modeled with a quadratic polynomial of the form $\text{height} = at^2 + bt + c$ such as we have been studying in this module. Projectile motion is also called a parabolic trajectory because of the shape of the path of a projectile's motion, as in the image of water in the fountain below.

Parabolic WaterTrajectory

Parabolic motion and it's related equations allow us to launch satellites for telecommunications, and rockets for space exploration. Recently, police departments have even begun using projectiles with GPS to track fleeing suspects in vehicles, rather than pursuing them by high-speed chase ((Note: "Cops' Latest Tool in High-speed Chases: GPS Projectiles." *CBSNews*. CBS Interactive, n.d. Web. 14 June 2016.)).

In this section we will solve simple quadratic polynomials that represent the parabolic motion of a projectile. The real mathematical model for the path of a rocket or a police GPS projectile may have different coefficients or more variables, but the concept remains the same. We will also learn to interpret the meaning of the variables in a polynomial that models projectile motion.

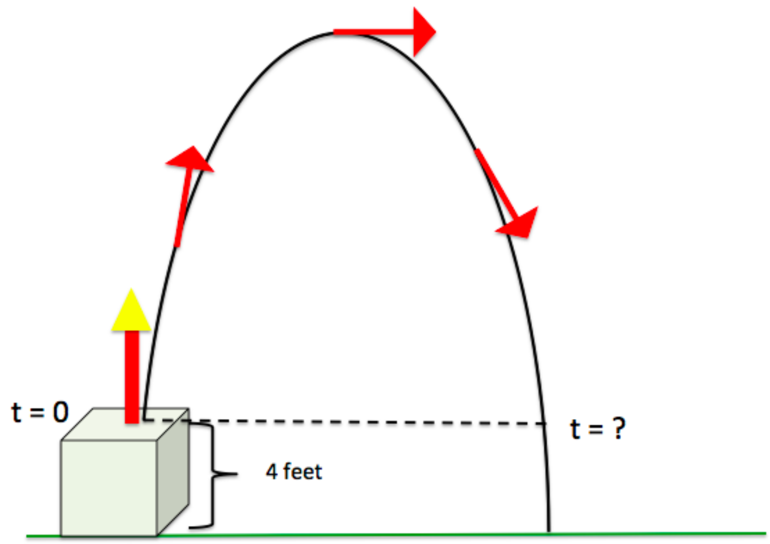
Example

A small toy rocket is launched from a 4-foot pedestal. The height (*h,* in feet) of the rocket *t* seconds after taking off is given by the formula $h = -2t^2 + 7t + 4$. How long will it take the rocket to hit the ground?
Show Solution

In the next example we will solve for the time that the rocket is at a given height other than zero.

Example

Use the formula for the height of the rocket in the previous example to find the time when the rocket is 4 feet from hitting the ground on it's way back down. Refer to the image.
$h=-2t^2+7t+4$



Show Solution

The video that follows presents another example of solving a quadratic equation that represents parabolic motion.

Watch this video online: https://youtu.be/hsWSzu3KcPU

In this section we introduced the concept of projectile motion, and showed that it can be modeled with a quadratic polynomial. While the models used in these examples are simple, the concepts and interpretations are the same. The methods used to solve quadratic polynomials that don't factor easily are many and well known, it is likely you will come across more in your studies.

Summary

You can find the solutions, or roots, of quadratic equations by setting one side equal to zero, factoring the polynomial, and then applying the Zero Product Property. The Principle of Zero Products states that if $ab=0$, then either $a=0$ or $b=0$, or both *a* and *b* are 0. Once the polynomial is factored, set each factor equal to zero and solve them separately. The answers will be the set of solutions for the original equation.

Not all solutions are appropriate for some applications. In many real-world situations, negative solutions are not appropriate and must be discarded.

MODULE 7: RATIONAL EXPRESSIONS AND EQUATIONS

INTRODUCTION

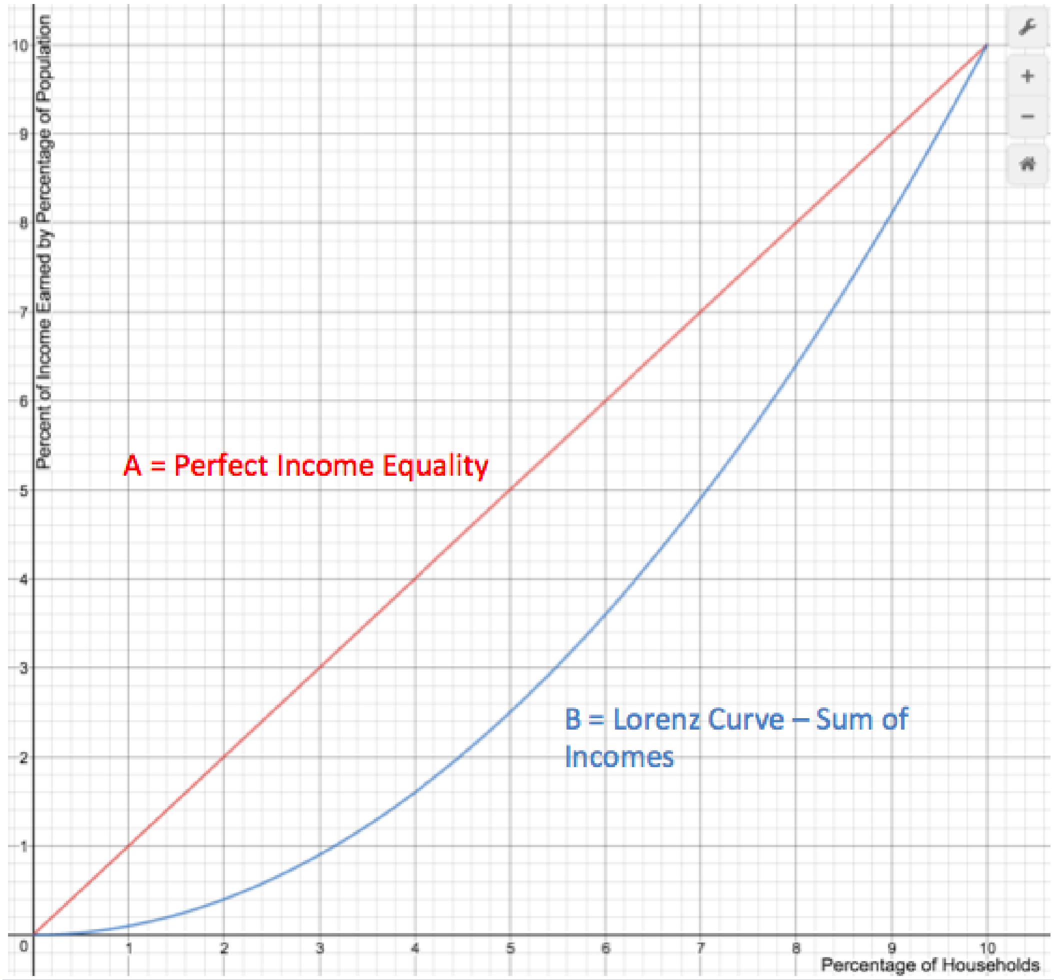
Why study rational expressions and equations?

If you were around in 2011, you probably heard about Occupy Wall Street, the protest against social and economic inequality in the US. According to Wikipedia, the main issues raised by Occupy Wall Street were social and economic inequality, greed, corruption, and the perceived undue influence of corporations on government—particularly from the financial services sector. The OWS slogan, "We are the 99%", refers to income inequality and wealth distribution in the U.S. between the wealthiest 1% and the rest of the population. To achieve their goals, protesters acted on consensus-based decisions made in general assemblies which emphasized direct action over petitioning authorities for redress.

Occupy Wall Street Ad

Economists are also interested in how wealth is distributed in economies. The Lorenz curve is a mathematical representation of the distribution of income or of wealth. Max O. Lorenz developed it in 1905 to represent inequality of the wealth distribution.

The curve in the graph below shows the proportion of overall income or wealth assumed by the bottom *x*% of the people. It is often used to represent income distribution, where it shows for the bottom *x*% of households, what percentage (*y*%) of the total income they have. The percentage of households is plotted on the *x*-axis, the percentage of income is on the *y*-axis. It can also be used to show distribution of assets. In such use, many economists consider it to be a measure of social inequality.

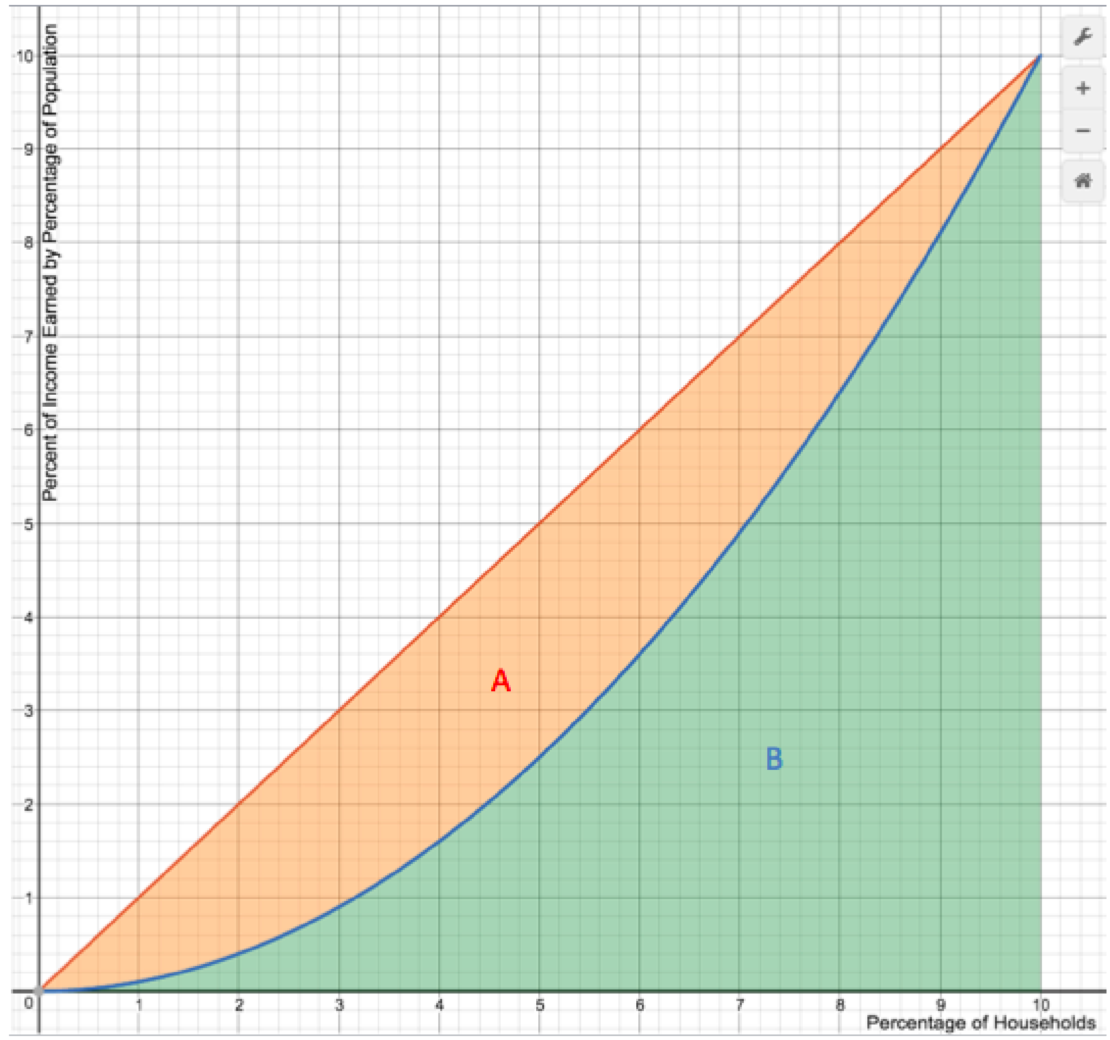


Lorenz Curve

Points on the Lorenz curve represent statements like “the bottom 20% of all households have 10% of the total income.” A perfectly equal income distribution would be one in which every person has the same income. In this case, the bottom *N*% of society would always have *N*% of the income. This can be depicted by the straight line $y = x$; called the “line of perfect equality.”

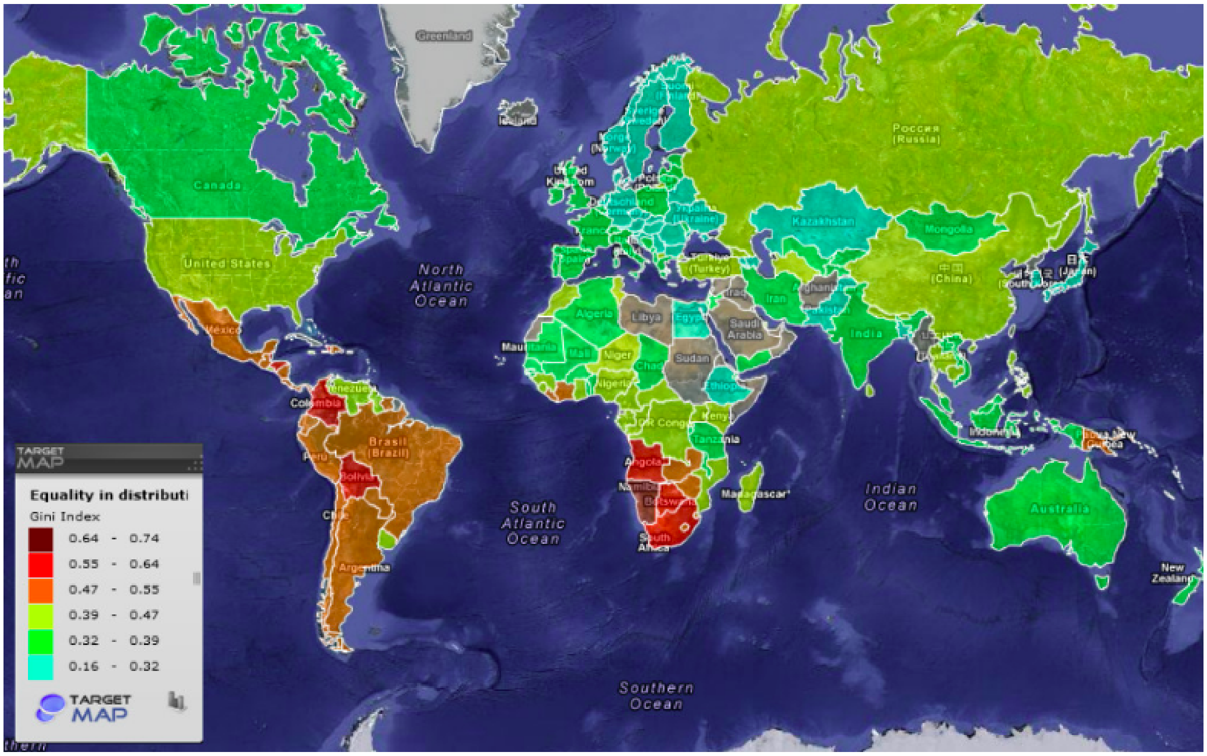
By contrast, a perfectly unequal distribution would be one in which one person has all the income and everyone else has none. In that case, the curve would be at $y = 0\%$ for all $x < 100\%$, and $y = 100\%$ when $x = 100\%$. This curve is called the “line of perfect inequality.”

