Performance Analysis of Cloud Computing Centers Using $M/G/m/m + r$ Queueing Systems: Supplementary Materials

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APPENDIX A
ON THE STRUCTURE OF EC2 SERVICES

The Amazon Elastic Compute Cloud (EC2) offers three types of services [1]: Reserved services are reserved and paid for in advance, so they are guaranteed by the provider and thus experience virtually no queueing; Spot services are also allocated in advance, but they go to the customer who offer a higher bid than Spot price set by Amazon; Spot Prices fluctuate periodically depending on the supply of and demand for Spot Instance capacity. Finally, On-Demand services provide no advance reservations and no long term commitment, which is why the clients’ tasks may experience non-negligible queueing delays and, possibly, blocking. Within the on-demand category, a request may target a specific infrastructure instance, a platform, or a software application, with probability of $\alpha$, $\beta$, and $\theta$, respectively (see Fig. 7); the same story is held within each tuple. Because we assumed separate arrivals of individual task requests, all the probabilities, $\alpha$, $\beta$, and $\theta$, as well as branching probabilities within each tuple are independent of each other. Assuming that the service time for each type of request (tuple) follows a simple exponential or Erlang distribution, the aggregate service time of the cloud center would follow a hyper-exponential or hyper-Erlang distribution – one in which the coefficient of variation, CoV, defined as the ratio of standard deviation and mean value, exceeds one [2]. Consequently, even in the above-mentioned scenario, which is a rather simple one, only a general assumption of distribution can substantiate the actual task service time; in practical calculations, we will have to choose a specific probability distribution that allows widely varying values for the coefficient of variation CoV.

APPENDIX B
PROOF OF ERGODICITY OF THE AMGM CHAIN

Let $S = \{E_0, E_1, E_2, \ldots, E_{m+r}\}$ be the set of all states in the Markov chain, and let $p_{ij}(n)$ indicate the probability of transition from state $i$ to $j$ in exactly $n$ steps. Also, let $f_i(n)$ denote the probability that first return to state $i$ occurs in exactly $n$ steps after leaving state $i$. Then, the probability of ever returning to state $i$ is

$$f_i = \sum_{n=1}^{\infty} f_i(n) \quad (19)$$

Using this notation, we can classify states in the Markov chain as follows.

A state $i$ is called recurrent or persistent if $f_i = 1$, and transient otherwise. Considering states for which $f_i = 1$, we may then define the mean recurrence time of $i$ as

$$M_i \triangleq \sum_{n=1}^{\infty} nf_i(n) \quad (20)$$

which is merely the average time to return to $i$, then, recurrent states may be classified on the basis of their mean recurrence time:

Recurrent state $i$ is called null if and only if $p_{ii}(n) \to 0$ as $n \to 0$; if this holds, then $p_{ji}(n) \to 0$ for all $j$.

**Theorem B.1: (Nullity)** Recurrent state $i$ is null if and only if $p_{ii}(n) \to 0$ as $n \to 0$; if this holds, then $p_{ji}(n) \to 0$ for all $j$.

**Proof:** See [3].

Let us now define the period $d(i)$ of a state $i$ as the greatest common divisor of the epochs at which return is possible: $d(i) = \gcd\{n : p_{ii}(n) > 0\}$. Then,
State $i$ is called periodic if $d(i) > 1$, and aperiodic otherwise, i.e., if $d(i) = 1$.

Finally, we define communicability as follows.

State $i$ communicates with $j$, written $i \rightarrow j$, if the chain may ever visit state $j$ with positive probability, starting from $i$. That is, $i \rightarrow j$ if $p_{ij}(n) > 0$ for some $n \geq 0$. We say $i$ and $j$ intercommunicate if $i \rightarrow j$ and $j \rightarrow i$, in which case we write $i \leftrightarrow j$.

It can be seen that $\leftrightarrow$ is an equivalence relation, hence the state space $S$ can be partitioned into the equivalence classes of $\leftrightarrow$; within each equivalence class all states are of the same type.

A set $C$ of states is called

(a) closed, if $p_{ij} = 0$ for all $i \in C$, $j \notin C$.
(b) irreducible, if $i \leftrightarrow j$ for all $i$, $j \in C$.

Finally, we can define ergodicity of a Markov chain as follows:

A Markov chain is called ergodic if it is irreducible, recurrent non-null, and aperiodic.

We will now show that aMGM chain satisfies these properties.

