Illustrative Mathematics

Integrated Math 3

In Integrated Math 3, exponential functions are considered over a domain of real numbers, necessitating work with fractional exponents. The logarithm is defined as the inverse of exponentiation and from this definition students consider the properties of logarithmic functions in addition to using them to solve for unknown exponents. Students extend their previous work with quadratics and polynomials to achieve a more general understanding of polynomials. In the geometric domain, when lines intersect circles, segments, angles, and arcs are created, with attending relationships that are proven and used to solve problems. This work with circles along with previous knowledge of sin, cos, and tan as operations combine in a study of the unit circle and sin(x), cos(x), and tan(x) as functions. Finally, students do further work in statistics where they revisit and extend their understanding of variability in data and of ways to describe variability in data. The normal distribution is studied, and students explore the reasoning that allows them to draw conclusions based on data from statistical studies.

M3.1 Exponential Functions 2

- **Create and analyze a simple exponential function arising from a real-world or mathematical context.**
- **Evaluate and interpret exponential functions at non-integer inputs.**
- **Understand functions of the form $f(t) = P(1 + r/n)^{nt}$ and solve problems with different compounding intervals.**
- **Understand informally how the base e is used in functions to model a quantity that compounds continuously.**
- **Write exponential expressions in different forms.**
- **Explain what the parameters of an exponential function mean in different contexts.**
- **Use the properties of exponents to write expressions in equivalent forms.**
- **Build exponential functions to model real world contexts.**
- **Analyze situations that involve geometric sequences and series.**
- **Derive the formula for the sum of a finite geometric series.**

In previous units students have worked with geometric sequences and understand they change by a constant ratio over a constant interval. They are able to write both recursive and closed equations for them. Students understand the difference between a linear and exponential function, can recognize situations and tables described by each, and know that an exponential function will always overtake a linear function. They know that an exponential function grows increasingly rapidly in one direction, and approaches a value asymptotically in the other direction. They have solved exponential equations of the form abx = c by graphing. They can construct an exponential function given a graph, description of a relationship, or two input output pairs with integer inputs (including in a table). Given an expression defining an exponential function, they can interpret its parameters in a context. They can also fit a simple exponential function to a scatterplot. Every exponential function until now has only involved integer inputs.

In this unit students broaden their view of exponential functions to include the entire real number line as a possible domain. They learn about functions with base e. An approach using compound interest that shows e arising as the natural base for a quantity being compounded continuously can serve as a way to develop understanding appropriate to this level. First, students must understand functions of the form $f(x) = P(1 + r/n)^{nt}$ which show a given compounding frequency, n.

Students examine some different forms of exponential functions and learn to interpret the parameters in terms of a context. They learn the concept of doubling time and see functions expressed in a form that shows the doubling time; they work algebraically with functions expressed in a form like $f(x) = A(1 + r/n)^{nt}$ that shows the compounding period; and they work with functions written with the base e, $g(x) = Ae^{rt}$, in many continuous growth contexts. Students build functions in those forms in order model real-world contexts. Contexts may include Moore's law for computer processor speeds, population growth, and temperature change. Students should also consider whether an exponential or a linear model is appropriate in various contexts.

Students analyze situations that involve summing an exponential sequence (which is generally called a geometric series) These arise naturally in some saving and banking problems as well as in some interesting geometric contexts. Students will build on previous work with geometric sequences to derive a formula for the sum of a geometric series.

A later unit introduces logarithms as the solutions to exponential equations. Ultimately, students are proficient at graphing, analyzing, solving, and modeling exponential situations. Students are comfortable with exponentials from a functions viewpoint. This lays the foundations for success in calculus.

M3.1.0 Pre-unit diagnostic assessment

M3.1.1 Understanding exponential growth and decay

Create and analyze a simple exponential function arising from a real-world or mathematical context.

Students have been introduced to exponential functions in A2. For some students this introduction could have occurred a year or two previously. Therefore this unit starts with an activity that re-introduces exponential growth and decay in an engaging real-world or mathematical context. The context can be modeled by an exponential function with domain contained in the integers, thus providing a review of previous experience as needed for students.

M3.1.2 Exponential functions on the real numbers

Evaluate and interpret exponential functions at non-integer inputs.

In A2 students considered exponential functions at integer inputs only. Now that they understand how to determine the value of b^x for any rational number x, they can approximate b^x to any degree of accuracy for any real number x. In this section they broaden their view of exponential functions to include the entire real number line as a possible domain. They evaluate functions and interpret their values at real inputs in terms of a context, in preparation for the more sophisticated work in the following sections.