The Gini coefficient is the ratio of the area between the line of perfect equality and the observed Lorenz curve to the area between the line of perfect equality and the line of perfect inequality. The higher the coefficient, the more unequal the distribution is. This is given by the ratio $G = \frac{A}{A+B}$, where *A* and *B* are the areas of regions as marked in the graph below.



Gini Index

The closer the Gini index index is to 1, the less equally wealth is distributed amongst an economy. The closer it is to 0, the more equally wealth is distributed. Below is a map of the world with Gini Index color coded.



World map color coded by Gini Index

The Gini Index is a rational expression, or ratio. In this module, we will define and apply mathematical operations to rational expressions. We will also solve rational equations.

Learning Outcomes

Operations With Rational Expressions

- Define and simplify rational expressions
- Multiply and divide rational expressions
- Add and subtract rational expressions

Rational Equations

- Solve rational equations
- Solve proportional problems
- Solve work equations
- Solve variation problems

IDENTIFY AND SIMPLIFY RATIONAL EXPRESSIONS

Learning Objectives

- Recognize and define a rational expression
- Determine the domain of a rational expression
- Simplify a rational expression

Rational expressions are fractions that have a polynomial in the numerator, denominator, or both. Although rational expressions can seem complicated because they contain variables, they can be simplified using the techniques used to simplify expressions such as $\frac{4x^3}{12x^2}$ combined with techniques for factoring polynomials.

Keep Calm and be Rational

To introduce the first important feature of rational expressions and equations, we will show an example.

Evaluate $\frac{x}{x-2}$ for $x = 2$

Substitute $x = 2$

$\frac{2}{2-2}$

$= \frac{2}{0}$

Remember that you can't divide by zero, so this means that for the expression $\frac{x}{x-2}$, x cannot be 2 because it will result in an undefined ratio. In general, finding values for a variable that will not result in division by zero is called finding the domain.

Domain of a rational expression or equation

The domain of a rational expression or equation is a collection of the values for the variable that will not result in an undefined mathematical operation such as division by zero. For a = any real number, we can notate the domain in the following way:
x is all real numbers where $x \neq a$

The reason you cannot divide any number *c* by zero $\left(\frac{c}{0} = ?\right)$ is that you would have to find a number that when you multiply it by 0 you would get back $c\,(? \cdot 0 = c)$. There are no numbers that can do this, so we say "division by zero is undefined". In simplifying rational expressions you need to pay attention to what values of the variable(s) in the expression would make the denominator equal zero. These values cannot be included in the domain, so they're called excluded values. Discard them right at the start, before you go any further.

(Note that although the *denominator* cannot be equivalent to 0, the *numerator* can—this is why you only look for excluded values in the denominator of a rational expression.)

For rational expressions, the domain will exclude values for which the value of the denominator is 0. The following two examples illustrate finding the domain of an expression.

Example

Identify the domain of the expression. $\frac{3x+2}{x-4}$
Show Solution

In the next example we will identify the domain of a rational expression that contains polynomials in the numerator and denominator.

Example

Identify the domain of the expression. $\frac{x+7}{x^2+8x-9}$
Show Solution

As with many other mathematical expressions and equations, it can be very helpful to simplify rational expressions. We simplified rational expressions with monomial terms in the exponents module. Here we will combine what we know about factoring polynomials with factoring rational expressions that have monomial terms. The goal is to be able to simplify an expression such as this:

$\frac{x^2+x-2}{x-1}$

Before we offer more examples, we will show you a technique for simplifying rational expressions that will make things a bit easier. The idea is that a number or variable divided by itself is equal to one, so we can factor a rational expression and identify common factors between the numerator and denominator.

$\frac{5x^2}{10x} = \frac{5\cdot x\cdot x}{5\cdot 2\cdot x}$

The common factors between the numerator and denominator are 5 and x, so we can "cancel" them to show that $\frac{5}{5} = 1 \text{ and } \frac{x}{x} = 1.$

$$\frac{5\cdot x\cdot x}{5\cdot 2\cdot x} = \frac{\cancel{5}\cdot \cancel{x}\cdot x}{\cancel{5}\cdot 2\cdot \cancel{x}} = \frac{x}{2}$$

The next example provides a reminder of how to simplify a monomial with variables and exponents. We will then use this idea to simplify a rational expression and define it's domain.

Example

Simplify $\frac{5x^2}{25x}$.
Show Solution

We can summarize the process as follows: Factor the numerator, factor the denominator, identify factors that are common to the numerator and denominator, cancel them to represent division, and simplify.

When simplifying rational expressions, it is a good habit to always consider the domain first. This will come in handy when you begin *solving* rational equations a bit later on. When finding the domain of an expression, you always start with the original expression because variable terms may be factored out as part of the simplification process.

In the examples that follow, the numerator and the denominator are polynomials with more than one term, but the same principles of simplifying will once again apply. Factor the numerator and denominator to simplify the rational expression.

Example

Simplify and state the domain for the expression. $\frac{x+3}{x^2+12x+27}$

Show Solution

Example

Simplify and state the domain for the expression. $\frac{x^2+10x+24}{x^3-x^2-20x}$

Show Solution

In the following video we present another example of finding the domain of a rational expression.

Visit this page in your course online to check your understanding.

Steps for Simplifying a Rational Expression

To simplify a rational expression, follow these steps:

- Determine the domain. The excluded values are those values for the variable that result in the expression having a denominator of 0.
- Factor the numerator and denominator.
- Find common factors for the numerator and denominator and simplify.

Summary

An additional consideration for rational expressions is to determine what values are excluded from the domain. Since division by 0 is undefined, any values of the variables that result in a denominator of 0 must be excluded. Excluded values must be identified in the original equation, not from its factored form.Rational expressions are fractions containing polynomials. They can be simplified much like numeric fractions. To simplify a rational expression, first determine common factors of the numerator and denominator, and then remove them by rewriting them as expressions equal to 1.

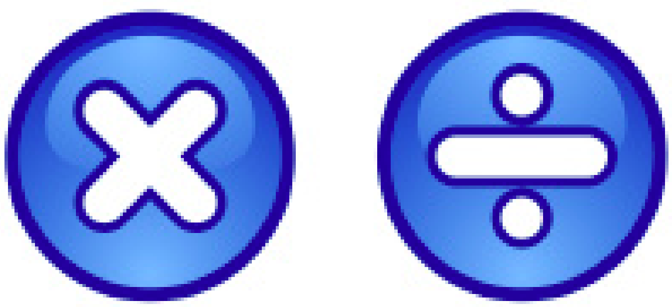
OPERATIONS ON RATIONAL EXPRESSIONS

Learning Objectives

- Multiply and divide rational expressions
- Add and subtract rational expressions
  - Add and subtract rational expressions with like denominators

- Add and subtract rational expressions with unlike denominators using a greatest common denominator
- Add and subtract rational expressions that share no common factors
- Add and subtract more than two rational expressions

Just as you can multiply and divide fractions, you can multiply and divide **rational expressions**. In fact, you use the same processes for multiplying and dividing rational expressions as you use for multiplying and dividing numeric fractions. The process is the same even though the expressions look different!



Multiply and Divide

Multiply Rational Expressions

Remember that there are two ways to multiply numeric fractions.

One way is to multiply the numerators and the denominators and then simplify the product, as shown here.

$$\frac{4}{5}\cdot\frac{9}{8}=\frac{36}{40}=\frac{3\cdot3\cdot2\cdot2}{5\cdot2\cdot2\cdot2}=\frac{3\cdot3\cdot\cancel{2}\cdot\cancel{2}}{5\cdot\cancel{2}\cdot\cancel{2}\cdot2}=\frac{3\cdot3}{5\cdot2}\cdot1=\frac{9}{10}$$

A second way is to factor and simplify the fractions *before* performing the multiplication.

$$\frac{4}{5}\cdot\frac{9}{8}=\frac{2\cdot2}{5}\cdot\frac{3\cdot3}{2\cdot2\cdot2}=\frac{\cancel{2}\cdot\cancel{2}\cdot3\cdot3}{\cancel{2}\cdot5\cdot\cancel{2}\cdot2}=1\cdot\frac{3\cdot3}{5\cdot2}=\frac{9}{10}$$

Notice that both methods result in the same product. In some cases you may find it easier to multiply and then simplify, while in others it may make more sense to simplify fractions before multiplying.

The same two approaches can be applied to rational expressions. In the following examples, both techniques are shown. First, let's multiply and then simplify.

Example

Multiply. $\frac{5a^2}{14}\cdot\frac{7}{10a^3}$

State the product in simplest form.
Show Solution

Okay, that worked. But this time let's simplify first, then multiply. When using this method, it helps to look for the **greatest common factor**. You can factor out *any* common factors, but finding the greatest one will take fewer steps.

Example

Multiply. $\frac{5a^2}{14} \cdot \frac{7}{10a^3}$

State the product in simplest form.
Show Solution

Both methods produced the same answer.

Also, remember that when working with rational expressions, you should get into the habit of identifying any values for the variables that would result in division by 0. These excluded values must be eliminated from the domain, the set of all possible values of the variable. In the example above, $\dfrac{5a^2}{14} \cdot \dfrac{7}{10a^3}$, the domain is all real numbers where *a* is not equal to 0. When $a = 0$, the denominator of the fraction $\frac{7}{10a^3}$ equals 0, which will make the fraction undefined.

Some rational expressions contain quadratic expressions and other multi-term polynomials. To multiply these rational expressions, the best approach is to first factor the polynomials and then look for common factors. (Multiplying the terms before factoring will often create complicated polynomials…and then you will have to factor these polynomials anyway! For this reason, it is easier to factor, simplify, and then multiply.) Just take it step by step, like in the examples below.

Example

Multiply. $\dfrac{a^2 - a - 2}{5a} \cdot \dfrac{10a}{a+1}, \quad a \neq -1, 0$

State the product in simplest form.
Show Solution

Example

Multiply. $\frac{a^2+4a+4}{2a^2-a-10} \cdot \frac{a+5}{a^2+2a}, \ a \neq -2, 0, \frac{5}{2}$

State the product in simplest form.
Show Solution

Note that in the answer above, you cannot simplify the rational expression any further. It may be tempting to express the 5's in the numerator and denominator as the fraction $\frac{5}{5}$, but these 5's are terms because they are being added or subtracted. Remember that only common factors, not terms, can be regrouped to form factors of 1!

In the following video we present another example of multiplying rational expressions.

Watch this video online: https://youtu.be/Hj6gF1SNttk

Divide Rational Expressions

You've seen that you multiply rational expressions as you multiply numeric fractions. It should come as no surprise that you also divide rational expressions the same way you divide numeric fractions. Specifically, to divide rational expressions, keep the first rational expression, change the division sign to multiplication, and then take the reciprocal of the second rational expression.

Let's begin by recalling division of numerical fractions.

$\frac{2}{3} \div \frac{5}{9} = \frac{2}{3} \cdot \frac{9}{5} = \frac{18}{15} = \frac{6}{5}$

Use the same process to divide rational expressions. You can think of division as multiplication by the reciprocal, and then use what you know about multiplication to simplify.

Reciprocal Architecture

You do still need to think about the domain, specifically the variable values that would make either denominator equal zero. But there's a new consideration this time—because you divide by multiplying by the reciprocal of one of the rational expressions, you also need to find the values that would make the *numerator* of that expression equal zero. Have a look.

Example

Identify the domain of the expression. $\frac{5x^2}{9} \div \frac{15x^3}{27}$
Show Solution

Knowing how to find the domain may seem unimportant here, but it will help you when you learn how to solve rational equations. To divide, multiply by the reciprocal.

Example

Divide. $\frac{5x^2}{9} \div \frac{15x^3}{27}$

State the quotient in simplest form.
Show Solution

Example

Divide. $\frac{3x^2}{x+2} \div \frac{6x^4}{(x^2+5x+6)}$

State the quotient in simplest form, and express the domain of the expression.
Show Solution

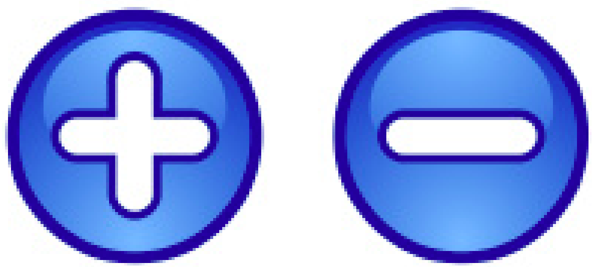
Notice that once you rewrite the division as multiplication by a reciprocal, you follow the same process you used to multiply rational expressions.

In the video that follows, we present another example of dividing rational expressions.

Watch this video online: https://youtu.be/B1tigfgs268

Add and Subtract Rational Expressions

In beginning math, students usually learn how to add and subtract whole numbers before they are taught multiplication and division. However, with fractions and rational expressions, multiplication and division are sometimes taught first because these operations are easier to perform than addition and subtraction. Addition and subtraction of rational expressions are not as easy to perform as multiplication because, as with numeric fractions, the process involves finding common denominators. By working carefully and writing down the steps along the way, you can keep track of all of the numbers and variables and perform the operations accurately.



Add and Subtract

Adding and Subtracting Rational Expressions with Like Denominators

Adding rational expressions with the same denominator is the simplest place to start, so let's begin there.

To add fractions with like denominators, add the numerators and keep the same denominator. Then simplify the sum. You know how to do this with numeric fractions.

$\frac{2}{9}+\frac{4}{9}=\frac{6}{9}$

$\frac{6}{9}=\frac{3\cdot 2}{3\cdot 3}=\frac{3}{3}\cdot\frac{2}{3}=1\cdot\frac{2}{3}=\frac{2}{3}$

Follow the same process to add rational expressions with like denominators. Let's try one.

Example

Add $\dfrac{2x^2}{x+4}+\dfrac{8x}{x+4}$, and define the domain.

State the sum in simplest form.
Show Solution

Caution! Remember to define the domain of a sum or difference *before* simplifying. You may lose important information when you simplify. In the example above, the domain is $x \neq -4$. If we were to have defined the domain after simplifying, we would find that the domain is all real numbers which is incorrect.

To subtract rational expressions with like denominators, follow the same process you use to subtract fractions with like denominators. The process is just like the addition of rational expressions, except that you subtract instead of add.

Example

Subtract $\frac{4x+7}{x+6}-\frac{2x+8}{x+6}$, and define the domain.

