APPENDIX

The computation of the force feedback includes a subtlety: the computation of $\lambda$ used in equation (16) is submitted to delay artifact that should be removed in order to have a transparent haptic feedback. To explain this, we will reuse the example described by Fig. 8 where three virtual mechanisms (including the coupling) are modeled. After the update of the constraint problem solution in the haptic loop, we solve the constraints using equation (11):

\[
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_C = 0
\end{bmatrix}
= \begin{bmatrix}
W_{11} & \ldots & \text{sym} \\
W_{21} & W_{22} & \ldots \\
W_{C1} & W_{C2} & W_{CC}
\end{bmatrix}
\begin{bmatrix}
\Delta \lambda_1 \\
\Delta \lambda_2 \\
\Delta \lambda_C
\end{bmatrix}
+ \begin{bmatrix}
\delta_1^p \\
\delta_2^p \\
\delta_C^p
\end{bmatrix}
\]

The problem is that the value of $\delta_C^p$ reflects the difference of position between the position of the spring $x_s$ and the position of the needle (more exactly the position of the syringe on the figure) in the simulation. Due to the delay (the simulation is running at low rates), this positional difference is not always negligible. Yet, in the haptic loop, it is necessary to take into account this modification of $\delta_C^p$ (using equation 17) in order to evaluate correctly $\lambda_1$ and $\lambda_2$. Note that these values are modified, because the non-diagonal terms like $[W_{C1}]$ and $[W_{C2}]$ are non-zero. But when it comes to sending the force to the device, this should only reflect the equilibrium with values $\lambda_1$ and $\lambda_2$. So, the force that has been used to correct the positional difference should be removed (somehow, we should consider that we have always $\delta_C^p = 0$). For that, we exploit the fact that the compliance represented by matrix $[W]$ (especially in its non-diagonal terms) reflects this equilibrium. Thus, given the value of $\lambda_1$ and $\lambda_2$, we compute:

\[
f_h = -\lambda_C = -[W_{CC}]^{-1} ([W_{C1}]\lambda_1 + [W_{C2}]\lambda_2)
\]

We emphasize that the values of needle compliance are calculated exactly (by a direct solver instead of the warping approximation) to obtain accurate results. Note that this approach also removes the gravity forces exerted on the needle from the haptic rendering. If this force is to be rendered, it can be easily added separately.

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1. $\delta_C = 0$ because the virtual fixed joint mechanism imposes a null difference of position between objects. Matrix $[W]$ is symmetric.

2. Here the use of the term equilibrium refers to a static view of the problem. From dynamics point of view we could talk about a instantaneous coupling between the base of the needle and the forces that are applied on the other virtual mechanisms.