M3.1.3 Changing compounding intervals, continuous compounding, and the base e

- **Understand functions of the form $f(t) = P(1 + r/n)^{nt}$ which use a given compounding frequency, n.**
- **Solve problems given different compounding intervals.**
- **Understand (informally) how the base e arises in the context of compounding intervals as n becomes arbitrarily large.**

The base e is very commonly used in scientific and other modeling applications. The fundamental reason that e is a useful base for an exponential function is beyond the scope of this course. However, an approach using compound interest that shows e arising as the natural base for a quantity being compounded continuously can serve as a way to develop understanding at this level. First, students must understand functions of the form $f(t) = P(1 + r/n)^{nt}$ which show a given compounding interval, n. This section examines that approach.

Tasks

A-SSE The Bank Account

F-BF Compounding with a 5% Interest Rate

M3.1.4 Interpreting exponential functions

- **Solve problems involving exponential functions in many different contexts.**
- **Write exponential expressions in different forms.**

- **Explain what the parameters of an exponential function mean in different contexts.**
- **Use the properties of exponents to write expressions in equivalent forms.**

In Unit A2, students worked with exponential functions in the form $f(t) = ab^t$ or $f(t) = a(1 + r)^t$ and interpreted the parameters a, b, and r in terms of a context. In this unit they see more complicated forms. In the previous section, students developed an understanding of different compounding intervals and continuous compounding using base e. The purpose of this section is to examine some of these different forms and learn to interpret the parameters in terms of a context. Students learn the concept of doubling time and see functions expressed in a form that shows the doubling time; they work algebraically with functions expressed in a form like $f(x) = A(1 + r/n)^{nt}$ that shows the compounding period; and they work with functions written with the base e, $g(x) = Ae^{rt}$, in many continuous growth contexts. In this section they do not build functions in any of these forms.

M3.1.5 Modeling with exponential functions

Build exponential functions to model real world contexts.

Having seen the purpose of various different expressions for exponential functions, students now start to build functions in those forms in order model real-world contexts. Contexts may include Moore's law for computer processor speeds, population growth, and temperature change. Students should also consider whether an exponential or a linear model is appropriate in various contexts.

Tasks

F-LE In the Billions and Exponential Modeling

F-LE Boiling Water

M3.1.6 Geometric sequences and series

- **Analyze situations that involve geometric sequences and series A-SSE.**
- **Derive the formula for the sum of a finite geometric series.**

Analyze situations that involve summing a exponential sequence (which is generally called a geometric series) These arise naturally in some saving and banking problems as well as in some interesting geometric contexts.

Tasks

A-SSE A Lifetime of Savings
A-SSE Course of Antibiotics
A-SSE Triangle Series
A-SSE Cantor Set

M3.1.7 Summative assessment

M3.2 Logarithms

• Understand the definition of a logarithm as the solution to an exponential equation.
• Practice evaluating logarithmic expressions and converting between the exponential form of an equation and the logarithmic form.
• Solve exponential equations using logarithms.
• Understand the natural logarithm as a special case.
• Graph exponential and logarithmic functions, both by hand and using technology.
• (Optional) Understand and explain the properties of logarithms.
• (Optional) Use properties of logarithms to solve problems.
• (Optional) Solve problems using the properties of logarithms.

Before this unit, students can construct exponential functions given a graph, table, or description, and interpret exponential functions expressed in different forms in terms of a context. They can graph exponential functions and note key features such as end behavior, asymptotes, and intercepts. They recognize real-world contexts that can be modeled by exponential functions, such as savings accounts and population growth, and they construct functions to model them. For example, they might describe the amount in a savings account with a 1000*initialbalancethatearns*10A(t) = 1000(1.0083)^{12t}.*Theycansolveproblemsinvolvingexponentialequationslike*20,000 = 1000(1.0083)^{12t}$ graphically, but not algebraically.

In this unit, students understand the logarithm defined operationally as the inverse of exponentiation, and have opportunities to practice interpreting logarithm notation and evaluating logarithms. They use logarithms to solve for an unknown exponent in situations modeled by exponential functions. These functions include ones expressed with base e, necessitating the introduction of the natural logarithm. Students graph logarithmic functions along with the exponential functions that are their inverses, developing an understanding of logarithmic functions as the inverses of exponential functions.

