

## 8.EE How Many Solutions?

### Task

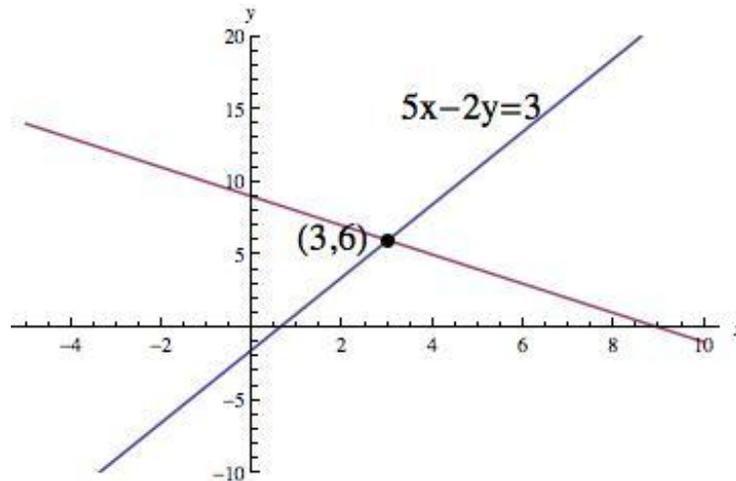
Consider the equation  $5x - 2y = 3$ . If possible, find a second linear equation to create a system of equations that has:

- a. Exactly 1 solution.
- b. Exactly 2 solutions.
- c. No solutions.
- d. Infinitely many solutions.

Bonus Question: In each case, how many such equations can you find?

### Solution

a. While it is possible to solve this problem purely algebraically, thinking about how the question relates to the graphs of the equations makes the problem much easier to solve. To have exactly one solution, we want the graph of  $5x - 2y = 3$  and the graph of the equation we come up with to intersect at exactly one point, as shown below.



To do this, we find a point on the line  $5x - 2y = 3$  and create another line that also contains that point but is not the same line. We can pick any value for  $x$  and find the corresponding value for  $y$ . Arbitrarily, let us consider  $x = 3$ . By substitution, we see that

$$\begin{aligned} 5(3) - 2y &= 3 \implies \\ 15 - 2y &= 3 \implies \\ y &= 6 \end{aligned}$$

So  $(3,6)$  is a point on the graph of  $5x - 2y = 3$ . Now we find any values  $a, b$ , or  $c$  in the equation  $ax + by = c$  that "work" with  $x = 3$  and  $y = 6$  and are not the same values as in the given equation. There are many possible ways to do this. For example, if we choose  $a = 1$  and  $b = 1$ , then

$$x + y = 3 + 6 = 9$$

so  $c$  must be 9. in other words,  $(3, 6)$  is a solution to the equation

$$x + y = 9$$

Taking the two equations together as a system of equations,

$$\begin{aligned} 5x - 2y &= 3 \\ x + y &= 9 \end{aligned}$$

we can verify algebraically that there is exactly one solution to this system, namely  $(3, 6)$ .

b. To have exactly two solutions, we would want a second line that intersects the graph

of  $5x - 2y = 3$  at exactly two points. However, this is not possible. Since two points determine one and only one line, we must conclude that if two lines intersect at two points, they must actually be the same line.

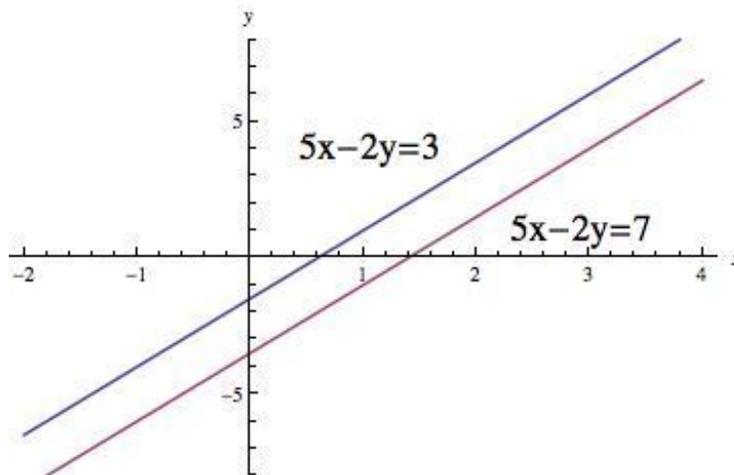
c. To have no solutions, we want our new line and the graph of  $5x - 2y = 3$  to not intersect anywhere, meaning that the two lines are parallel. Consider any equation of the form  $5x - 2y = c$  where  $c \neq 3$ , for example  $5x - 2y = 7$ . Then our corresponding system of equations is

$$5x - 2y = 3$$

$$5x - 2y = 7$$

From this we can see that no matter what values of  $x$  and  $y$  we substitute into the two equations,  $5x - 2y$  can never equal 3 and 7 simultaneously, meaning that no point  $(x, y)$  can be on both lines at the same time. So, this system of equations has no solutions.

Graphically we can think of two lines with the same slope but different  $y$ -intercepts.



d. To have infinitely many solutions, we want our equation and  $5x - 2y = 3$  to intersect everywhere. In other words, they will be the same line. One way to denote this is to simply use the same equation,  $5x - 2y = 3$ , or just multiply both sides of the equation by a constant; let's say we multiply each term by 2. Then for the system of equations

$$5x - 2y = 3$$

$$10x - 4y = 6$$

any  $x$  and  $y$  pair that satisfies the first equation will satisfy the second, since taking two numbers that are equal and multiplying them both by 2 will result in two equal numbers. So this system has infinitely many solutions, as the equations both correspond to the same line and lines have infinitely many points.

(Bonus) In parts (a), (c), and (d), there are infinitely many equations that can be found. In part (a), the point/solution  $(3, 6)$  was arbitrary, and we could have picked any point, on the given line and drawn a line through the point at any slope. In part (c), we could have chosen any constant (other than 3) to take the place of the 3, and it would have still resulted in a system with no solutions. In part (d), we could have multiplied the terms of the given equation by any constant (besides 0) and it would have described the same line, though the equation would look different, resulting in infinitely many solutions to the corresponding system of equations. However, the graphs of the different equations in (d) would all look identical.



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