State the difference in simplest form.
Show Solution

In the video that follows, we present more examples of adding rational expressions with like denominators. Additionally, we review finding the domain of a rational expression.

Watch this video online: https://youtu.be/BeaHKtxB868

Adding and Subtracting Rational Expressions with Unlike Denominators

What do they have in common?

Before adding and subtracting rational expressions with *unlike* denominators, you need to find a common denominator. Once again, this process is similar to the one used for adding and subtracting numeric fractions with unlike denominators. Remember how to do this?

$$\frac{5}{6}+\frac{8}{10}+\frac{3}{4}$$

Since the denominators are 6, 10, and 4, you want to find the **least common denominator** and express each fraction with this denominator before adding. (BTW, you can add fractions by finding *any* common denominator; it does not have to be the least. You focus on using the *least* because then there is less simplifying to do. But either way works.)

Finding the least common denominator is the same as finding the **least common multiple** of 4, 6, and 10. There are a couple of ways to do this. The first is to list the multiples of each number and determine which multiples they have in common. The least of these numbers will be the least common denominator.

Number	**Multiples**															
4	8	12	16	20	24	28	32	36	40	44	48	52	56	**60**	64	
6	12	18	24	30	36	42	48	54	**60**	66	68					
10	20	30	40	50	**60**	70	80									

The other method is to use **prime factorization**, the process of finding the prime factors of a number. This is how the method works with numbers.

Example

Use prime factorization to find the least common multiple of 6, 10, and 4.
Show Solution

Both methods give the same result, but prime factorization is faster. Your choice!

Now that you have found the least common multiple, you can use that number as the least common denominator of the fractions. Multiply each fraction by the fractional form of 1 that will produce a denominator of 60:

$\frac{5}{6}\cdot\frac{10}{10}=\frac{50}{60}$

$\frac{8}{10}\cdot\frac{6}{6}=\frac{48}{60}$

$\frac{3}{4}\cdot\frac{15}{15}=\frac{45}{60}$

Now that you have like denominators, add the fractions:

$\frac{50}{60}+\frac{48}{60}+\frac{45}{60}=\frac{143}{60}$

In the next example, we show how to find the least common multiple of a rational expression with a monomial in the denominator.

Example

Add $\frac{2n}{15m^2}+\frac{3n}{21m}$, and give the domain.

State the sum in simplest form.
Show Solution

That took a while, but you got through it. Adding rational expressions can be a lengthy process, but taken one step at a time, it can be done.

Now let's try subtracting rational expressions. You'll use the same basic technique of finding the least common denominator and rewriting each rational expression to have that denominator.

Example

Subtract $\frac{2}{t+1}-\frac{t-2}{t^2-t-2}$, define the domain.

State the difference in simplest form.
Show Solution

The video that follows contains an example of adding rational expressions whose denominators are not alike. The denominators of both expressions contain only monomials.

Watch this video online: https://youtu.be/Wk8ZZhE9ZjI

The video that follows contains an example of subtracting rational expressions whose denominators are not alike. The denominators are a trinomial and a binomial.

Watch this video online: https://youtu.be/MMlNtCrkakI

Add rational expressions whose denominators have no common factors

So far all the rational expressions you've added and subtracted have shared some factors. What happens when they don't have factors in common?

No Common Factors

In the next example, we show how to find a common denominator when there are no common factors in the expressions.

Example

Subtract $\frac{3y}{2y-1}-\frac{4}{y-5}$, and give the domain.

State the difference in simplest form.
Show Solution

In the video that follows, we present an example of adding two rational expression whose denominators are binomials with no common factors.

Watch this video online: https://youtu.be/CKGpiTE5vIg

You may need to combine more than two rational expressions. While this may seem pretty straightforward if they all have the same denominator, what happens if they do not?

In the example below, notice how a common denominator is found for three rational expressions. Once that is done, the addition and subtraction of the terms looks the same as earlier, when you were only dealing with two terms.

Example

Simplify $\frac{2x^2}{x^2-4}+\frac{x}{x-2}-\frac{1}{x+2}$, and give the domain.

State the result in simplest form.
Show Solution

In the video that follows we present an example of subtracting 3 rational expressions with unlike denominators. One of the terms being subtracted is a number, so the denominator is 1.

Visit this page in your course online to check your understanding.

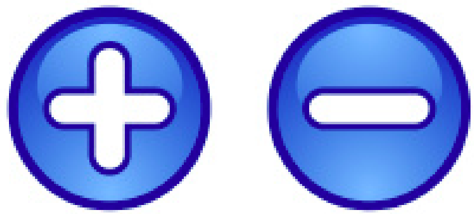
Example

Simplify $\frac{y^2}{3y} - \frac{2}{x} - \frac{15}{9}$, and give the domain.

State the result in simplest form.
Show Solution

In this last video, we present another example of adding and subtracting three rational expressions with unlike denominators.

Watch this video online: https://youtu.be/43xPStLm39A



Add and Subtract

Summary

In the next sections we will introduce some techniques for solving rational equations. The methods shown here will help you when you are solving rational equations. To add and subtract rational expressions that share common factors, you first identify which factors are missing from each expression, and build the LCD with them. To add and subtract rational expressions with no common factors, the LCD will be the product of all the factors of the denominators.

Summary

Rational expressions are multiplied and divided the same way as numeric fractions. To multiply, first find the greatest common factors of the numerator and denominator. Next, regroup the factors to make fractions equivalent to one. Then, multiply any remaining factors. To divide, first rewrite the division as multiplication by the reciprocal of the denominator. The steps are then the same as for multiplication.

When expressing a product or quotient, it is important to state the excluded values. These are all values of a variable that would make a denominator equal zero at any step in the calculations.

SOLVE RATIONAL EQUATIONS

Learning Objectives

- Solve rational equations
    - Solve rational equations by clearing denominators
    - Identify extraneous solutions in a rational equation
    - Solve for a variable in a rational formula
- Applications of rational equations
    - Identify the components of a work equation
    - Solve a work equation
    - Define and write a proportion
    - Solve proportional problems involving scale drawings
    - Define direct variation, and solve problems involving direct variation
    - Define inverse variation and solve problems involving inverse variation
    - Define joint variation and solve problems involving joint variation

Equations that contain rational expressions are called **rational equations**. For example, $\frac{2x+1}{4}=\frac{x}{3}$ is a rational equation. Rational equations can be useful for representing real-life situations and for finding answers to real problems. In particular, they are quite good for describing a variety of proportional relationships.

One of the most straightforward ways to solve a rational equation is to eliminate denominators with the common denominator, then use properties of equality to isolate the variable. This method is often used to solve linear equations that involve fractions as in the following example:

Solve $\frac{1}{2}x-3=2-\frac{3}{4}x$ by clearing the fractions in the equation first.

Multiply both sides of the equation by 4, the common denominator of the fractional coefficients.

$$\frac{1}{2}x-3=2-\frac{3}{4}x$$
$$4\left(\frac{1}{2}x-3\right)=4\left(2-\frac{3}{4}x\right)$$

$$4\left(\frac{1}{2}x\right)-4(3)=4(2)+4\left(-\frac{3}{4}x\right)$$
$$2x-12=8-3x$$
$$\underline{+3x}\qquad\underline{+3x}$$
$$5x-12=8$$
$$\underline{+12}\quad\underline{+12}$$
$$5x=20$$
$$x=4$$

We could have found a common denominator and worked with fractions, but that often leads to more mistakes. We can apply the same idea to solving rational equations. The difference between a linear equation and a rational equation is that rational equations can have polynomials in the numerator and denominator of the fractions. This means that clearing the denominator may sometimes mean multiplying the whole rational equation by a polynomial. In the next example, we will clear the denominators of a rational equation with a terms that has a polynomial in the numerator.

Example

Solve the equation $\frac{x+5}{8}=\frac{7}{4}$.
Show Solution

In the next example, we show how to solve a rational equation with a binomial in the denominator of one term. We will use the common denominator to eliminate the denominators from both fractions. Note that the LCD is the product of both denominators because they don’t share any common factors.

Example

Solve the equation $\frac{8}{x+1} = \frac{4}{3}$.
Show Solution

You could also solve this problem by multiplying each term in the equation by 3 to eliminate the fractions altogether. Here is how it would look.

Example

Solve the equation $\frac{x}{3} + 1 = \frac{4}{3}$.
Show Solution

In the video that follows we present two ways to solve rational equations with both integer and variable denominators.

Watch this video online: https://youtu.be/R9y2D9VFw0I

Excluded Values and Extraneous Solutions

Some rational expressions have a variable in the denominator. When this is the case, there is an extra step in solving them. Since division by 0 is undefined, you must exclude values of the variable that would result in a denominator of 0. These values are called **excluded values**. Let’s look at an example.

Example

Solve the equation $\frac{2x-5}{x-5} = \frac{15}{x-5}$.
Show Solution

In the following video we present an example of solving a rational equation with variables in the denominator.

Watch this video online: https://youtu.be/gGA-dF_aQQQ

You’ve seen that there is more than one way to solve rational equations. Because both of these techniques manipulate and rewrite terms, sometimes they can produce solutions that don’t work in the original form of the equation. These types of answers are called **extraneous solutions**. That’s why it is always important to check all solutions in the original equations—you may find that they yield untrue statements or produce undefined expressions.

Example

Solve the equation $\frac{16}{m+4} = \frac{m^2}{m+4}$.
Show Solution

Rational formulas

Rational formulas can be useful tools for representing real-life situations and for finding answers to real problems. Equations representing direct, inverse, and joint variation are examples of rational formulas that can model many real-life situations. As you will see, if you can find a formula, you can usually make sense of a situation.

When solving problems using rational formulas, it is often helpful to first solve the formula for the specified variable. For example, work problems ask you to calculate how long it will take different people working at different speeds to finish a task. The algebraic models of such situations often involve rational equations derived from the work formula, $W = rt$. The amount of work done (*W*) is the product of the rate of work (*r*) and the time spent working (*t*). Using algebra, you can write the work formula 3 ways:

$W = rt$

Find the time (t): $t = \frac{W}{r}$ *(divide both sides by r)*

Find the rate (r): $r = \frac{W}{t}$ *(divide both sides by t)*

Example

The formula for finding the density of an object is $D = \frac{m}{v}$, where *D* is the density, *m* is the mass of the object and *v* is the volume of the object. Rearrange the formula to solve for the mass (*m*) and then for the volume (*v*).
Show Solution

Now let's look at an example using the formula for the volume of a cylinder.

Example

The formula for finding the volume of a cylinder is $V = \pi r^2 h$, where *V* is the volume, *r* is the radius and *h* is the height of the cylinder. Rearrange the formula to solve for the height (*h*).
Show Solution

In the following video we give another example of solving for a variable in a formula, or as they are also called, a literal equation.

Watch this video online: https://youtu.be/ecEUUbRLDQs

Applications of Rational Equations

Rational equations can be used to solve a variety of problems that involve rates, times and work. Using rational expressions and equations can help you answer questions about how to combine workers or machines to complete a job on schedule.

A Good Day's Work

Work

A “work problem” is an example of a real life situation that can be modeled and solved using a rational equation. Work problems often ask you to calculate how long it will take different people working at different speeds to finish a task. The algebraic models of such situations often involve rational equations derived from the work formula, $W = rt$. (Notice that the work formula is very similar to the relationship between distance, rate, and time, or $d = rt$.) The amount of work done (*W*) is the product of the rate of work (*r*) and the time spent working (*t*). The work formula has 3 versions.

$$W = rt$$

$$t = \frac{W}{r}$$

$$r = \frac{W}{t}$$

Some work problems include multiple machines or people working on a project together for the same amount of time but at different rates. In that case, you can add their individual work rates together to get a total work rate. Let’s look at an example.

Example

Myra takes 2 hours to plant 50 flower bulbs. Francis takes 3 hours to plant 45 flower bulbs. Working together, how long should it take them to plant 150 bulbs?
Show Solution

Watch this video online: https://youtu.be/SzSasnDF7Ms

Other work problems go the other way. You can calculate how long it will take one person to do a job alone when you know how long it takes people working together to complete the job.

Example

Joe and John are planning to paint a house together. John thinks that if he worked alone, it would take him 3 times as long as it would take Joe to paint the entire house. Working together, they can complete the job in 24 hours. How long would it take each of them, working alone, to complete the job?
Show Solution

In the video that follows, we show another example of finding one person’s work rate given a combined work rate.

Watch this video online: https://youtu.be/kbRSYb8UYqU

As shown above, many work problems can be represented by the equation $\frac{t}{a} + \frac{t}{b} = 1$, where *t* is the time to do the job together, *a* is the time it takes person A to do the job, and *b* is the time it takes person B to do the job. The 1 refers to the total work done—in this case, the work was to paint 1 house.

The key idea here is to figure out each worker’s individual rate of work. Then, once those rates are identified, add them together, multiply by the time *t*, set it equal to the amount of work done, and solve the rational equation.

We present another example of two people painting at different rates in the following video.

Watch this video online: https://youtu.be/SzSasnDF7Ms

Proportions

Matryoshka, or nesting dolls.

A proportion is a statement that two ratios are equal to each other. There are many things that can be represented with ratios, including the actual distance on the earth that is represented on a map. In fact, you probably use proportional reasoning on a regular basis and don't realize it. For example, say you have volunteered to provide drinks for a community event. You are asked to bring enough drinks for 35-40 people. At the store you see that drinks come in packages of 12. You multiply 12 by 3 and get 36 – this may not be enough if 40 people show up, so you decide to buy 4 packages of drinks just to be sure.

This process can also be expressed as a proportional equation and solved using mathematical principles. First, we can express the number of drinks in a package as a ratio:

$$\frac{12\text{ drinks}}{1\text{ package}}$$

Then we express the number of people who we are buying drinks for as a ratio with the unknown number of packages we need. We will use the maximum so we have enough.

$$\frac{40\text{ people}}{x\text{ packages}}$$

We can find out how many packages to purchase by setting the expressions equal to each other:

$$\frac{12\text{ drinks}}{1\text{ package}}=\frac{40\text{ people}}{x\text{ packages}}$$

To solve for x, we can use techniques for solving linear equations, or we can cross multiply as a shortcut.

$$\frac{12\text{ drinks}}{1\text{ package}}=\frac{40\text{ people}}{x\text{ packages}}$$

$$x\cdot\frac{12\text{ drinks}}{1\text{ package}}=\frac{40\text{ people}}{x\text{ packages}}\cdot x$$

$$12x=40$$

$$x=\frac{40}{12}=\frac{10}{3}=3.33$$

We can round up to 4 since it doesn't make sense to by 0.33 of a package of drinks. Of course, you don't write out your thinking this way when you are in the grocery store, but doing so helps you to be able to apply the concepts to less obvious problems. In the following example we will show how to use a proportion to find the number of people on teh planet who don't have access to a toilet.