The Common Core State Standards do not explicitly require students to know and use the properties of logarithms. Student intending to pursue STEM careers in college should go deeper and learn these topics. Two optional sections at the end of the unit develop and apply the properties of logarithms.

After this unit, logarithms become a natural part of the toolkit in working with situations modeled by exponential functions. Applications abound in calculus, engineering, and the sciences. There are many sets of data that reveal their structure when plotted on logarithmic scales.

M3.2.0 Pre-unit diagnostic assessment

Diagnose students' ability to
• use reasoning and exponent properties to solve for an unknown exponents;
• graph a relatively simple exponential function, showing correct end behavior and intercepts;
• solve an exponential function at a given value graphically;
• summarize an exponential relation by writing an equation, given a table or several points.

M3.2.1 Motivate the need to undo exponentiation

Generate a need to find an unknown exponent which is not easy to guess and check.

The goal of this section is to help students see why a logarithm might be useful in their mathematical toolkit. Students are presented with a situation where they develop an exponential model F-BF.A.1 and need to find an unknown exponent that yields a specified value. Teachers can revisit and assess solving such a problem by graphing with technology A-REI.D.11 or by guess and check. They can then point out that students can solve all other kinds of equations that they know about by rewriting them in a helpful, equivalent form (for example, $x^3 = 1000$ can be rewritten as $x = 1000^{1/3}$) and suggest that there should be a way to do that to find an unknown exponent (as in, for example, $3^x = 1000$) F-LE.A.4. Teachers can either introduce the logarithm at this point, or leave students in suspense until the next section.

M3.2.2 Understand the definition of a logarithm

• **Understand the definition of a logarithm as the solution to an exponential equation.**
• **Practice evaluating log expressions and converting between the exponential form of an equation and the logarithmic form.**

The logarithm can be defined operationally. Just as $\sqrt[3]{2}$ is defined to be the positive real number that when multiplied by itself three times is equal to 2, the solution to $2^x = k$ is defined to be $x = \log_2(k)$. In this section students develop this definition through an exploratory activity. They analyze a number of true statements about logarithms without having been told the meaning of the notation, make conjectures about the pattern they fit using their knowledge of exponents, and express their meaning in terms of an equivalent exponential equation. They come up with a definition of the logarithm by precisely describing what they see and generalizing it. They then practice interpreting and converting logarithmic expressions.

M3.2.3 Use logarithms to solve problems

• **Solve exponential equations using logarithms.**
• **Understand the natural logarithm as a special case.**

Once students understand what a logarithm is, they need ample opportunity to apply that understanding to solve problems in various contexts. All of the problems in this section should involve exponential functions with base 2, 10, or e and should be solvable by reasoning directly from the definition of the logarithm, using the equivalence between $b^x = y$ and $x = \log_b y$. In particular many of them involve modeling continuous growth using an exponential function with base e, so this is the section where the natural logarithm is introduced. Students do not need to know the property $\log(a^b) = b\log(a)$ or solve equations using this property ("taking logs of both sides"). Note on calculating logarithms: some scientific calculators have buttons for base 10 logarithms and natural logarithms, but not base 2 logarithms. There are many online calculators that can calculate the latter, such as Desmos or the Google calculator, which is activated by typing \log_2(x)$ into the search engine.
Tasks

F-LE Algae Blooms
F-LE Newton's Law of Cooling
F-LE Moore's Law and Computers
F-LE Snail Invasion

M3.2.4 The logarithm function

• • Graph exponential and logarithmic functions, both by hand and using technology.
• Verify that f(x) = 10x and g(x) = log10(x) are inverses of one another.

The previous sections focus on the definition of the logarithm as a notation for an unknown exponent and on solving equations involving exponentials and logarithms. In this section students start to view the logarithm as a function. This is analogous to the progression from learning about the square root symbol to studying the square root function. Students create graphs of logarithm functions by viewing them as the inverse of exponential functions, with inputs and outputs reversed. They apply their previous understanding of the nature of exponential functions to draw conclusions about the behavior of the graphs of logarithmic functions. For example, students who understand debt as an exponential function of time view the same situation as time being a logarithmic function of debt.
Tasks
F-BF Exponentials and Logarithms II
F-LE Exponential Kiss

M3.2.5 Going deeper: properties of logarithms (optional)

• Understand and explain the properties of logarithms.
• Use properties of logarithms to solve problems.

