8.EE Extending the Definitions of Exponents, Variation 1

Task

Marco and Seth are lab partners studying bacterial growth. They were surprised to find that the population of the bacteria doubled every hour.

a. The table shows that there were 2,000 bacteria at the beginning of the experiment. What was the size of population of bacteria after 1 hour? After 2, 3 and 4 hours? Enter this information into the table:

Hours into study				0	1	2	3	4
Population (thousands)				2				

b. If you know the size of the population at a certain time, how do you find the population one hour later?

c. Marco said he thought that they could use the equation $P = 2t + 2$ to find the population at time t. Seth said he thought that they could use the equation $P = 2 \cdot 2^t$. Decide whether either of these equations produces the correct populations for $t = 1, 2, 3, 4$.

d. Assuming the population doubled every hour before the study began, what was the population of the bacteria 1 hour *before* the students started their study? What about 3 hours before?

e. If you know the size of the population at a certain time, how do you find the population one hour *earlier*?

f. What number would you use to represent the time 1 hour before the study started? 2 hours before? 3 hours before? Finish filling in the table if you haven't already.

g. Now use Seth's equation to find the population of the bacteria 1 hour before the study started. Use the equation to find the population of the bacteria 3 hours before. Do these values produce results consistent with the arithmetic you did earlier?

h. Use the context to explain why it makes sense that $2^{-n} = (\frac{1}{2})^n = \frac{1}{2^n}$. That is, describe why, based on the population growth, it makes sense to define 2 raised to a negative integer exponent as repeated multiplication by $\frac{1}{2}$.

IM Commentary

This is an instructional task meant to generate a conversation around the meaning of negative integer exponents. While it may be unfamiliar to some students, it is good for them to learn the convention that negative time is simply any time before $t = 0$.

Students will struggle to put their explanation for part (h) together. A teacher might want to have the students do parts (a) - (g) as a precursor to providing an explanation like the one given in the solution for part (h).

The Standards for Mathematical Practice focus on the nature of the learning experiences by attending to the thinking processes and habits of mind that students need to develop in order to attain a deep and flexible understanding of mathematics. Certain tasks lend themselves to the demonstration of specific practices by students. The practices that are observable during exploration of a task depend on how instruction unfolds in the classroom. While it is possible that tasks may be connected to several practices, only one practice connection will be discussed in depth. Possible secondary practice connections may be discussed but not in the same degree of detail.

This task leads students systematically through the use of repeated reasoning to understand algorithms and make generalizations about patterns. (MP.8) This task could be used as an introduction to the properties of integer exponents when related to negative exponents. To lead the students to this generalization a teacher could separate each of the tasks listed *a-g* and have students work in groups to complete each task with discussion and written consensus about the results each time. After the students have worked through *g*, all results of *a-g* are posted for students to see; the teacher could have students individually write an explanation for *h*. Students could then have small and/or whole group discussion to finalize an answer with the teacher guiding the discussion to the correct answer and explicitly pointing out, if needed, the repeated reasoning being used and the generalization of the pattern that is developed. The core of this task – generalizing to negative exponents - is also about making use of structure (MP. 7). Students see the structure of adding 1 to the exponent corresponding to multiplying by 2. Then they see that this works for negative exponents as well.

Solution

a. What was the size of population of bacteria after 1 hour? After 2, 3 and 4 hours? Enter this information into the table:

Hours into study	0	1	2	3	4
Population (thousands)	2	4	8	16	32

b. You multiply it by 2, since it doubled.

c. The values predicted by Seth's equation agree exactly with those in the table above; Seth's equation works because it predicts a doubling of the population every hour. Marco’s doesn’t because it doesn’t double the new population you have – instead it is doubling the time. Marco's equation predicts a linear growth of only two thousand bacteria per hour.

d. Since the population is multiplied by 2 every hour we would have to divide by 2 (which is the same as multiplying by $\frac{1}{2}$) to work backwards. The population 1 hour before the study started would be

$$\frac{1}{2} \cdot 2 = 1 \text{ thousand},$$

and the population 3 hours before the study started would be

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 2 = 0.25 \text{ thousand} = 250.$$

e. Since the population is multiplied by 2 every hour we would have to divide by 2 (or multiply by $\frac{1}{2}$) to work backwards.

f. Time before the study started would be negative time; for example one hour before the study began was $t = -1$.

Hours into study	-3	-2	-1	0	1	2	3	4
Population (thousands)	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0.25$	$\frac{1}{2} \cdot 1 = 0.5$	1	2	4	8	16	32

g. Since one hour before the study started would be $t = -1$, we would simply plug this value into Seth's equation:

$$2 \cdot (2)^{-1} = 2 \cdot \left(\frac{1}{2}\right) = 1 \text{ thousand.}$$

Three hours before would be $t = -3$. Using the equation:

$$2 \cdot (2)^{-3} = \frac{2}{2^3} = 0.25 \text{ thousand,}$$

giving us the same answers as we got through reasoning.

h. Since the bacteria double every hour, we multiply the population by two for every hour we go forward in time. So if we want to know what the population will be 8 hours after the experiment started, we need to multiply the population at the start ($t = 0$) by 2 eight times. This explains why we raise 2 to the number of hours that have passed to find the new population; repeatedly doubling the population means we repeatedly multiply the population at $t = 0$ by 2.

In this context, negative time corresponds to time *before* the experiment started. To figure out what the population was before the experiment started we have to “undouble” (or multiply by $\frac{1}{2}$) for every hour we have to go back in time. So if we want to know what the population was 8 hours before the experiment started, we need to multiply the population at the start ($t = 0$) by $\frac{1}{2}$ eight times. The equation indicates that we should raise 2 to a power that corresponds to the number of hours we need to go back in time. For every hour we go back in time, we multiply by $\frac{1}{2}$. So it makes sense in this context that raising 2 to the -8 power (or any negative integer power) is the same thing as repeatedly multiplying $\frac{1}{2}$ 8 times (or the opposite of the power you raised 2 to). In other words, it makes sense in this context that

$$2^{-n} = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}.$$

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