Example

As of March, 2016 the world's population was estimated at 7.4 billion. ((Note: "Current World Population." World Population Clock: 7.4 Billion People (2016). Accessed June 21, 2016. http://www.worldometers.info/world-population/. "Current World Population." World Population Clock: 7.4 Billion People (2016). Accessed June 21, 2016. http://www.worldometers.info/world-population/. "Current World Population." World Population Clock: 7.4

Billion People (2016). Accessed June 21, 2016. http://www.worldometers.info/world-population/.)). According to water.org, 1 out of every 3 people on the planet lives without access to a toilet. Find the number of people on the planet that do not have access to a toilet.
Show Solution

In the next example, we will use the length of a person's femur to estimate their height. This process is used in forensic science and anthropology, and has been found in many scientific studies to be a very good estimate.

Example

It has been shown that a person's height is proportional to the length of their femur ((Note: Obialor, Ambrose, Churchill Ihentuge, and Frank Akapuaka. "Determination of Height Using Femur Length in Adult Population of Oguta Local Government Area of Imo State Nigeria." Federation of American Societies for Experimental Biology, April 2015. Accessed June 22, 2016. http://www.fasebj.org/content/29/1_Supplement/LB19.short.)). Given that a person who is 71 inches tall has a femur length of 17.75 inches, how tall is someone with a femur length of 16 inches?
Show Solution

Another way to describe the ratio of femur length to height that we found in the last example is to say there's a 1:4 ratio between femur length and height, or 1 to 4.

Ratios are also used in scale drawings. Scale drawings are enlarged or reduced drawings of objects, buildings, roads, and maps. Maps are smaller than what they represent and a drawing of dendritic cells in your brain is most likely larger than what it represents. The scale of the drawing is a ratio that represents a comparison of the length of the actual object and it's representation in the drawing. The image below shows a map of the us with a scale of 1 inch representing 557 miles. We could write the scale factor as a fraction $\frac{1}{557}$ or as we did with the femur-height relationship, 1:557.

Map with scale factor

In the next example we will use the scale factor given in the image above to find the distance between Seattle Washington and San Jose California.

Example

Given a scale factor of 1:557 on a map of the US, if the distance from Seattle, WA to San Jose, CA is 1.5 inches on the map, define a proportion to find the actual distance between them.

Show Solution

In the next example, we will find a scale factor given the length between two cities on a map, and their actual distance from each other.

Example

Two cities are 2.5 inches apart on a map. Their actual distance from each other is 325 miles. Write a proportion to represent and solve for the scale factor for one inch of the map.
Show Solution

The video that follows show another example of finding an actual distance using the scale factor from a map.

Watch this video online: https://youtu.be/id3sp4wvmVg

In the video that follows, we present an example of using proportions to obtain the correct amount of medication for a patient, as well as finding a desired mixture of coffees.

Watch this video online: https://youtu.be/yGid1a_x38g

Variation

So many cars, so many tires.

Direct Variation

Variation equations are examples of rational formulas and are used to describe the relationship between variables. For example, imagine a parking lot filled with cars. The total number of tires in the parking lot is dependent on the total number of cars. Algebraically, you can represent this relationship with an equation.

$\text{number of tires} = 4 \cdot \text{number of cars}$

The number 4 tells you the rate at which cars and tires are related. You call the rate the **constant of variation**. It's a constant because this number does not change. Because the number of cars and the number of tires are linked by a constant, changes in the number of cars cause the number of tires to change in a proportional, steady way. This is an example of **direct variation**, where the number of tires varies directly with the number of cars.

You can use the car and tire equation as the basis for writing a general algebraic equation that will work for all examples of direct variation. In the example, the number of tires is the output, 4 is the constant, and the number of cars is the input. Let's enter those generic terms into the equation. You get $y=kx$. That's the formula for all direct variation equations.

$\text{number of tires} = 4 \cdot \text{number of cars}$
$\text{output} = \text{constant} \cdot \text{input}$

> Example
>
> Solve for *k*, the constant of variation, in a direct variation problem where $y=300$ and $x=10$.
> Show Solution

In the video that follows, we present an example of solving a direct variation equation.

Watch this video online: https://youtu.be/DLPKiMD_ZZw

Inverse Variation

Another kind of variation is called **inverse variation**. In these equations, the output equals a constant divided by the input variable that is changing. In symbolic form, this is the equation $y=\frac{k}{x}$.

One example of an inverse variation is the speed required to travel between two cities in a given amount of time.

Let's say you need to drive from Boston to Chicago, which is about 1,000 miles. The more time you have, the slower you can go. If you want to get there in 20 hours, you need to go 50 miles per hour (assuming you don't stop driving!), because $\frac{1{,}000}{20}=50$. But if you can take 40 hours to get there, you only have to average 25 miles per hour, since $\frac{1{,}000}{40}=25$.

The equation for figuring out how fast to travel from the amount of time you have is $speed=\frac{miles}{time}$. This equation should remind you of the distance formula $d=rt$. If you solve $d=rt$ for *r*, you get $r=\frac{d}{t}$, or $speed=\frac{miles}{time}$.

In the case of the Boston to Chicago trip, you can write $s=\frac{1{,}000}{t}$. Notice that this is the same form as the inverse variation function formula, $y=\frac{k}{x}$.

> Example
>
> Solve for *k*, the constant of variation, in an inverse variation problem where $x=5$ and $y=25$.
> Show Solution

In the next example, we will find the water temperature in the ocean at a depth of 500 meters. Water temperature is inversely proportional to depth in the ocean.

Water temperature in the ocean varies inversely with depth.

Example

The water temperature in the ocean varies inversely with the depth of the water. The deeper a person dives, the colder the water becomes. At a depth of 1,000 meters, the water temperature is 5° Celsius. What is the water temperature at a depth of 500 meters?
Show Solution

In the video that follows, we present an example of inverse variation.

Watch this video online: https://youtu.be/y9wqI6Uo6_M

Joint Variation

A third type of variation is called **joint variation**. Joint variation is the same as direct variation except there are two or more quantities. For example, the area of a rectangle can be found using the formula $A = lw$, where *l* is the length of the rectangle and *w* is the width of the rectangle. If you change the width of the rectangle, then the area changes and similarly if you change the length of the rectangle then the area will also change. You can say that the area of the rectangle "varies jointly with the length and the width of the rectangle."

The formula for the volume of a cylinder, $V = \pi r^2 h$ is another example of joint variation. The volume of the cylinder varies jointly with the square of the radius and the height of the cylinder. The constant of variation is π.

Example

The area of a triangle varies jointly with the lengths of its base and height. If the area of a triangle is 30 inches ^{2} when the base is 10 inches and the height is 6 inches, find the variation constant and the area of a triangle whose base is 15 inches and height is 20 inches.
Show Solution

Finding *k* to be $\frac{1}{2}$ shouldn't be surprising. You know that the area of a triangle is one-half base times height, $A = \frac{1}{2}bh$. The $\frac{1}{2}$ in this formula is exactly the same $\frac{1}{2}$ that you calculated in this example!

In the following video, we show an example of finding the constant of variation for a jointly varying relation.

Watch this video online: https://youtu.be/JREPATMScbM

Direct, Joint, and Inverse Variation

k is the constant of variation. In all cases, $k \neq 0$.

- Direct variation: $y = kx$
- Inverse variation: $y = \frac{k}{x}$
- Joint variation: $y = kxz$

Summary

Rational formulas can be used to solve a variety of problems that involve rates, times, and work. Direct, inverse, and joint variation equations are examples of rational formulas. In direct variation, the variables have a direct relationship—as one quantity increases, the other quantity will also increase. As one quantity decreases, the other quantity decreases. In inverse variation, the variables have an inverse relationship—as one variable increases, the other variable decreases, and vice versa. Joint variation is the same as direct variation except there are two or more variables.

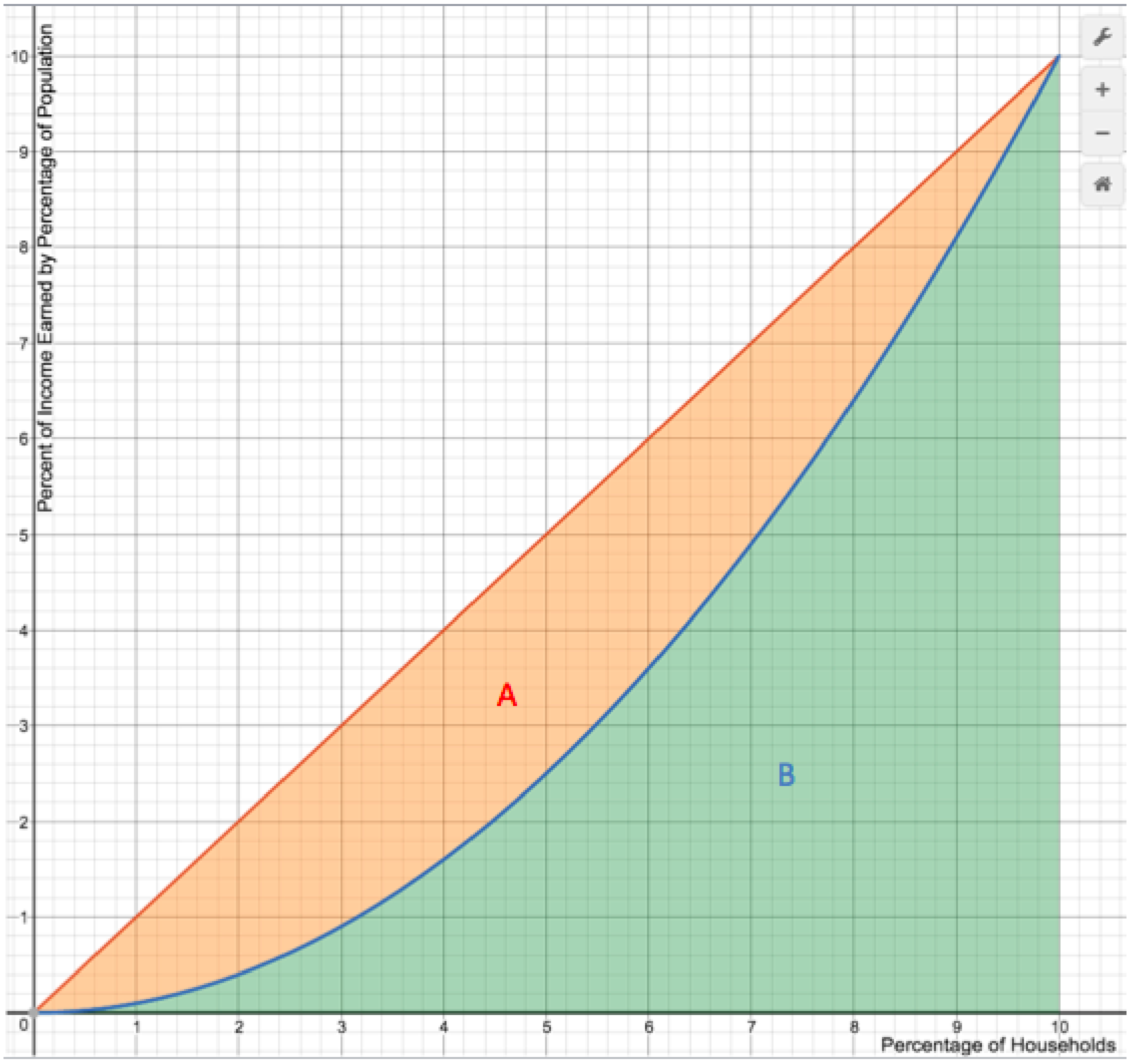
Summary

You can solve rational equations by finding a common denominator. By rewriting the equation so that all terms have the common denominator, you can solve for the variable using just the numerators. Or, you can multiply both sides of the equation by the least common multiple of the denominators so that all terms become polynomials instead of rational expressions.

An important step in solving rational equations is to reject any extraneous solutions from the final answer. Extraneous solutions are solutions that don't satisfy the original form of the equation because they produce untrue statements or are excluded values that make a denominator equal to 0.

CONCLUSION

In the beginning of this module, we presented the Gini coefficient, also called the Gini Index. The Gini coefficient is the ratio of how far an economy departs from equal distribution of wealth to a perfectly equal distribution of wealth, as shown in the graph below.



Gini Index

It's great to have a tool to describe an issue like income inequality that ripples through social, economic, and political realms so pervasively. But, some may argue, the real power of the tool is how it is used to instruct those who make decisions that affect the ever widening rift between the top 1% and the rest of us. According to Andrew Fieldhouse, blogger for the Economic Policy Institute,

> "Changes in tax and transfer policies are one of the more easily quantifiable contributors to income inequality, say compared with policies (or lack thereof) related to labor protections, collective bargaining, minimum wage erosion and trade." ((Note: "How Much Can Tax Policy Curb Income Inequality Growth? Maybe a Lot." Economic Policy Institute. Accessed June 29, 2016. http://www.epi.org/blog/tax-policy-curb-income-inequality-growth/.))

He states that market-based income inequality, as measured by the "Gini" index, rose 23.1 percent between 1979 and 2007. Fieldhouse argues that by raising top tax rates, policy makers could help to slow the growth of income inequality without hampering growth of the economy. He also claims that tax policy lacks what could be valuable guidance from economic indicators such as the Gini Index. Using mathematics to inform policies also has the added benefit of being politically neutral. Math does not care what side of the aisle the policy may be coming from. When used wisely, mathematical facts offer an objective picture.

MODULE 8: ROOTS AND RATIONAL EXPONENTS

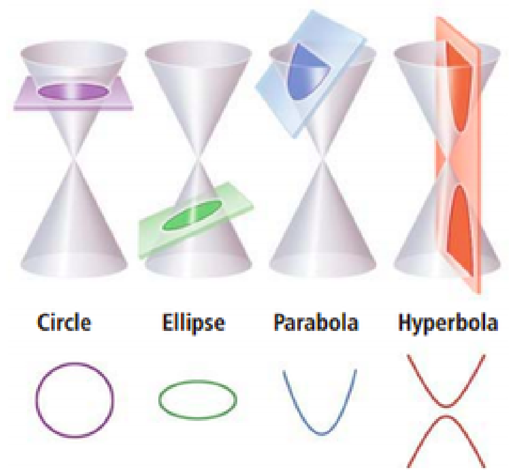
INTRODUCTION

Why learn about roots and rational exponents?

One of the most well known uses of roots and rational exponents is found in quadratic equations. Quadratic comes from *quadratus,* the Latin word for square. When we factor a quadratic equation, we are completing the square.

Humans have been working with quadratic equations for over two thousand years in many areas around the world, including India, Egypt, China, and Greece. The circle, ellipse, hyperbola, and parabola are all examples of quadratic curves, which have been studied since the ancient Greeks. Of these, however, only the circle was relevant to the real world until the beginning of the Renaissance in the 16th century.

The Renaissance period was a rediscovery of Greek philosophy which, as you probably realize by now, includes a significant amount of mathematical principles. In addition to the rediscovery of old ideas, Renaissance thinkers were discovering new facts about the world around them, some of which challenged the ideas of the ancient Greeks.