The sections previous to this one are sufficient to meet the standards about logarithms in CCSSM including the (+) standards. Students pursuing STEM careers will need to go further. This section and the next one show the bridging material needed for STEM readiness. The properties of logarithms arise naturally from the properties of exponents and the fact that logarithmic functions are inverse to exponential functions. In this section students are given opportunities to notice patterns when combining logarithmic expressions with operations (for example, $\log(A) + \log(B)$ is equal to $\log(AB)$), conjecture that these patterns could be useful shortcuts, and then justify why the patterns always work.

M3.2.6 Going deeper: using the properties of logarithms (optional)

Solve problems using the properties of logarithms.

Students solve problems involving exponential functions with bases other than 2, e, or 10, or involving more than one exponential function with different bases, so that it is natural to use the property $\log(a^b) = b\log(a)$. The problems here are more complex than the ones in Section 3 and are suitable for students who are preparing for STEM majors in college.

Tasks

F-LE Rumors

F-LE Comparing Exponentials

M3.2.7 Summative assessment

Assess students' ability to • solve equations with unknown exponents using logarithms;
• graph a simple logarithmic function by hand showing intercepts and end behavior;
• demonstrate understanding of the inverse nature of exponential and logarithmic functions.

M3.3 Polynomials and Rational Functions

• Add, subtract, and multiply polynomials and express them in standard form using the properties of operations.
• Prove and make use of polynomial identities.
• Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
• Use the remainder theorem to find factors of polynomials.
• Use various strategies including graphing and factoring to solve problems in contexts that can be modeled by polynomials in one variable.
• Build a rational function that describes a relationship between two quantities.
• Graph rational functions, interpret features of the graph in terms of a context, and use the graphs to solve problems.
• Express rational functions in different forms to see different aspects of the situation they model.
• Solve simple rational equations and understand why extraneous roots can arise.

Coming into this unit, students understand that one can do arithmetic on quadratic expressions, and have generalized that understanding to polynomial expressions. They have solved quadratic equations with complex solutions using the methods of factoring, completing the square, and the quadratic formula. They have graphed quadratic functions and understand the relationship between zeros and factors of quadratics. They are familiar with modeling situations from which quadratic relations arise.

Students have modeled contexts with simple rational functions. They have graphed these functions and interpreted features of the graphs in context, including vertical asymptotes and end behavior. They can relate the domain of a function to its graph.

In this unit, students extend their previous work with quadratics and polynomials (in unit A4) to achieve a more general understanding of polynomials. They work with polynomials as a system where you can add, subtract, and multiply, analogous to the integers. They graph polynomial functions and they understand and use the relationship between factors, zeros, and intercepts on the graph. They model with polynomial functions.

Students study the graphs of simple rational functions. They consider contexts which can be modeled with rational functions and interpret vertical and horizontal asymptotes in terms of the context. They rewrite simple rational expressions in different forms to see different aspects of the context, and they find approximate solutions using graphical methods to rational equations that arise from the context. They also solve simple rational equations algebraically and study how and why extraneous solutions may arise.

After this unit, students going into STEM fields will see more examples, including polynomial approximations to other functions. Power series expansions (like Taylor series) show up in calculus. The characteristic polynomial of a matrix is a useful tool in university level linear algebra. Students going into higher mathematics will encounter polynomials as objects that help form abstract algebraic structures such as rings and fields.

Students may encounter simple rational relationships in real life. If they take a calculus course with a rigorous algebra component, they should do more work with more complicated rational expressions and equations beforehand.

M3.3.0 Pre-unit diagnostic assessment

Assess students' ability to
• graph a quadratic by factoring to find zeros, and correctly interpreting end behavior;
• solve a quadratic equation with a method suitable to the given form of the equation;
• write a simple rational equation that models a situation A-CED.A.1, and use the equation to solve problems;

M3.3.1 What is a polynomial?

• Add, subtract, and multiply polynomials and express them in standard form using the properties of operations. • Prove and make use of polynomial identities.

In this section students become familiar with the arithmetic of polynomials.

They add, subtract, and multiply them, and they use the properties of operations, particularly the distributive law, to express them as a sum of powers with coefficients. They recognize that every polynomial can be put in this form. They use polynomials to express and verify numerical patterns. The emphasis in this section should not be on formal definitions or formal proofs of closure properties. Rather the emphasis is on preparing students for the manipulations they will be using the coming sections, when they start to study polynomial functions.

Tasks

A-APR Powers of 11

A-APR Trina's Triangles

A-APR Non-Negative Polynomials

M3.3.2 Graphing polynomials

• Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

• Use the remainder theorem to find factors of polynomials.

