APPENDIX

**Derivation of Equation (3)** From the geometry of Fig. 2 we have:

\[
\mathbf{n}(d, r) = \sin \theta_o \cdot \mathbf{u} + \cos \theta_o \cdot \mathbf{o}
\]

\[
= \cos \theta_o (\tan \theta_o \cdot \mathbf{u} + \mathbf{o}).
\]

(9)

(10)

It follows that the vector

\[
\hat{\mathbf{n}}(d, r) = \tan \theta_o \cdot \mathbf{u} + \mathbf{o}
\]

(11)

is in the direction of the surface normal. To obtain an expression for this vector in terms of the incoming and outgoing light directions, we use Snell’s law. Let \(\theta_\delta = \theta_o - \theta_i\) be the angle between these two directions. Then we have

\[
\sin \theta_o = \frac{r \sin \theta_i}{r}
\]

\[
\sin \theta_o = r \sin(\theta_o - \theta_\delta)
\]

\[
\sin \theta_o = r(\sin \theta_o \cos \theta_\delta - \cos \theta_o \sin \theta_\delta)
\]

\[
\tan \theta_o = \frac{r \sin \theta_\delta}{r \cos \theta_\delta - 1} = \frac{r||\mathbf{i}(d) \wedge \mathbf{o}||}{r(\mathbf{o} \cdot \mathbf{i}(d)) - 1}.
\]

(12)

(13)

(14)

(15)

**Proof of Theorem 1** The proof uses two basic intuitions. First, given an arbitrary value for the refractive index, each viewpoint can be thought of as defining a 3D constraint curve, representing all assignments of distances and normals to a pixel \(q\) that are compatible with Snell’s law. Hence, an assignment that is consistent with both viewpoints corresponds to the intersection of two such constraint curves. Second, and most important, for an arbitrarily-shaped surface, these 3D curves will be in general position with respect to each other and, therefore, will not have a common intersection. From these two facts we conclude that when the refractive index has an arbitrary value, there will be no distance and normal assignment that is consistent with both viewpoints. Hence, such consistency can only be achieved for isolated refractive index values. We formalize these intuitions below.

Let \(r^*\) be the true refractive index of the surface and let \(r \neq r^*\) be an arbitrary value of this index. Without loss of generality, we assume that the function \(C(q)\) is known for all pixels \(q\) and is continuous. Let \(q\) be an arbitrary pixel in the first viewpoint and let \(d^*\) be its true distance to the surface. Given value \(r\) for the refractive index, every distance \(d\) defines a unique normal, \(\mathbf{n}(d, r)\), compatible with Snell’s law (Fig. 3 and Eq. (3)). Suppose we represent unit vectors with
two angles: an angle $\theta$, corresponding to the angle between the vector and the ray through $q$; and an angle $\phi$, corresponding to the angle between the vector and the normal of $q$’s refraction plane. In this representation, the distance and normal assignments to $q$ that are compatible with Snell’s law define a curve $\gamma$ in $(d, \theta, \phi)$-space. This curve will always lie on the plane $\phi = 0$ since, by definition, the normal $n(d, r)$ always lies on the refraction plane of pixel $q$.

Now let $q'(d)$ be the projection of $p(d)$ in the second viewpoint (Fig. 1). Since $C(q'(d))$ is known, there is only one normal, $n'(d, r)$, that can be assigned to $p(d)$ and is compatible with Snell’s law in the second viewpoint. This normal will lie on the refraction plane of pixel $q'$. Generically, the refraction planes of pixels $q$ and $q'$ are distinct. Hence, the normal $n'(d, r)$ may not lie on the refraction plane of pixel $q$ and, as $d$ varies, $n'(d, r)$ will trace a general curve $\gamma'$ in $(d, \theta, \phi)$-space, i.e., a curve that is not restricted to the plane $\phi = 0$.

We now show that $\gamma$ and $\gamma'$ do not intersect. First note that the two curves cannot intersect in the neighborhood of the “true” distance $d^*$ because $n'(d^*, r) \neq n(d^*, r)$.

Now consider distances away from $d^*$. We show that $\gamma'$ and $\gamma$ generically will not intersect there either. In particular, the normal $n'(d, r)$ is completely determined by point $C(q'(d))$ which, in turn, is determined by the normal of the true surface point projecting to pixel $q'(d)$. Since $q'(d)$ lies on the epipolar line of $q$ for all values of $d$, it follows that curve $\gamma'$ is completely determined by the surface normal of points at the intersection, $C$, of this epipolar plane with the true surface. For $\gamma'$ and $\gamma$ to intersect there must be a point on $C$ outside the neighborhood of $p(d^*)$ whose surface normal is identical to $p(d^*)$. This condition, however, cannot be satisfied for an open 2D set of points on a generic surface. It follows that $\gamma'$ and $\gamma$ are non-intersecting for almost all points on the surface and, hence, for almost all pixels in the surface’s projection.

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4Observe that $n'(d^*, r^*) = n(d^*, r^*)$ since Snell’s law is satisfied for both viewpoints in the true scene. Now, since there is a 1-1 correspondence between refractive indices and normals when $d^*$ is fixed, and since the refraction planes of $q$ and $q'(d)$ have only one normal in common, it follows that $n'(d^*, r) \neq n(d^*, r)$.