Circle, ellipse, parabola and hyperbola.

For example, in 1543 Copernicus published his theory that the Earth rotated around the Sun, which was the exact opposite of what most people in Western culture believed at the time. Copernicus thought that the orbit of the Earth was a circle, which was regarded as the most perfect possible curve because of its symmetry.

In 1609 Kepler published his first two laws of planetary motion. Building off of Tycho Brahe's work, Kepler discovered that instead of circles planets move around the sun in *ellipses.* Other objects, Kepler found, may also follow a *parabola* or *hyberbola* which, along with ellipses, belong to a group of curves known as conic sections.

The invention of the telescope in the 17th century allowed us to learn even more about the planets. Using a telescope he created, Galileo was able to observe the moon, discover Jupiter's four satellites, verify the phases of Venus, and prove the validity of Copernicus' theory of heliocentrism. His telescope used lenses, the shape of which was formed by two intersecting hyperbolae. Newton's reflecting telescope includes a mirror for which each cross section takes the shape of a parabola. Even the Hubble Space Telescope uses paraboloidal mirrors. As unbelievable as it may seem, our world of modern communications, from Facebook to GPS to planetary exploration, is made possible thanks to quadratic equations!

Learning Outcomes

Roots and Rational Exponents

- Find square roots
- Simplify square roots with variables

- Simplify cube roots
- Use rational exponents

Operations With Radicals

- Multiply and divide radical expressions
- Add and subtract radicals
- Rationalize denominators

Radical and Quadratic Equations

- Solve radical equations
- Use square roots and completing the square
- Use the quadratic formula

IDENTIFY AND SIMPLIFY ROOTS

Learning Objectives

- Square Roots
  - Use square root notation to write principal square roots
  - Simplify principal square roots using factorization
- Cube Roots
  - Use cube root notation to write cube roots
  - Simplify cube roots using factorization
- Simplify Square Roots
  - Simplify square roots with variables
  - Determine when a simplified root needs an absolute value
- Rational Exponents
  - Convert between radical and exponent notation
  - Use the laws of exponents to simplify expressions with rational exponents
  - Use rational exponents to simplify radical expressions

We know how to square a number:

$5^2 = 25$ and $(-5)^2 = 25$

Taking a square root is the opposite of squaring so we can make these statements:

- 5 is the nonngeative square root of 25
- -5 is the negative square root of 25

Find the square roots of the following numbers:

1. 36
2. 81
3. -49
4. 0

1. We want to find a number whose square is 36. $6^2 = 36$ therefore, the nonnegative square root of 36 is 6 and the negative square root of 36 is -6
2. We want to find a number whose square is 81. $9^2 = 81$ therefore, the nonnegative square root of 81 is 9 and the negative square root of 81 is -9
3. We want to find a number whose square is -49. When you square a real number, the result is always positive. Stop and think about that for a second. A negative number times itself is positive, and a positive number times itself is positive. Therefore, -49 does not have square roots, there are no real number solutions to this question.
4. We want to find a number whose square is 0. $0^2 = 0$ therefore, the nonnegative square root of 0 is 0. We do not assign 0 a sign, so it has only one square root, and that is 0.

The notation that we use to express a square root for any real number, a, is as follows:

Writing a Square Root

The symbol for the square root is called a **radical symbol.** For a real number, *a* the square root of *a* is written as $\sqrt{a}$
The number that is written under the radical symbol is called the **radicand**.
By definition, the square root symbol, $\sqrt{\ }$ always means to find the nonnegative root, called the **principal root**.
$\sqrt{-a}$ is not defined, therefore $\sqrt{a}$ is defined for $a > 0$

Let's do an example similar to the example from above, this time using square root notation. Note that using the square root notation means that you are only finding the principal root – the nonnegative root.

Example

Simplify the following square roots:

1. $\sqrt{16}$
2. $\sqrt{9}$
3. $\sqrt{-9}$
4. $\sqrt{5^2}$

Show Solution

The last problem in the previous example shows us an important relationship between squares and square roots, and we can summarize it as follows:

The square root of a square

For a nonnegative real number, a, $\sqrt{a^2} = a$

In the video that follows, we simplify more square roots using the fact that $\sqrt{a^2} = a$ means finding the principal square root.

Watch this video online: https://youtu.be/B3riJsl7uZM

What if you are working with a number whose square you do not know right away? We can use factoring and the product rule for square roots to find square roots such as $\sqrt{144}$, or $\sqrt{225}$.

The Product Rule for Square Roots

Given that a and b are nonnegative real numbers, $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$

In the examples that follow we will bring together these ideas to simplify square roots of numbers that are not obvious at first glance:

- square root of a square,
- the product rule for square roots
- factoring

Example

Simplify $\sqrt{144}$
Show Solution

Example

Simplify $\sqrt{225}$
Show Solution

Caution! The square root of a product rule applies when you have multiplication ONLY under the square root. You cannot apply the rule to sums:

$\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$

Prove this to yourself with some real numbers: let a = 64 and b = 36, then use the order of operations to simplify each expression.

$$\sqrt{64+36}=\sqrt{100}=10$$

$$\sqrt{64}+\sqrt{36}=8+6=14$$

$$10 \neq 14$$

So far, you have seen examples that are perfect squares. That is, each is a number whose square root is an integer. But many radical expressions are not perfect squares. Some of these radicals can still be simplified by finding perfect square factors. The example below illustrates how to factor the radicand, looking for pairs of factors that can be expressed as a square.

Example

Simplify. $\sqrt{63}$
Show Solution

The final answer $3\sqrt{7}$ may look a bit odd, but it is in simplified form. You can read this as “three radical seven” or “three times the square root of seven.”

Shortcut This Way

In the next example, we take a bit of a shortcut by making use of the common squares we know, instead of using prime factors. It helps to have the squares of the numbers between 0 and 10 fresh in your mind to make simplifying radicals faster.

- $0^2 = 0$
- $2^2 = 4$
- $3^2 = 9$
- $4^2 = 16$
- $5^2 = 25$
- $6^2 = 36$
- $7^2 = 49$
- $8^2 = 64$
- $9^2 = 81$
- $10^2 = 100$

Example

Simplify. $\sqrt{2,000}$
Show Solution

In this last video, we show examples of simplifying radicals that are not perfect squares.

Watch this video online: https://youtu.be/oRd7aBCsmfU

Cube Roots

Rubik's Cune

While square roots are probably the most common radical, you can also find the third root, the fifth root, the 10th root, or really any other *n*th root of a number. Just as the square root is a number that, when squared, gives the radicand, the **cube root** is a number that, when cubed, gives the radicand.

Find the cube roots of the following numbers:

1. 27
2. 8
3. -8
4. 0

1. We want to find a number whose cube is 27. $3 \cdot 9 = 27$ and $9 = 3^2$, so $3/cdot3/cdot3 = 3^3 = 27$
2. We want to find a number whose cube is 8. $2 \cdot 2 \cdot 2 = 8$ the cube root of 8 is 2.
3. We want to find a number whose cube is -8. We know 2 is the cube root of 8, so maybe we can try -2. $-2 \cdot -2 \cdot -2 = -8$, so the cube root of -8 is -2. This is different from square roots because multiplying three negative numbers together results in a negative number.
4. We want to find a number whose cube is 0. $0 \cdot 0 \cdot 0$, no matter how many times you multiply 0 by itself, you will always get 0.

The cube root of a number is written with a small number 3, called the **index**, just outside and above the radical symbol. It looks like $\sqrt[3]{}$. This little 3 distinguishes cube roots from square roots which are written without a small number outside and above the radical symbol.

Caution! Be careful to distinguish between $\sqrt[3]{x}$, the cube root of *x*, and $3\sqrt{x}$, three *times* the *square* root of *x*. They may look similar at first, but they lead you to much different expressions!

We can also use factoring to simplify cube roots such as $\sqrt[3]{125}$. You can read this as "the third root of 125" or "the cube root of 125." To simplify this expression, look for a number that, when multiplied by itself two times (for a total of three identical factors), equals 125. Let's factor 125 and find that number.

Example

Simplify. $\sqrt[3]{125}$
Show Solution

The prime factors of 125 are $5 \cdot 5 \cdot 5$, which can be rewritten as 5^3. The cube root of a cubed number is the number itself, so $\sqrt[3]{5^3} = 5$. You have found the cube root, the three identical factors that when multiplied together give 125. 125 is known as a **perfect cube** because its cube root is an integer.

Here's an example of how to simplify a radical that is not a perfect cube.

Example

Simplify. $\sqrt[3]{32m^5}$
Show Solution

In the example below, we use the following idea:

$$\sqrt[3]{(-1)^3} = -1$$

to simplify the radical. You do not have to do this, but it may help you recognize cubes more easily when they are nonnegative.

Example

Simplify. $\sqrt[3]{-27x^4y^3}$
Show Solution

In the video that follows, we show more examples if simplifying cube roots.

Watch this video online: https://youtu.be/9Nh-Ggd2VJo

You could check your answer by performing the inverse operation. If you are right, when you cube $-3xy\sqrt[3]{x}$ you should get $-27x^4y^3$.

$$\left(-3xy\sqrt[3]{x}\right)\left(-3xy\sqrt[3]{x}\right)\left(-3xy\sqrt[3]{x}\right)$$
$$-3\cdot-3\cdot-3\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y\cdot\sqrt[3]{x}\cdot\sqrt[3]{x}\cdot\sqrt[3]{x}$$
$$-27\cdot x^3\cdot y^3\cdot\sqrt[3]{x^3}$$
$$-27x^3y^3\cdot x$$
$$-27x^4y^3$$

You can find the odd root of a negative number, but you cannot find the even root of a negative number. This means you can simplify the radicals $\sqrt[3]{-81}$, $\sqrt[5]{-64}$, and $\sqrt[7]{-2187}$, but you cannot simplify the radicals $\sqrt{-100}$, $\sqrt[4]{-16}$, or $\sqrt[6]{-2,500}$.

Let's look at another example.

Example

Simplify. $\sqrt[3]{-24a^5}$
Show Solution

The steps to consider when simplifying a radical are outlined below.

Simplifying a radical

When working with exponents and radicals:

- If *n* is odd, $\sqrt[n]{x^n}=x$.
- If *n* is even, $\sqrt[n]{x^n}=|x|$. (The absolute value accounts for the fact that if *x* is negative and raised to an even power, that number will be positive, as will the *n*th principal root of that number.)

Example

Simplify. $\sqrt{100x^2y^4}$
Show Solution

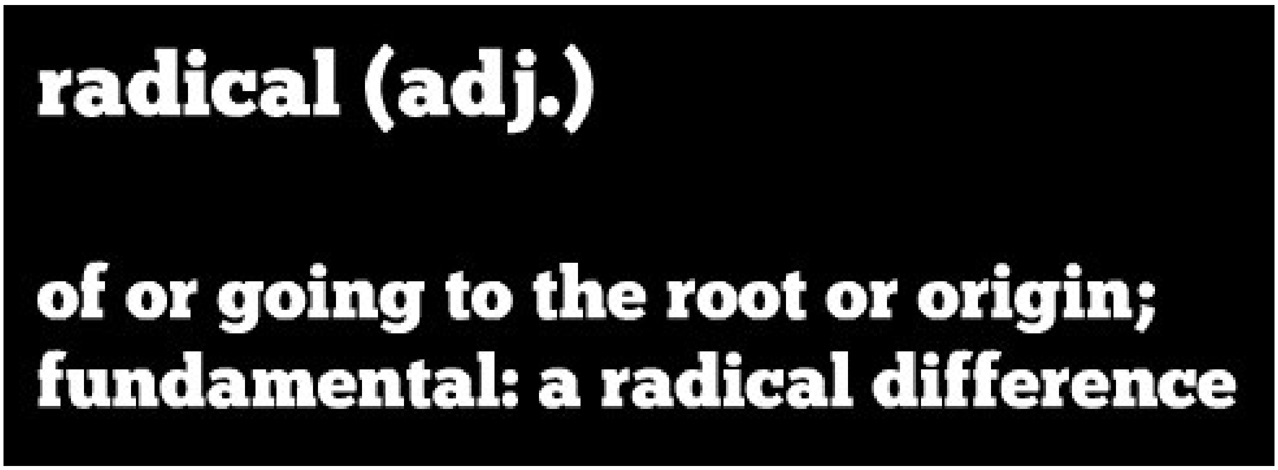
You can check your answer by squaring it to be sure it equals $100x^2y^4$.

In the last video, we share examples of finding cube roots with negative radicands.

Watch this video online: https://youtu.be/BtJruOpmHCE

Simplify Square Roots with Variables

Radical expressions are expressions that contain radicals. Radical expressions come in many forms, from simple and familiar, such as $\sqrt{16}$, to quite complicated, as in $\sqrt[3]{250x^4y}$. Using factoring, you can simplify these radical expressions, too.



Radical

Simplifying Square Roots

Radical expressions will sometimes include variables as well as numbers. Consider the expression $\sqrt{9x^6}$. Simplifying a radical expression with variables is not as straightforward as the examples we have already shown with integers.

Consider the expression $\sqrt{x^2}$. This looks like it should be equal to *x*, right? Let's test some values for *x* and see what happens.

In the chart below, look along each row and determine whether the value of *x* is the same as the value of $\sqrt{x^2}$. Where are they equal? Where are they not equal?

After doing that for each row, look again and determine whether the value of $\sqrt{x^2}$ is the same as the value of $|x|$.

x	x^2	$\sqrt{x^2}$	$\|x\|$
-5	25	5	5
-2	4	2	2
0	0	0	0
6	36	6	6
10	100	10	10

Notice—in cases where *x* is a negative number, $\sqrt{x^2} \neq x$! (This happens because the process of squaring the number loses the negative sign, since a negative times a negative is a positive.) However, in all cases $\sqrt{x^2} = |x|$. You need to consider this fact when simplifying radicals that contain variables, because by definition $\sqrt{x^2}$ is always nonnegative.

Taking the Square Root of a Radical Expression

When finding the square root of an expression that contains variables raised to a power, consider that $\sqrt{x^2} = |x|$.
Examples: $\sqrt{9x^2} = 3|x|$, and $\sqrt{16x^2y^2} = 4|xy|$

Let's try it.
The goal is to find factors under the radical that are perfect squares so that you can take their square root.

Example

Simplify. $\sqrt{9x^6}$
Show Solution

Variable factors with even exponents can be written as squares. In the example above, $x^6 = x^3 \cdot x^3 = |x^3|^2$ and

$y^4 = y^2 \cdot y^2 = \left(|y^2|\right)^2$.

Let's try to simplify another radical expression.

Example

Simplify. $\sqrt{49x^{10}y^8}$
Show Solution

You find that the square root of $49x^{10}y^8$ is $7\left|x^5\right|y^4$. In order to check this calculation, you could square $7\left|x^5\right|y^4$, hoping to arrive at $49x^{10}y^8$. And, in fact, you would get this expression if you evaluated $\left(7\left|x^5\right|y^4\right)^2$.