The last activity in the previous section prepared students to start viewing polynomials in one variable as defining functions. In this section they study the graphs of polynomial functions. They see that the long run behavior of a polynomial is determined by its highest degree term. The use the relationship between factors and zeros to sketch the graph of a polynomial or to choose an appropriate viewing window for a graph produced by technology. The learn The Remainder Theorem and use it to find factors of polynomials.

Tasks

F-IF Graphs of Power Functions

F-IF Running Time

A-APR Graphing from Factors I

A-APR Graphing from Factors II

A-APR Graphing from Factors III

A-APR Zeroes and factorization of a quadratic polynomial I

M3.3.3 Modeling with polynomials

Use various strategies including graphing and factoring to solve problems in contexts that can be modeled by polynomials in one variable.

The main focus of modeling in this course is situations that can be modeled by linear, exponential, and quadratic functions. However, some contexts naturally give rise to polynomial models. In this section students make use of all that they have learned about polynomial functions to solve problems in such contexts.

Tasks

A-CED, A-REI Introduction to Polynomials - College Fund

M3.3.4 Rational Functions

- **Build a rational function that describes a relationship between two quantities.**
- **Graph rational functions.**
- **Interpret the graph of a rational function in terms of a context.**

In this section students study simple rational functions. The emphasis is on rational functions that arise naturally out of a real-world context, and on interpreting features of their graphs in terms of that context. Students experiment with graphs using technology to learn the relationship between features of the graph and the structure of the expression defining the function.

Tasks

F-BF,IF The Canoe Trip, Variation 1

F-IF Average Cost

F-IF Graphing Rational Functions

M3.3.5 Modeling with rational functions

- **Graph rational functions and use the graphs to solve problems.**
- **Express rational functions in different forms to see different aspects of the situation they model.**
- **Solve simple rational equations and understand why extraneous roots can arise.**

In the previous section students encountered simple rational functions. Here, they extend that work. They consider contexts which can be modeled with rational functions. They rewrite simple rational expressions in different forms to see different aspects of the context, and they find approximate solutions using graphical methods to rational equations that arise from the context. They also solve simple rational equations algebraically and study how and why extraneous solutions may arise.

M3.3.6 End Assessment

Assess students' ability to
• graph a polynomial by factoring to find zeros, and correctly interpreting end behavior;
• apply the remainder theorem to solve a mathematical problem;
• model with a polynomial A-CED.A.1, and use the model to solve a problem;
• rewrite a rational expression in a different form to solve a problem;
• write a rational equation that models a situation A-CED.A.1, and use the equation to solve problems;
• demonstrate understanding that when solving a rational equation, extraneous roots may emerge as a result of the solution process.

M3.4 Circles

• Use the Pythagorean Theorem to derive an equation for a circle of given center and radius.
• Use similarity to derive the fact that the length of the arc of a circle intercepted by an angle is proportional to the radius of the circle.
• Derive a formula for the area of a sector.
• Identify and describe relationships between central and inscribed angles and their arcs.
• Prove that an inscribed angle that subtends a diameter is a right angle, and its converse.
• Identify and describe relationships and ratios of lengths for intersecting chords.
• Prove that a radius and a tangent to a circle at the same point are perpendicular.
• Prove properties of angles of inscribed polygons.
• Use relationships about inscribed angles to solve problems about inscribed polygons.
• Use circles, cones, tangent segments, chords, and related figures, and their properties to describe objects.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

In grade 4, students learned to view an angle as indicating an amount of “turn,” e.g., “An angle that turns through 1/360 of a circle is called a ‘one-degree angle,’ and can be used to measure angles.”4.MD.5 In unit G1, students established a precise definition for a circle, e.g., that a circle is the locus of all points at a given distance from a given point. They made straightedge and compass constructions of circles, perpendicular bisectors, angle bisectors, and midpoints. From unit G3, they know that all circles are similar and they have gained experience using triangle congruence criteria. In unit G4, students were introduced to right triangle trigonometry. Unit G5 concerned solid geometry, including one task (Use Cavalieri's Principle to Compare Aquarium Volumes) that involved working with a cross-section of a sphere. Students begin this unit by working with circles on the coordinate plane, and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and a formula for the area of a sector of a circle. They examine properties of central and inscribed angles in a circle, and their subtended arcs. Students examine properties of tangent lines and radii, and intersecting chords. They apply these properties in a variety of contexts, including real-world contexts. After this unit, students extend the domain of trigonometric functions to the unit circle in the coordinate plane, working with radian measure, and again making use of the Pythagorean Theorem.

