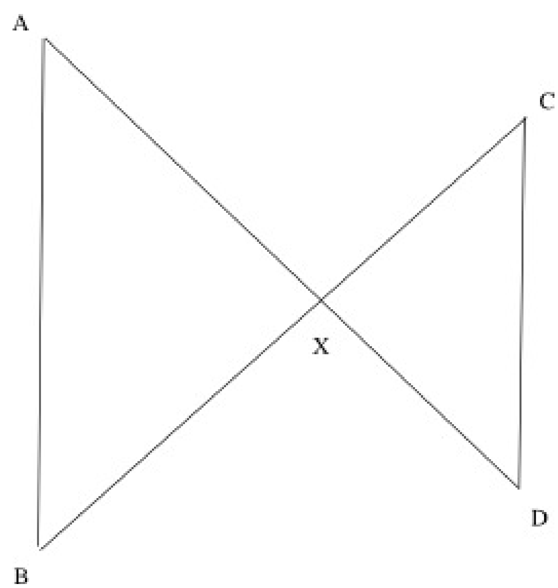
G-SRT Are They Similar?

Alignments to Content Standards: G-SRT.A.2

Task

In the picture given below, line segments AD and BC intersect at X. Line segments AB and CD are drawn, forming two triangles AXB and CXD.



In each part (a)-(d) below, some additional assumptions about the picture are given. In each problem, determine whether the given assumptions are enough to prove that the two triangles are similar; and if so, what the correct correspondence of vertices is. If the two triangles must be similar, prove this result by describing a sequence of similarity transformations that maps one triangle to the other. If not, explain why not.

a. The lengths AX and XD satisfy the equation $2AX = 3XD$.

b. The lengths AX, BX, CX, and DX satisfy the equation $\frac{AX}{BX} = \frac{DX}{CX}$.

c. Lines AB and CD are parallel.

d. Angle XAB is congruent to angle XCD.

IM Commentary

In this problem, students are given a picture of two triangles that appear to be similar, but whose similarity cannot be proven without further information. Asking students to provide a sequence of similarity transformations that maps one triangle to the other focuses them on the work of standard G-SRT.2, using the definition of similarity in terms of similarity transformations. Teachers will need to remind students to show that their sequences of similarity transformations have the intended effect; that is, that all parts of one triangle get mapped to the corresponding parts of the other triangle.

A teacher who wishes to remove the focus on using similarity transformations may instead simply ask students to prove or disprove that the triangles are similar in each problem using properties of parallel lines and the definition of similarity. This shifts the focus of this task from G-SRT.2 to G-SRT.5.

As noted in the solution, almost any attempt to draw a counterexample works. But on this part, a teacher may simply wish to have students observe that they do not seem to have enough information to prove similarity, rather than showing that it is impossible to prove similarity.

The Standards for Mathematical Practice focus on the nature of the learning experiences by attending to the thinking processes and habits of mind that students need to develop in order to attain a deep and flexible understanding of mathematics. Certain tasks lend themselves to the demonstration of specific practices by students. The practices that are observable during exploration of a task depend on how instruction unfolds in the classroom. While it is possible that tasks may be connected to several practices, only one practice connection will be discussed in depth. Possible secondary practice connections may be discussed but not in the same degree of detail.

This task is linked to Standard for Mathematical Practice #3 “Construct viable arguments and critique the reasoning of others.” This task gives students an opportunity to create viable arguments, given different initial conditions. The task asks students to consider which conditions are enough to constitute a proof and which are not, therefore providing a fertile environment for students to construct arguments leading from assumptions to the desired end result, or explain why no argument could exist. By looking at different assumptions within the same figure, this task emphasizes to students the importance of specific assumptions make to the arguments that can be created and therefore what can be proven.

Edit this solution

Solution

a. We are given that 2AX = 3XD. This is not enough information to prove similarity. To see that in a simple way draw an arbitrary triangle △AXB. Extend AX and choose a point D on the extended line so that 2AX=3XD. Extend BX and choose a point C on the extended line so that 2BX=XC. Now triangles AXB and CXD satisfy the given conditions but are not similar. (If you are extremely unlucky, AXB and CXD might be similar by a different correspondence of sides. If this happens, rotate the line BC a little bit. The lengths of AX, XD, BX, XC remain the same but the triangles are no longer similar.)

b. We are given that $\frac{AX}{BX} = \frac{DX}{CX}$. Rearranging this proportion gives $\frac{AX}{DX} = \frac{BX}{CX}$. Let $k = \frac{AX}{DX}$. Suppose we rotate the triangle DXC 180 degrees about point X: Since AD is a straight line, DX and AX align upon rotation of 180 degrees, as do CX and BX, and so angles DXC and AXB coincide after this rotation. (Alternatively, one could observe that DXC and AXB are vertical angles, and hence congruent, giving a second argument the angles line up precisely). Then dilate the triangle DXC by a factor of k about the center X. This dilation moves the point D to A, since k(DX) = AX, and moves C to B, since k(CX) = BX. Then since the dilation fixes X, and dilations take line segments to line segments, we see that the triangle DXC is mapped to triangle AXB. So the original triangle DXC is similar to AXB. (Note that we state the similarity so that the vertices of each triangle are written in corresponding order.)

c. Again, rotate triangle DXC so that angle DXC coincides with angle AXB. Then the image of the side CD under this rotation is parallel to the original side CD, so the new side CD is still parallel to side AB. Now, apply a dilation about point X that moves the vertex C to point B. This dilation moves the line CD to a line through B parallel to the previous line CD. We already know that line AB is parallel to CD, so the dilation must move the line CD onto the line AB. Since the dilation moves D to a point on the ray XA and on the line AB, D must move to A. Therefore, the rotation and dilation map the triangle DXC to the triangle AXB. Thus DXC is similar to AXB.

d. Suppose we draw the bisector of angle AXC, and reflect the triangle CXD across this angle bisector. This maps the segment XC onto the segment XA; and since reflections preserve angles, it also maps segment XD onto segment XB. Since angle XCD is congruent to angle XAB, we also know that the image of side CD is parallel to side AB. Therefore, if we apply a dilation about the point X that takes the new point C to A, then the new line CD will be mapped onto the line AB, by the same reasoning used in (c). Therefore, the new point D is mapped to B, and thus the triangle XCD is mapped to triangle XAB. So triangle XCD is similar to triangle XAB. (Note that this is not the same correspondence we had in parts (b) and (c)!)