In the video that follows we show several examples of simplifying radicals with variables.

Watch this video online: https://youtu.be/q7LqsKPoAKo

Example

Simplify. $\sqrt{a^3b^5c^2}$
Show Solution

In the next section, we will explore cube roots, and use the methods we have shown here to simplify them. Cube roots are unique from square roots in that it is possible to have a negative number under the root, such as $\sqrt[3]{-125}$.

Rational Exponents

Roots can also be expressed as fractional exponents. The square root of a number can be written with a radical symbol or by raising the number to the $\frac{1}{2}$ power. This is illustrated in the table below.

Exponent Form	**Root Form**	**Root of a Square**	**Simplified**

$25^{\frac{1}{2}}$	$\sqrt{25}$	$\sqrt{5^2}$	5
$16^{\frac{1}{2}}$	$\sqrt{16}$	$\sqrt{4^2}$	4
$100^{\frac{1}{2}}$	$\sqrt{100}$	$\sqrt{10^2}$	10

Use the example below to familiarize yourself with the different ways to write square roots.

Example

Fill in the missing cells in the table.

Exponent Form	Root Form	Root of a Square	Simplified
$36^{\frac{1}{2}}$			
	$\sqrt{81}$		
		$\sqrt{12^2}$	

Show Solution

In the following video, we show another example of filling in a table to connect the different notation used for roots.

Watch this video online: https://youtu.be/eGJgmo2CpN4

We can extend the concept of writing $\sqrt{x} = x^{\frac{1}{2}}$ to cube roots. Remember, cubing a number raises it to the power of three. Notice that in these examples, the denominator of the rational exponent is the number 3.

Radical Form	Exponent Form	Integer
$\sqrt[3]{8}$	$8^{\frac{1}{3}}$	2
$\sqrt[3]{8}$	$125^{\frac{1}{3}}$	5
$\sqrt[3]{1000}$	$1000^{\frac{1}{3}}$	10

These examples help us model a relationship between radicals and rational exponents: namely, that the *n*th root of a number can be written as either $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Radical Form	Exponent Form
$\sqrt{x}$	$x^{\frac{1}{2}}$
$\sqrt[3]{x}$	$x^{\frac{1}{3}}$
$\sqrt[4]{x}$	$x^{\frac{1}{4}}$

Radical Form	Exponent Form
$\dots$	$\dots$
$\sqrt[n]{x}$	$x^{\frac{1}{n}}$

Convert Between Radical and Exponent Notation

When faced with an expression containing a rational exponent, you can rewrite it using a radical. In the table above, notice how the denominator of the rational exponent determines the index of the root. So, an exponent of $\frac{1}{2}$ translates to the square root, an exponent of $\frac{1}{5}$ translates to the fifth root or $\sqrt[5]{}$, and $\frac{1}{8}$ translates to the eighth root or $\sqrt[8]{}$.

Example

Write $\sqrt[3]{81}$ as an expression with a rational exponent.
Show Solution

Example

Express $(2x)^{\frac{1}{3}}$ in radical form.
Show Solution

Remember that exponents only refer to the quantity immediately to their left unless a grouping symbol is used. The example below looks very similar to the previous example with one important difference—there are no parentheses! Look what happens.

Example

Express $2x^{\frac{1}{2}}$ in radical form.
Show Solution

The next example is intended to help you practice placing a rational exponent on the appropriate terms in an expression that is written in radical form

Example

Express $4\sqrt[3]{xy}$ with rational exponents.
Show Solution

In the next video, we show examples of converting between radical and exponent form.

Watch this video online: https://youtu.be/5cWkVrANBWA

When converting from radical to rational exponent notation, the degree of the root becomes the denominator of the exponent. If you start with a square root, you will have an exponent of $\frac{1}{2}$ on the expression in the radical (the radicand). On the other hand, if you start with an exponent of $\frac{1}{3}$ you will use a cube root. The following statement summarizes this idea.

Writing Fractional Exponents

Any radical in the form $\sqrt[n]{a}$ can be written using a fractional exponent in the form $a^{\frac{1}{n}}$.

Simplifying Radical Expressions Using Rational Exponents and the Laws of Exponents

Let's explore some radical expressions now and see how to simplify them. Let's start by simplifying this expression, $\sqrt[3]{a^6}$.

One method of simplifying this expression is to factor and pull out groups of a^3, as shown below in this example.

Example

Simplify. $\sqrt[3]{a^6}$
Show Solution

You can also simplify this expression by thinking about the radical as an expression with a rational exponent, and using the principle that any radical in the form $\sqrt[n]{a^x}$ can be written using a fractional exponent in the form $a^{\frac{x}{n}}$.

Example

Simplify. $\sqrt[3]{a^6}$
Show Solution

Note that rational exponents are subject to all of the same rules as other exponents when they appear in algebraic expressions.

Both simplification methods gave the same result, a^2. Depending on the context of the problem, it may be easier to use one method or the other, but for now, you'll note that you were able to simplify this expression more quickly using rational exponents than when using the "pull-out" method.

Let's try another example.

Example

Simplify. $\sqrt[4]{81x^8y^3}$
Show Solution

Again, the alternative method is to work on simplifying under the radical by using factoring. For the example you just solved, it looks like this.

Example

Simplify. $\sqrt[4]{81x^8y^3}$
Show Solution

The following video shows more examples of how to simplify a radical expression using rational exponents.

Watch this video online: https://youtu.be/CfxhFRHUq_M

Summary

The square root of a number is the number which, when multiplied by itself, gives the original number. Principal square roots are always positive and the square root of 0 is 0. You can only take the square root of values that are nonnegative. The square root of a perfect square will be an integer. Other square roots can be simplified by identifying factors that are perfect squares and taking their square root.

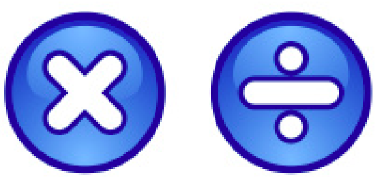
A radical expression is a mathematical way of representing the *n*th root of a number. Square roots and cube roots are the most common radicals, but a root can be any number. To simplify radical expressions, look for exponential factors within the radical, and then use the property $\sqrt[n]{x^n} = x$ if *n* is odd, and $\sqrt[n]{x^n} = |x|$ if *n* is even to pull out quantities. All rules of integer operations and exponents apply when simplifying radical expressions.

A radical can be expressed as an expression with a fractional exponent by following the convention $\sqrt[n]{a^m} = a^{\frac{m}{n}}$. Rewriting radicals using fractional exponents can be useful in simplifying some radical expressions. When working with fractional exponents, remember that fractional exponents are subject to all of the same rules as other exponents when they appear in algebraic expressions.

OPERATIONS ON RADICAL EXPRESSIONS

Learning Objectives

- Multiply and divide radical expressions
  - Use properties of exponents to multiply and divide radical expressions
- Add and subtract radical expressions
  - Identify radicals that can be added or subtracted
  - Add radical expressions
  - Subtract radical expressions
- Rationalize denominators
  - Define irrational and rational denominators
  - Remove radicals from a single term denominator



Multiply and Divide

You can do more than just simplify **radical expressions**. You can multiply and divide them, too. Multiplying radicals is very simple if the index on all the radicals match. The prodcut rule of radicals can be generalized as follows

A Product Raised to a Power Rule

For any numbers *a* and *b* and any integer *x*: $(ab)^x = a^x \cdot b^x$
For any numbers *a* and *b* and any positive integer *x*: $(ab)^{\frac{1}{x}} = a^{\frac{1}{x}} \cdot b^{\frac{1}{x}}$
For any numbers *a* and *b* and any positive integer *x*: $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$

The Product Raised to a Power Rule is important because you can use it to multiply radical expressions. Note that the roots are the same—you can combine square roots with square roots, or cube roots with cube roots, for example. But you can't multiply a square root and a cube root using this rule.

In the following example, we multiply two square roots

Example

Simplify. $\sqrt{18} \cdot \sqrt{16}$
Show Solution

Using the Product Raised to a Power Rule, you can take a seemingly complicated expression, $\sqrt{18} \cdot \sqrt{16}$, and turn it into something more manageable, $12\sqrt{2}$.

You may have also noticed that both $\sqrt{18}$ and $\sqrt{16}$ can be written as products involving perfect square factors. How would the expression change if you simplified each radical first, *before* multiplying?

Example

Simplify. $\sqrt{18} \cdot \sqrt{16}$
Show Solution

In both cases, you arrive at the same product, $12\sqrt{2}$. It does not matter whether you multiply the radicands or simplify each radical first.

You multiply radical expressions that contain variables in the same manner. As long as the roots of the radical expressions are the same, you can use the Product Raised to a Power Rule to multiply and simplify. Look at the two examples that follow. In both problems, the Product Raised to a Power Rule is used right away and then the expression is simplified.

Example

Simplify. $\sqrt{12x^3} \cdot \sqrt{3x}$, $x \geq 0$
Show Solution

In this video example, we multiply more square roots with and without variables.

Visit this page in your course online to check your understanding.

Example

Multiply $2\sqrt[3]{18} \cdot -7\sqrt[3]{15}$
Show Solution

We will show one more example of multiplying cube root radicals, this time we will include a variable.

Example

Multiply $\sqrt[3]{4x^3} \cdot \sqrt[3]{2x^2}$
Show Solution

In the next video, we present more examples of multiplying cube roots.

Watch this video online: https://youtu.be/cxRXofdeIIM

Dividing Radical Expressions

You can use the same ideas to help you figure out how to simplify and divide radical expressions. Recall that the Product Raised to a Power Rule states that $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$. Well, what if you are dealing with a quotient instead of a product?

There is a rule for that, too. The **Quotient Raised to a Power Rule** states that $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$. Again, if you imagine that the exponent is a rational number, then you can make this rule applicable for roots as well:

$$\left(\frac{a}{b}\right)^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}$$

Therefore

$$\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}.$$

A Quotient Raised to a Power Rule

For any real numbers *a* and *b* (*b* ≠ 0) and any positive integer *x*: $\left(\frac{a}{b}\right)^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}$

For any real numbers *a* and *b* (*b* ≠ 0) and any positive integer *x*: $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$

As you did with multiplication, you will start with some examples featuring integers before moving on to radicals with variables.

Example

Simplify. $\sqrt{\frac{48}{25}}$
Show Solution

As with multiplication, the main idea here is that sometimes it makes sense to divide and then simplify, and other times it makes sense to simplify and then divide. Whichever order you choose, though, you should arrive at the same final expression.

Now let’s turn to some radical expressions containing variables. Notice that the process for dividing these is the same as it is for dividing integers.

Example

Simplify. $\frac{\sqrt{30x}}{\sqrt{10x}}, x > 0$
Show Solution

As you become more familiar with dividing and simplifying radical expressions, make sure you continue to pay attention to the roots of the radicals that you are dividing. For example, you can think of this expression:

$$\frac{\sqrt{8y^2}}{\sqrt{225y^4}}$$

As equivalent to:

$$\sqrt{\frac{8y^2}{225y^4}}$$

This is because both the numerator and the denominator are square roots.

Notice that you cannot express this expression:

$$\frac{\sqrt{8y^2}}{\sqrt[4]{225y^4}}$$

In this format:

$$\sqrt[4]{\frac{8y^2}{225y^4}}.$$

This is becuase the numerator is a square root and the denominator is a fourth root. In this last video, we show more examples of simplifying a quotient with radicals.

Watch this video online: https://youtu.be/QwUsRWCNt24

Add and Subtract Radical Expressions

Adding and subtracting radicals is much like combining like terms with variables. We can add and subtract expressions with variables like this:

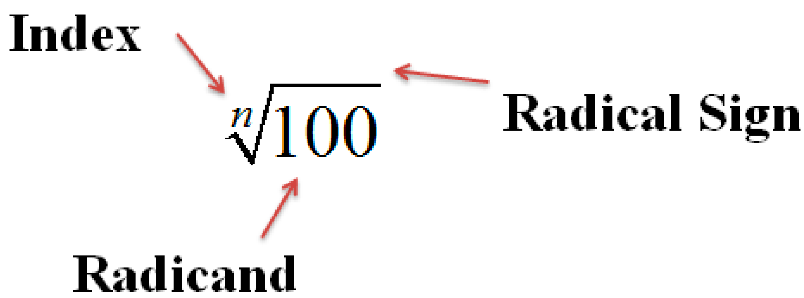
$5x+3y-4x+7y=x+10y$

There are two keys to combining radicals by addition or subtraction: look at the **index**, and look at the **radicand**. If these are the same, then addition and subtraction are possible. If not, then you cannot combine the two radicals.

Keys

Remember the index is the degree of the root and the radicand is the term or expression under the radical. In the diagram below, the index is n, and the radicand is 100. The radicand is placed under the root symbol and the index is placed outside the root symbol to the left:

$\sqrt[n]{}$ **indicates an nth root.**



Index and radicand

Practice identifying radicals that are compatible for addition and subtraction by looking at the index and radicand of the roots in the following example.

Example

Identify the roots that have the same index and radicand.
$10\sqrt{6}$
$-1\sqrt[3]{6}$
$\sqrt{25}$
$12\sqrt{6}$
$\frac{1}{2}\sqrt[3]{25}$
$-7\sqrt[3]{6}$
Show Solution

Let’s use this concept to add some radicals.

Example

Add. $3\sqrt{11}+7\sqrt{11}$
Show Solution

It may help to think of radical terms with words when you are adding and subtracting them. The last example could be read “three square roots of eleven plus 7 square roots of eleven”.

This next example contains more addends. Notice how you can combine *like* terms (radicals that have the same root and index) but you cannot combine *unlike* terms.

Example

Add. $5\sqrt{2}+\sqrt{3}+4\sqrt{3}+2\sqrt{2}$
Show Solution

Notice that the expression in the previous example is simplified even though it has two terms: $7\sqrt{2}$ and $5\sqrt{3}$. It would be a mistake to try to combine them further! (Some people make the mistake that $7\sqrt{2}+5\sqrt{3}=12\sqrt{5}$. This is incorrect because $\sqrt{2}$ and $\sqrt{3}$ are not like radicals so they cannot be added.)

Example

Add. $3\sqrt{x}+12\sqrt[3]{xy}+\sqrt{x}$
Show Solution

Sometimes you may need to add *and* simplify the radical. If the radicals are different, try simplifying first—you may end up being able to combine the radicals at the end, as shown in these next two examples.

Example

Add and simplify. $2\sqrt[3]{40}+\sqrt[3]{135}$
Show Solution

Example

Add and simplify. $x\sqrt[3]{xy^4}+y\sqrt[3]{x^4y}$
Show Solution

Subtracting Radicals

Subtraction of radicals follows the same set of rules and approaches as addition—the radicands and the indices (plural of index) must be the same for two (or more) radicals to be subtracted.