M3.4.0 Diagnostic pre-unit assessment

Diagnose students' ability to

- **find the area of a circle and of a sector of a circle;**
- **find measure of a sector angle when the sector partitions the circle into congruent pieces;**
- **construct a perpendicular bisector;**
- **find the distance between two given points on the coordinate plane.**

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.1 Equations of circles

Use the Pythagorean Theorem to derive a general equation for a circle of given center and radius.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.2 Arc lengths, sectors, and radians

• Use similarity to derive the fact that length of an intercepted arc of an angle is proportional to the length of the radius of the circle.
• Define radian measure of an angle.
• Derive a formula for the area of a sector.
• Use the formula in solving problems.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.3 Central and inscribed angles

• Identify and describe relationships between central and inscribed angles and their arcs.
• Prove that an inscribed angle that subtends a diameter is a right angle,

and, conversely, that a right angle inscribed in a circle must subtend a diameter.
• Apply these relationships in various contexts.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.4 Mid-Unit Assessment

Assess students' ability to

• solve area problems involving sectors;
• use properties of central angles and inscribed angles to find arc length.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.5 Radii, tangent lines, chords, and secants

• Prove that a radius and a tangent line that intersect at the same point are perpendicular.
• Identify and describe relationships and ratios of lengths for intersecting

chords.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.6 Inscribed and circumscribed polygons

• Prove that the opposite angles in a cyclic quadrilateral that contains the center of the circle are supplementary.
• Construct the inscribed and circumscribed circles of a triangle.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.7 Applications

Use circles, cones, tangent segments, chords, and related figures, and their properties to describe objects.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.4.8 Summative Assessment

Assess students' ability to apply knowledge of circle properties.

Students begin this unit by working with circles on the coordinate plane and use the Pythagorean Theorem to derive the equation of a circle. They learn about radian measure and use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius of the circle. They develop a formula for the area of a sector of a circle and examine properties of central and inscribed angles in a circle along with their subtended arcs. Students also examine properties of tangent lines, radii, and intersecting chords and apply these properties to solve problems in a variety of contexts, including real-world situations.

M3.5 Trigonometric Functions

- **Understand some real-world situations that demonstrate periodic behavior.**
- **Define coordinates on the unit circle as the sine and cosine of an angle.**
- **Understand radian measure and convert between radians and degree.**
- **Graph basic trigonometric functions using radians as the x-axis scale.**
- **Understand the relationship between parameters in a trigonometric function and the graph.**
- **Model with trigonometric functions, including fitting them to data.**
- **Prove the Pythagorean Identity** $\sin^2\theta + \cos^2\theta = 1$.
- **Use the unit circle to prove trigonometric identities and relate them to symmetries of the graphs of sine and cosine.**
- **Use the Pythagorean Identity to calculate trigonometric ratios.**

Students will already be familiar with right triangle trigonometry. They will have used sine, cosine and tangent to solve the angles and sides of a right triangle. Students will also have already seen that similar triangles preserve angles, as well as side ratios. For instance, they will have seen that a 3-4-5 right triangle has the same angles as a 6-8-10 triangle. They may have seen the “special right triangles” in the context of similar triangles. Students will also be familiar with Pythagorean Theorem, and will have used it to solve sides of right triangles. They will be familiar with the graphs of more than one type of function, including linear, quadratic, and exponential. Depending on timing, they may also have seen graphs of rational and polynomial functions.

In this unit, students make a big transition to thinking of trigonometric ratios as functions rather than a relationship between angles and side ratios in a right triangle. There are two important steps in this transition. First, students re-envision trigonometric ratios on the unit circle, and then use the idea of an angle as a movement around a circle to extend the definition to angles of any measure. Second, they use the unit circle to understand a new way of measuring angles, radian measure, which uses distance around the circumference of the circle to measure an angle, rather than an arbitrary division of the circle into 360 degrees. From now on students will be thinking of sine and cosine both as functions with numerical inputs, as well as ratios related to angles.

Students learn the basic shape of the graph of a trigonometric function, and then examine graphs of functions with parameters controlling the period, amplitude, and phase shift. They study the effect of varying these parameters and fit trigonometric functions to data. Addressing some (+) standards, students explore further the consequences of the unit circle definition of sine and cosine. They make a connection to the Pythagorean theorem. Then, they see how the symmetries of the circle give rise to symmetries of the graphs of sine and cosine, and represent these symmetries as identities.