Example

Subtract. $5\sqrt{13}-3\sqrt{13}$
Show Solution

Example

Subtract. $4\sqrt[3]{5a}-\sqrt[3]{3a}-2\sqrt[3]{5a}$
Show Solution

In the video example that follows, we show more examples of how to add and subtract radicals that don't need to be simplified beforehand.

Watch this video online: https://youtu.be/5pVc44dEsTI

The following video shows how to add and subtract radicals that can be simplified beforehand.

Watch this video online: https://youtu.be/tJk6_7lbrlw

Rationalize Denominators

Although radicals follow the same rules that integers do, it is often difficult to figure out the value of an expression containing radicals. For example, you probably have a good sense of how much $\frac{4}{8}$, 0.75 and $\frac{6}{9}$ are, but what about the quantities $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{5}}$? These are much harder to visualize.

You can use a technique called **rationalizing a denominator** to eliminate the radical. The point of rationalizing a denominator is to make it easier to understand what the quantity really is by removing radicals from the denominators.

Recall that the numbers 5, $\frac{1}{2}$, and 0.75 are all known as rational numbers—they can each be expressed as a ratio of two integers ($\frac{5}{1}$, $\frac{1}{2}$, and $\frac{3}{4}$ respectively). Some radicals are irrational numbers because they cannot be represented as a ratio of two integers. As a result, the point of rationalizing a denominator is to change the expression so that the denominator becomes a rational number.

Here are some examples of irrational and rational denominators.

Irrational		Rational
$\frac{1}{\sqrt{2}}$	=	$\frac{\sqrt{2}}{2}$
$\frac{2+\sqrt{3}}{\sqrt{3}}$	=	$\frac{2\sqrt{3}+3}{3}$

Now let's examine how to get from irrational to rational denominators.

Let's start with the fraction $\frac{1}{\sqrt{2}}$. Its denominator is $\sqrt{2}$, an irrational number. This makes it difficult to figure out what the value of $\frac{1}{\sqrt{2}}$ is.

You can rename this fraction without changing its value, if you multiply it by 1. In this case, set 1 equal to $\frac{\sqrt{2}}{\sqrt{2}}$. Watch what happens.

$$\frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2 \cdot 2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

The denominator of the new fraction is no longer a radical (notice, however, that the numerator is).

So why choose to multiply $\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$? You knew that the square root of a number times itself will be a whole number. In algebraic terms, this idea is represented by $\sqrt{x} \cdot \sqrt{x} = x$. Look back to the denominators in the multiplication of $\frac{1}{\sqrt{2}} \cdot 1$. Do you see where $\sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2$?

Here are some more examples. Notice how the value of the fraction is not changed at all—it is simply being multiplied by 1.

Example

Rationalize the denominator.

$\frac{-6\sqrt{6}}{\sqrt{3}}$

Show Solution

In the video example that follows, we show more examples of how to rationalize a denominator with an integer radicand.

Watch this video online: https://youtu.be/K7NdhPLVI7g

You can use the same method to rationalize denominators to simplify fractions with radicals that contain a variable. As long as you multiply the original expression by another name for 1, you can eliminate a radical in the denominator without changing the value of the expression itself.

Example

Rationalize the denominator.

$\frac{\sqrt{2y}}{\sqrt{4x}}, \text{ where } x \neq 0$

Show Solution

Example

Rationalize the denominator and simplify.

$\sqrt{\frac{100x}{11y}}, \text{ where } y \neq 0$

Show Solution

THE video that follows shows more examples of how to rationalize a denominator with a monomial radicand.

Watch this video online: https://youtu.be/EBUzRctmgyk

Summary

When you encounter a fraction that contains a radical in the denominator, you can eliminate the radical by using a process called rationalizing the denominator. To rationalize a denominator, you need to find a quantity that, when multiplied by the denominator, will create a rational number (no radical terms) in the denominator. When the denominator contains a single term, as in $\frac{1}{\sqrt{5}}$, multiplying the fraction by $\frac{\sqrt{5}}{\sqrt{5}}$ will remove the radical from the denominator.

Summary

The Product Raised to a Power Rule and the Quotient Raised to a Power Rule can be used to simplify radical expressions as long as the roots of the radicals are the same. The Product Rule states that the product of two or more numbers raised to a power is equal to the product of each number raised to the same power. The same is true of roots: $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$. When dividing radical expressions, the rules governing quotients are similar: $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$.

Combining radicals is possible when the index and the radicand of two or more radicals are the same. Radicals with the same index and radicand are known as like radicals. It is often helpful to treat radicals just as you would treat variables: like radicals can be added and subtracted in the same way that like variables can be added and subtracted. Sometimes, you will need to simplify a radical expression before it is possible to add or subtract like terms.

SOLVING RADICAL EQUATIONS

Learning Objectives

- Solve Radical Equations
  - Isolate square roots in equations and solve for a variable
  - Identify extraneous solutions to radical equations
- Square Roots and Completing the Square for Solving Radical Equations

- Use square roots to solve quadratic equations
- Complete the square to solve a quadratic equation
- Using the Quadratic Formula to Solve Quadratic Equations
  - Write a quadratic equation in standard form and identify the values of *a*, *b*, and *c* in a standard form quadratic equation.
  - Use the Quadratic Formula to find all real solutions.
  - Solve application problems requiring the use of the Quadratic Formula.

A basic strategy for solving radical equations is to isolate the radical term first, and then raise both sides of the equation to a power to remove the radical. (The reason for using powers will become clear in a moment.) This is the same type of strategy you used to solve other, non-radical equations—rearrange the expression to isolate the variable you want to know, and then solve the resulting equation.

Solutions to Radical Equations

The solutions of $x^2 = a$ are called the square roots of a.

- When a is positive, a > 0, $x^2 = a$ has two solutions, $+\sqrt{a}, -\sqrt{a}$. $+\sqrt{a}$ is the nonnegative square root of a, and $-\sqrt{a}$ is the negative square root of a.
- When a is negative, a < 0, $x^2 = a$ has no solutions.
- When a is zero, a = 0, $x^2 = a$ has one solution: a = 0

Just to drive home the importance of the concept that when a is negative, a < 0, $x^2 = a$ has no solutions, we will restate it in words. If you have a negative number under a square root sign as in this example,

$\sqrt{-3}$

There will be no real number solutions.

There are two key ideas that you will be using to solve radical equations. The first is that if $a = b$, then $a^2 = b^2$. (This property allows you to square both sides of an equation and remain certain that the two sides are still equal.) The second is that if the square root of any nonnegative number *x* is squared, then you get *x*: $(\sqrt{x})^2 = x$. (This property allows you to "remove" the radicals from your equations.)

Let's start with a radical equation that you can solve in a few steps: $\sqrt{x} - 3 = 5$.

Example

Solve. $\sqrt{x} - 3 = 5$
Show Solution

To check your solution, you can substitute 64 in for *x* in the original equation. Does $\sqrt{64} - 3 = 5$? Yes—the square root of 64 is 8, and $8 - 3 = 5$.

Notice how you combined like terms and then squared both *sides* of the equation in this problem. This is a standard method for removing a radical from an equation. It is important to isolate a radical on one side of the equation and simplify as much as possible *before* squaring. The fewer terms there are before squaring, the fewer additional terms will be generated by the process of squaring.

In the example above, only the variable *x* was underneath the radical. Sometimes you will need to solve an equation that contains multiple terms underneath a radical. Follow the same steps to solve these, but pay attention to a critical point—square both *sides* of an equation, not individual *terms*. Watch how the next two problems are solved.

Example

Solve. $\sqrt{x+8} = 3$
Show Solution

In the following video we show how to solve simple radical equations.

Visit this page in your course online to check your understanding.

Example

Solve. $1+\sqrt{2x+3}=6$
Show Solution

Solving Radical Equations

Follow the following four steps to solve radical equations.

1. Isolate the radical expression.
2. Square both sides of the equation: If $x=y$ then $x^2=y^2$.
3. Once the radical is removed, solve for the unknown.
4. Check all answers.

Identify Extraneous Solutions

Following rules is important, but so is paying attention to the math in front of you—especially when solving radical equations. Take a look at this next problem that demonstrates a potential pitfall of squaring both sides to remove the radical.

Example

Solve. $\sqrt{a-5}=-2$
Show Solution

Look at that—the answer $a=9$ does not produce a true statement when substituted back into the original equation. What happened?

Check the original problem: $\sqrt{a-5}=-2$. Notice that the radical is set equal to -2, and recall that the principal square root of a number can only be *positive*. This means that no value for *a* will result in a radical expression whose positive square root is -2! You might have noticed that right away and concluded that there were no solutions for *a*.

Incorrect values of the variable, such as those that are introduced as a result of the squaring process are called **extraneous solutions**. Extraneous solutions may look like the real solution, but you can identify them because they will not create a true statement when substituted back into the original equation. This is one of the reasons why checking your work is so important—if you do not check your answers by substituting them back into the original equation, you may be introducing extraneous solutions into the problem.
In the next video example, we solve more radical equations that may have extraneous solutions.

Visit this page in your course online to check your understanding.

Have a look at the following problem. Notice how the original problem is $x+4=\sqrt{x+10}$, but after both sides are squared, it becomes $x^2+8x+16=x+10$. Squaring both sides may have introduced an extraneous solution.

Example

Solve. $x+4=\sqrt{x+10}$
Show Solution

Example

Solve. $4+\sqrt{x+2}=x$
Show Solution

In the last video example we solve a radical equation with a binomial term on one side.

Visit this page in your course online to check your understanding.

Square Roots and Completing the Square

Quadratic equations can be solved in many ways. In the previous section we introduced the idea that solutions to radical equations in general can be found using these facts:

Solutions to Quadratic Equations

The solutions of $x^2=a$ are called the square roots of a.

- When a is positive, a > 0, $x^2=a$ has two solutions, $+\sqrt{a},-\sqrt{a}$. $+\sqrt{a}$ is the nonnegative square root of a, and $-\sqrt{a}$ is the negative square root of a.
- When a is negative, a < 0, $x^2=a$ has no solutions.
- When a is zero, a = 0, $x^2=a$ has one solution: a = 0

A shortcut way to write $\sqrt{a}$ or $-\sqrt{a}$ is $\pm\sqrt{a}$. The symbol $\pm$ is often read "positive or negative." If it is used as an operation (addition or subtraction), it is read "plus or minus."

Example

Solve using the Square Root Property. $x^2=9$
Show Solution

Notice that there is a difference here in solving $x^2=9$ and finding $\sqrt{9}$. For $x^2=9$, you are looking for *all numbers* whose square is 9. For $\sqrt{9}$, you only want the *principal* (nonnegative) square root. The negative of the principal square root is $-\sqrt{9}$; both would be $\pm\sqrt{9}$. *Unless there is a symbol in front of the radical sign, only the nonnegative value is wanted!*

In the example above, you can take the square root of both sides easily because there is only one term on each side. In some equations, you may need to do some work to get the equation in this form. You will find that this involves isolating x^2.

Example

Solve. $10x^2+5=85$
Show Solution

In the following video we show more examples of solving simple quadratic equations using square roots.

Watch this video online: https://youtu.be/Fj-BP7uaWrI

Sometimes more than just the *x* is being squared:

Example

Solve. $(x-2)^2-50=0$

Show Solution

In this video example, you will see more examples of solving quadratic equations using square roots.

Watch this video online: https://youtu.be/4H5qZ_-8YM4

Completing the Square to Solve a Quadratic Equation

Of course, quadratic equations often will not come in the format of the examples above. Most of them will have *x* terms. However, you may be able to factor the expression into a squared binomial—and if not, you can still use squared binomials to help you.

Some of the above examples have squared binomials: $(1+r)^2$ and $(x-2)^2$ are squared binomials. If you expand these, you get a perfect square trinomial.

Perfect square trinomials have the form $x^2+2xs+s^2$ and can be factored as $(x+s)^2$, or they have the form $x^2-2xs+s^2$ and can be factored as $(x-s)^2$. Let's factor a perfect square trinomial into a squared binomial.

Example

Factor $9x^2-24x+16$.
Show Solution

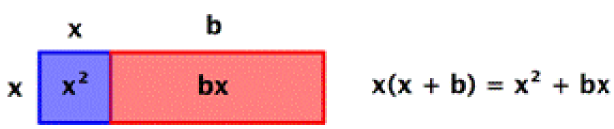
You can use the procedure in this next example to help you solve equations where you identify perfect square trinomials, even if the equation is not set equal to 0.

Example

Solve. $4x^2+20x+25=8$
Show Solution

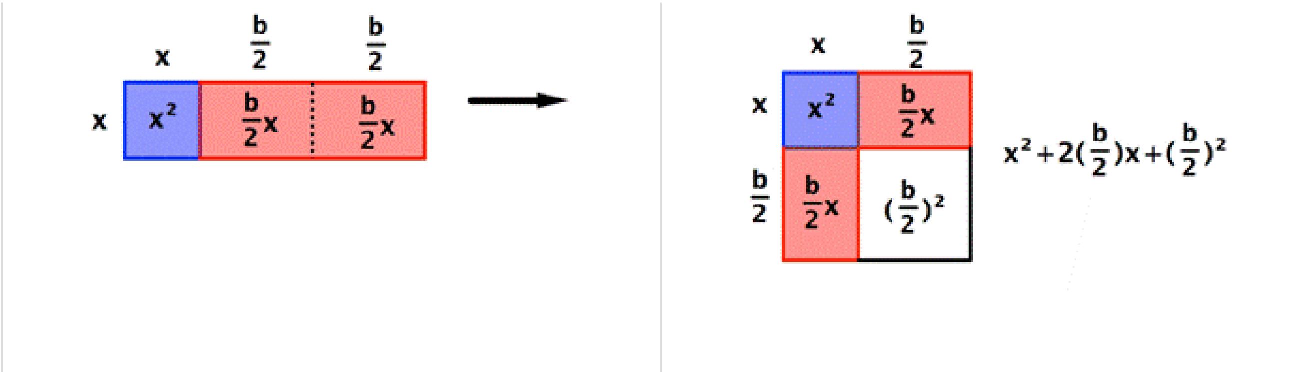
One way to solve quadratic equations is by **completing the square**. When you don't have a perfect square trinomial, you can *create* one by adding a constant term that is a perfect square to both sides of the equation. Let's see how to find that constant term.

"Completing the square" does exactly what it says—it takes something that is not a square and makes it one. This idea can be illustrated using an area model of the binomial x^2+bx.