In later math courses, students may see more complex uses of trigonometry. For instance, they may go on to learn that the derivative of the sine function is the cosine, and use radians to prove some of the properties of trigonometric derivatives.

M3.5.0 Pre-unit diagnostic assessment

Assess students' ability to
•using basic right triangle trigonometry to solve sides and angles of a right triangle;
•use Pythagorean Theorem to solve for the sides of a right triangle;
•view the vertical distance between the x-axis and a given point as the value of the point's y-coordinate, and the horizontal distance between the y-axis and a given point as the value of the point's x-coordinate.

M3.5.1 Introduction to periodic behavior

See a basic real-world model of periodic behavior and make sense of what data or graph it might generate.

Students have had a lot of exposure to graphing linear, quadratic, and exponential functions. They have yet to see a function that behaves periodically and understand how it might connect to a specific context. The ferris wheel provides a familiar scenario for students to see how the height of the cart will go up and down continuously, and to connect this information to a possible graph of the height. Students can make a rough sketch after watching the demo, or can use the more specific tools available in the Desmos activity to attempt to get a more accurate graph.

M3.5.2 Extending trigonometric functions to the real numbers

- **See coordinates on the unit circle as the sine and cosine of an angle.**
- **Understand radian measure and convert between radians and degrees.**
- **Graph basic trigonometric functions using radians as the x-axis scale.**

Students have understood trigonometric ratios in terms of right triangles, using them to solve for various sides and angles. In this section they make the big transition to thinking of trigonometric ratios as functions. There are two important steps in this transition. First, students re-envision trigonometric ratios on the unit circle, and then use the idea of an angle as a movement around a circle to extend the definition to angles of any measure. Second, they use the unit circle to understand a new way of measuring angles, radian measure, which uses distance around the circumference of the circle to measure an angle, rather than an arbitrary division of the circle into 360 degrees. From now on students will be thinking of sine and cosine as functions with numerical inputs, in addition to than ratios related to angles.

Tasks

F-TF.1 What exactly is a radian?

F-TF Bicycle Wheel

F-TF Trig Functions and the Unit Circle

M3.5.3 Modeling periodic behavior

- **Understand the relationship between parameters in a trigonometric function and the shape of the graph.**
- **Model with trigonometric functions, including fitting them to data.**

Students have learned the basic shape of the graph of a trigonometric function, and now begin examining graphs of functions with parameters controlling the period, amplitude, and phase shift. They study the effect of varying these parameters and fit trigonometric functions to data.

Tasks

F-TF, F-BF Exploring Sinusoidal Functions

M3.5.4 Identities and special values for trigonometric functions

- **Prove the Pythagorean Identity** $\sin^2\theta + \cos^2\theta = 1$.
- **Use the unit circle to prove trigonometric identities and relate them to symmetries of the graphs of sine and cosine.**
- **Use the Pythagorean Identity to calculate trigonometric ratios.**

In this section, which includes some (+) standards, students explore further the consequences of the unit circle definition of sine and cosine. They make a connection between the Pythagorean theorem. Then, they see how the symmetries of the circle give rise to symmetries of the graphs of sine and cosine, and represent these symmetries as identities.

Tasks

F-TF Trigonometric Ratios and the Pythagorean Theorem
F-TF Properties of Trigonometric Functions
F-TF, G-CO, Trigonometric Identities and Rigid Motions
F-TF Special Triangles 1

M3.5.5 Summative Assessment

Assess students' ability to

- **understand radians and angles on a unit circle;**
- **graph basic and transformed trigonometric functions;**
- **compare trigonometric functions algebraically and graphically;**
- **interpret and graph data within a given context.**

Tasks

F-TF Foxes and Rabbits 2

M3.6 Inferences

• Understand that statistical methods are used to draw conclusions from data.
• Understand that the validity of data-based conclusions depends on the quality of the data and how the data were collected.
• Critique and evaluate data-based claims that appear in popular media.
• Distinguish between observational studies, surveys and experiments.
• Explain why random selection is important in the design of observational studies and surveys.
• Explain why random assignment is important in the design of statistical experiments.
• Calculate and interpret the standard deviation as a measure of variability.
• Use the normal distribution as a model for data distributions that are approximately symmetric and bell-shaped.
• Use the least squares regression line to model linear relationships in bivariate numerical data.
• Understand sampling variability in the context of estimating a population or a population mean.
• Use data from a random sample to estimate a population proportion.
• Use data from a random sample to estimate a population mean.
• Calculate and interpret margin of error in context.
• Understand the relationship between sample size and margin of error.
• Given data from a statistical experiment, create a randomization distribution.
• Use a randomization distribution to determine if there is a significant difference between two experimental conditions.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of “statistical significance” as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

This unit builds on the foundation of units S1 and S2, as well as students’ work with statistics in grade 7. In particular, the concepts of sampling variability and distributions introduced in earlier units are critical to understanding the process of drawing conclusions from data, which is central to this unit. In this unit, students revisit and extend their understanding of variability in data and of ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability. Normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. Students calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students explore the reasoning that allows them to draw conclusions based on data from statistical studies. They learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Students develop the understanding that such estimates are subject to sampling variability. The notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of “statistical significance” as they use data from statistical experiments to determine if there is a significant difference between experimental conditions. The standards addressed in this unit (especially S-IC.B.4 and S-IC.B.5) are conceptually complex and will require several weeks to fully develop. Together with units S1, S2, and S3, this unit provides a capstone experience in statistics for grades 6–12 and a solid foundation for an AP Statistics course or a college level introductory statistics course.

M3.6.0 Pre-unit diagnostic assessment

Diagnose students' ability to

• construct and interpret a graphical display;
• calculate and interpret a sample proportion;
• calculate and interpret a sample mean;
• fit a line to bivariate data.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of "statistical significance" as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

M3.6.1 Drawing reasonable conclusions

• Understand that statistical methods are used to draw conclusions from data.
• Understand that the validity of data-based conclusions depends on the quality of the data and how the data were collected.

• Critique and evaluate data-based claims that appear in popular media.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of "statistical significance" as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

M3.6.2 Collecting data and types of statistical studies

• Distinguish between observational studies, surveys and experiments.
• Explain why random selection is important in the design of observational studies and surveys.
• Explain why random assignment is important in the design of statistical experiments.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of "statistical significance" as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

M3.6.3 Describing data distributions

• Calculate and interpret the standard deviation as a measure of variability.
• Use the normal distribution as a model for data distributions that are approximately symmetric and bell-shaped.
• Use the least squares regression line to model linear relationships in bivariate numerical data.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of “statistical significance” as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

M3.6.4 Mid-Unit Assessment

Assess students’ ability to

• given a description of a statistical study, identify the study type (observational study, survey, or experiment);
• determine what type of statistical study would produce data that could be used to answer a given question;
• distinguish between data distributions for which it would be reasonable to use the normal distribution as a model and those for which it would not be reasonable;
• find an area under a normal curve and interpret it in the context of modeling a data distribution.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of “statistical significance” as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

M3.6.5 Drawing conclusions based on data from a random sample

• Understand sampling variability in the context of estimating a population or a population mean.
• Use data from a random sample to estimate a population proportion.
• Use data from a random sample to estimate a population mean.
• Calculate and interpret margin of error in context.
• Understand the relationship between sample size and margin of error.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of "statistical significance" as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

M3.6.6 Drawing conclusions based on data from a statistical experiment

• Given data from a statistical experiment, create a randomization distribution.
• Use a randomization distribution to determine if there is a significant difference between two experimental conditions.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of "statistical significance" as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.

M3.6.7 Summative Assessment

Assess students' ability to

• distinguish between an observational study and an experiment;
• use an appropriate normal distribution to model a data distribution;
• estimate a population proportion and interpret a margin of error in context;
• given a simulated sampling distribution, estimate a margin of error;
• given data from a statistical experiment, create a randomization distribution and use it to determine if there is a significant difference between experimental conditions.

In this unit students revisit and extend their understanding of variability in data and of the ways to describe variability in data. Students calculate and interpret the standard deviation, first introduced at a conceptual level in unit S1, as a way to quantify variability. Students use distributions to describe variability and normal distributions are introduced as a way to model data distributions that are bell-shaped and approximately symmetric. They calculate and interpret areas under a normal curve in the context of modeling a data distribution. Students also explore the reasoning that allows them to draw conclusions based on data from statistical studies and learn the distinction between an observational study and a statistical experiment. They then use data from a random sample from a population to estimate the values of population characteristics such as a population mean or a population proportion. Of particular importance in this unit, students develop the understanding that such estimates are subject to sampling variability, and the notion of margin of error is introduced as a way of quantifying the uncertainty associated with an estimate of a population characteristic. Students also explore the important idea of "statistical significance" as they use data from statistical experiments to determine if there is a significant difference between experimental conditions.