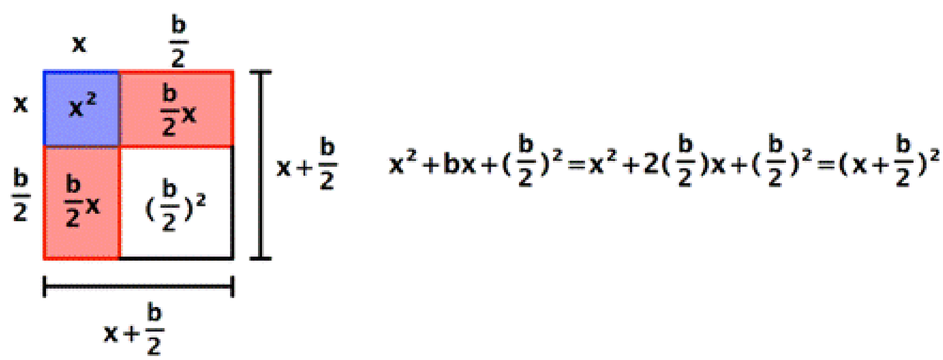
In this example, the area of the overall rectangle is given by $x\left(x+b\right)$.

Now let's make this rectangle into a square. First, divide the red rectangle with area *bx* into two equal rectangles each with area $\frac{b}{2}x$. Then rotate and reposition one of them. You haven't changed the size of the red area—it still adds up to bx.



The red rectangles now make up two sides of a square, shown in white. The area of that square is the length of the red rectangles squared, or $\left(\frac{b}{2}\right)^2$.

Here comes the cool part—do you see that when the white square is added to the blue and red regions, the whole shape is also now a square? In other words, you've "completed the square!" By adding the quantity $\left(\frac{b}{2}\right)^2$ to the original binomial, you've made a square, a square with sides of $x + \frac{b}{2}$.



Notice that the area of this square can be written as a squared binomial: $\left(x + \frac{b}{2}\right)^2$.

Finding a Value that will Complete the Square in an Expression

To complete the square for an expression of the form $x^2 + bx$:

- Identify the value of *b;*
- Calculate and add $\left(\frac{b}{2}\right)^2$.

The expression becomes $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$.

Example

Find the number to add to $x^2 + 8x$ to make it a perfect square trinomial.
Show Solution

Notice that $\left(\frac{b}{2}\right)^2$ is always positive, since it is the square of a number. When you complete the square, you are always adding a positive value.

In the following video, we show more examples of how to find a constant terms that will make a trinomial a perfect square.

Watch this video online: https://youtu.be/vt-pM1LEP1M

You can use completing the square to help you solve a quadratic equation that cannot be solved by factoring.

Let's start by seeing what happens when you complete the square in an equation. In the example below, notice that completing the square will result in adding a number to *both* sides of the equation—you have to do this in order to keep both sides equal!

Example

Rewrite $x^2+6x=8$ so that the left side is a perfect square trinomial.
Show Solution

Can you see that completing the square in an equation is very similar to completing the square in an expression? The main difference is that you have to add the new number ($+9$ in this case) to both sides of the equation to maintain equality.

Now let's look at an example where you are using completing the square to actually solve an equation, finding a value for the variable.

Example

Solve. $x^2–12x–4=0$
Show Solution

In this last video, we solve more quadratic equations by completing the square.

Watch this video online: https://youtu.be/IjCjbtrPWHM

You may have noticed that because you have to use both square roots, all the examples have two solutions. Here is another example that's slightly different.

Example

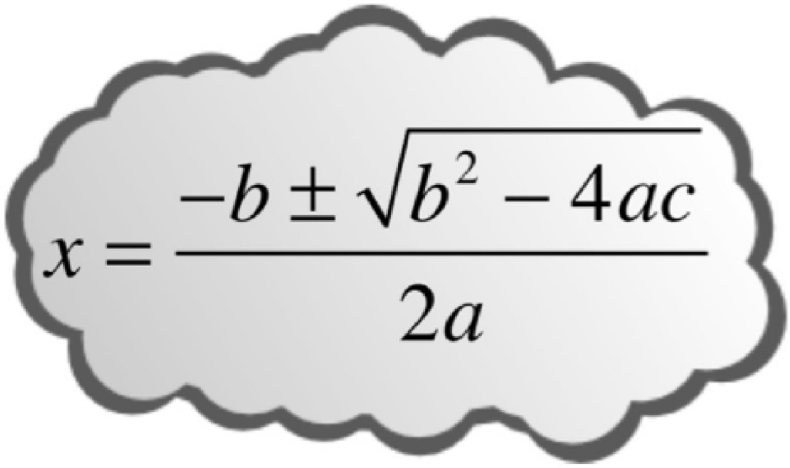
Solve $x^2+16x+17=-47$.
Show Solution

Take a closer look at this problem and you may see something familiar. Instead of completing the square, try adding 47 to both sides in the equation. The equation $x^2+16x+17=-47$ becomes $x^2+16x+64=0$. Can you factor this equation using grouping? (Think of two numbers whose product is 64 and whose sum is 16).

It can be factored as $(x+8)(x+8)=0$, of course! Knowing how to complete the square is very helpful, but it is not always the only way to solve an equation.

Use the Quadratic Formula to Solve Quadratic Equations

The Quadratic Formula



Quadratic formula

You can solve any quadratic equation by **completing the square**—rewriting part of the equation as a perfect square trinomial. If you complete the square on the generic equation $ax^2 + bx + c = 0$ and then solve for *x*, you find that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This equation is known as the Quadratic Formula.

This formula is very helpful for solving quadratic equations that are difficult or impossible to factor, and using it can be faster than completing the square. The Quadratic Formula can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$.

The form $ax^2 + bx + c = 0$ is called standard form of a quadratic equation. Before solving a quadratic equation using the Quadratic Formula, it's *vital* that you be sure the equation is in this form. If you don't, you might use the wrong values for *a*, *b*, or *c*, and then the formula will give incorrect solutions.

Example

Rewrite the equation $3x + 2x^2 + 4 = 5$ in standard form and identify *a*, *b*, and *c*.
Show Solution

Example

Rewrite the equation $2(x + 3)^2 - 5x = 6$ in standard form and identify *a*, *b*, and *c*.
Show Solution

Solving a Quadratic Equation using the Quadratic Formula

The Quadratic Formula will work with *any* quadratic equation, but *only* if the equation is in standard form, $ax^2 + bx + c = 0$. To use it, follow these steps.

- Put the equation in standard form first.
- Identify the coefficients, *a*, *b,* and *c.* Be careful to include negative signs if the *bx* or *c* terms are subtracted.
- Substitute the values for the coefficients into the Quadratic Formula.
- Simplify as much as possible.
- Use the $\pm$ in front of the radical to separate the solution into two values: one in which the square root is added, and one in which it is subtracted.
- Simplify both values to get the possible solutions.

That's a lot of steps. Let's try using the Quadratic Formula to solve a relatively simple equation first; then you'll go back and solve it again using another factoring method.

Example

Use the Quadratic Formula to solve the equation $x^2 + 4x = 5$.
Show Solution

You can check these solutions by substituting 1 and -5 into the original equation.

$$\begin{aligned} x &= 1 \\ x^2 + 4x &= 5 \\ (1)^2 + 4(1) &= 5 \\ 1 + 4 &= 5 \\ 5 &= 5 \end{aligned}$$	$$\begin{aligned} x &= -5 \\ x^2 + 4x &= 5 \\ (-5)^2 + 4(-5) &= 5 \\ 25 - 20 &= 5 \\ 5 &= 5 \end{aligned}$$

You get two true statements, so you know that both solutions work: $x = 1$ or -5. You've solved the equation successfully using the Quadratic Formula!

The power of the Quadratic Formula is that it can be used to solve *any* quadratic equation, even those where finding number combinations will not work.

In teh following video, we show an example of using the quadratic formula to solve an equation with two real solutions.

Watch this video online: https://youtu.be/xtwO-n8IRPw

Most of the quadratic equations you've looked at have two solutions, like the one above. The following example is a little different.

Example

Use the Quadratic Formula to solve the equation $x^2 - 2x = 6x - 16$.
Show Solution

Again, check using the original equation.

$$\begin{aligned} x^2 - 2x &= 6x - 16 \\ (4)^2 - 2(4) &= 6(4) - 16 \\ 16 - 8 &= 24 - 16 \\ 8 &= 8 \end{aligned}$$

In the following video we show an example of using the quadratic formula to solve a quadratic equation that has one repeated solution.

Watch this video online: https://youtu.be/OXwwzWcxFgE

In this video example we show that solutions to quadratic equations can have rational answers.

Watch this video online: https://youtu.be/xtwO-n8IRPw

Applying the Quadratic Formula

Quadratic equations are widely used in science, business, and engineering. Quadratic equations are commonly used in situations where two things are multiplied together and they both depend on the same variable. For example, when working with area, if both dimensions are written in terms of the same variable, you use a quadratic equation. Because the quantity of a product sold often depends on the price, you sometimes use a quadratic equation to represent revenue as a product of the price and the quantity sold. Quadratic equations are also used when gravity is involved, such as the path of a ball or the shape of cables in a suspension bridge.

A very common and easy-to-understand application is the height of a ball thrown at the ground off a building. Because gravity will make the ball speed up as it falls, a quadratic equation can be used to estimate its height any time before it hits the ground. *Note: The equation isn't completely accurate, because friction from the air will slow the ball down a little. For our purposes, this is close enough.*

Example

A ball is thrown off a building from 200 feet above the ground. Its starting velocity (also called *initial velocity*) is -10 feet per second. (The negative value means it's heading toward the ground.)

The equation $h=-16t^2-10t+200$ can be used to model the height of the ball after *t* seconds. About how long does it take for the ball to hit the ground?
Show Solution

The area problem below does not look like it includes a Quadratic Formula of any type, and the problem seems to be something you have solved many times before by simply multiplying. But in order to solve it, you will need to use a quadratic equation.

Example

Bob made a quilt that is 4 ft $\times$ 5 ft. He has 10 sq. ft. of fabric he can use to add a border around the quilt. How wide should he make the border to use all the fabric? (The border must be the same width on all four sides.)
Show Solution

In this last video, we show how to use the quadratic formula to solve an application involving a picture frame.

Watch this video online: https://youtu.be/AIIoxXQ-V50

Summary

A common method for solving radical equations is to raise both sides of an equation to whatever power will eliminate the radical sign from the equation. But be careful—when both sides of an equation are raised to an *even* power, the possibility exists that extraneous solutions will be introduced. When solving a radical equation, it is important to always check your answer by substituting the value back into the original equation. If you get a true statement, then that value is a solution; if you get a false statement, then that value is not a solution.

Completing the square is used to change a binomial of the form x^2+bx into a perfect square trinomial $x^2+bx+\left(\frac{b}{2}\right)^2$, which can be factored to $\left(x+\frac{b}{2}\right)^2$. When solving quadratic equations by completing the square, be careful to add $\left(\frac{b}{2}\right)^2$ to *both* sides of the equation to maintain equality. The Square Root Property can then be used to solve for *x*. With the Square Root Property, be careful to include both the principal square root and its opposite. Be sure to simplify as needed.

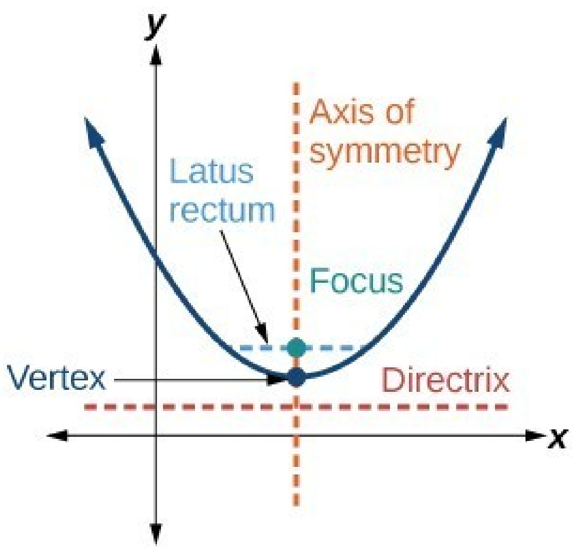
Quadratic equations can appear in different applications. The Quadratic Formula is a useful way to solve these equations, or any other quadratic equation! The Quadratic Formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, is found by completing the square of the quadratic equation .

CONCLUSION

In the beginning of this module, we proposed that one of the most well known uses of roots and rational exponents is found in quadratic equations.

Quadratic equations, when graphed on the Cartesian coordinate plane, form the shape of a parabola. Parabolas are unique and interesting shapes because no matter how you stretch, shrink, or enlarge them, they are all identical to each other. What does that mean?

A parabola is the set of all points (x, y) in a plane that are the same distance from a fixed line, called the **directrix**, and a fixed point (the **focus**) not on the directrix.

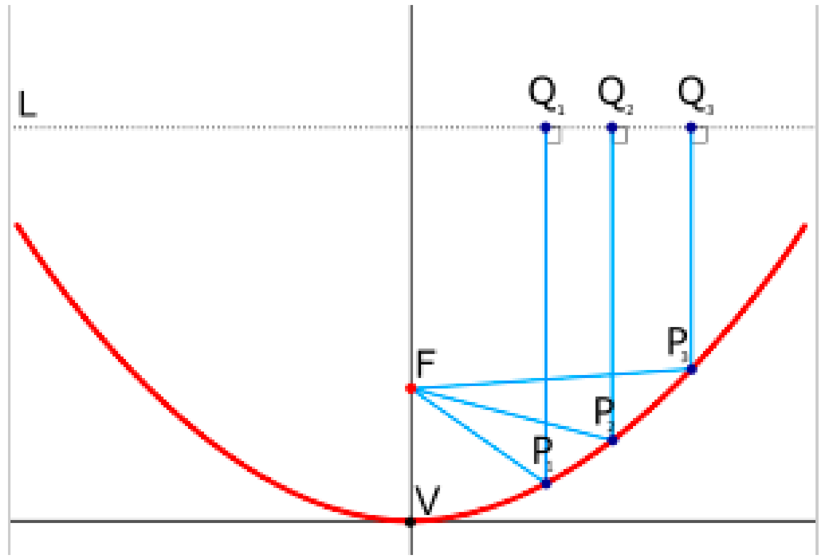


This feature of parabolas is what makes them so useful in so many ways, and what makes them identical to each other – this relationship holds no matter how you stretch or shrink them. Ellipses and circles do not share this quality. You can stretch and shrink them and they are not identical to each other. This unique fact about parabolas has some interesting impacts in our real lives.

For example, did you know that stage lighting and car headlamps are often parabolic in shape? The reason for this stems from the shape of the paraboloid.

According to Wikipedia, a parabolic (or paraboloid or paraboloidal) reflector (or dish or mirror) is a reflective surface used to collect or project energy such as light, sound, or radio waves.

Parabolic reflectors are used to collect energy from a distant source (for example sound waves or incoming star light) and bring it to a common focal point, as in the image below:



Parallel rays coming in to a parabolic mirror are focused at a point F. The vertex is V, and the axis of symmetry passes through V and F. For off-axis reflectors (with just the part of the paraboloid between the points P1 and P3), the receiver is still placed at the focus of the paraboloid, but it does not cast a shadow onto the reflector.

Since the principles of reflection are reversible, parabolic reflectors can also be used to project energy of a source at its focus outward in a parallel beam, used in devices such as spotlights and car headlights.