**Lemma B.2:** aMGM chain is irreducible.

*Proof:* aMGM is a finite Markov chain with state space of $S$. All we need to show is that $S$ is closed and does not include any proper close subset. Without loss of generality we examine the state $m + r$ in aMGM chain in order to establish the communicability class of the $m + r$. Since there is direct communication between state $m + r$ and all other states with a non-zero probability, we have:

$$\forall k \in S \Rightarrow m + r \rightarrow k$$

Now we consider $m + r - 1$; this state can communicate with $m + r$ with the probability of $p_{m+r-1,m+r}>0$; so we can write $m + r - 1 \rightarrow m + r$. Therefore $m + r - 1 \leftrightarrow m + r$ is held. Consequently with the same reasoning we have:

$$\{m + r - 2 \leftrightarrow m + r - 1\}, \{m + r - 3 \leftrightarrow m + r - 2\}, \{m + r - 4 \leftrightarrow m + r - 3\}, \ldots ; \{0 \leftrightarrow 1\}$$

Thus

$$\forall k \in S \Rightarrow m + r \leftrightarrow k$$

As a result, we just showed that $S$ is closed and does not include any proper closed subset. So aMGM chain is irreducible. □

**Lemma B.3:** aMGM chain is recurrent non-null.

*Proof:* First, we show that aMGM chain is recurrent. Based on recurrence definition we need to show

$$\forall k \in S, f_k = 1$$

Fig. 7. Amazon EC2 Structure, adapted from [1].
By induction on system capacity, we prove above statement. For the basis of induction, \( k = 1 \), aMGM chain, Fig. 8, has the minimum capacity and the following holds:

\[
P_{00} + p_{01} = 1 \quad \text{and} \quad p_{11} + p_{10} = 1
\]

Fig. 8. aMGM chain for Minimal Capacity (\( k=1 \)).

Now we examine \( f_1 \):

\[
f_1 = p_{11} + p_{10} \sum_{i=0}^{\infty} i! 0^i p_{00}^i = p_{11} + p_{10} \sum_{i=0}^{\infty} i! (1 - p_{00})
\]

\[
= p_{11} + p_{10} \sum_{i=0}^{\infty} i! - p_{00}^{i+1} = p_{11} + p_{10} \Rightarrow f_1 = 1
\]

With similar reasoning we can show that \( f_0 = 1 \) as well.

Now we assume that for \( k = z \) we have

\[
f_0 = f_1 = \ldots = f_z = 1
\]

Then we need to show that for \( k = z + 1 \) following is held:

\[
f_0 = f_1 = \ldots = f_z = f_{z+1} = 1
\]

For this configuration the aMGM chain would be exactly the same with Fig. 2 (in the main text) but here the last state is \( z + 1 \). We can merge all the states \{0, 1, \ldots, z\} to one state named \( z^* \); then accordingly aMGM chain would be composed of two states, Fig. 9, in which:

\[
\begin{align*}
p_{z+1, z^*} &= \sum_{i=0}^{z} p_{z+1, i} \\
p_{z^*, z^*} &= \sum_{i=0}^{z} p_{ii}
\end{align*}
\]

And

\[
\begin{align*}
p_{z+1, z^*+1} + p_{z+1, z^*} &= 1 \\
p_{z^*, z^*+1} + p_{z^*, z^*} &= 1
\end{align*}
\]

Fig. 9. Merged aMGM chain in Two States.

Like the basis of induction, \( k = 1 \), we can show that:

\( f_{z+1} = 1 \) \quad \text{and} \quad \( f_{z^*} = 1 \).

So far, we have shown that aMGM chain is recurrent. In order to show that it is non-null we need to have:

\[
\mathbb{M}_i = \sum_{n=1}^{\infty} n f_i^{(n)} < \infty
\]

In other words, the mean return time for any states should be a finite quantity.

**Lemma B.4:** (Ratio Test) Series \( \sum_{n=0}^{\infty} a_n \) is absolutely convergent if \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \) [4].

Having lemma B.4 in mind, we can proceed:

\[
\begin{align*}
\lim_{n \to \infty} \left| \frac{(n + 1) f_i^{(n+1)}}{n f_i^{(n)}} \right| &= \lim_{n \to \infty} \left| \frac{n+1}{n} \cdot \lim_{n \to \infty} \frac{f_i^{(n+1)}}{f_i^{(n)}} \right| \\
&= 1 \cdot \lim_{n \to \infty} \left| \frac{f_i^{(n+1)}}{f_i^{(n)}} \right| < 1
\end{align*}
\]

The last step is straight forward since

\[
p_{ii}(n+1) < p_{ii}(n)
\]

Therefore for each state, \( i \), \( \mathbb{M}_i < \infty \). As a result aMGM chain is both recurrent and non-null.

**Lemma B.5:** aMGM chain is aperiodic.

**Proof:** All states in aMGM chain have a self loop with a non-zero probability; that is

\[
\forall k \in S, \ p_{kk} > 0 \Rightarrow d(k) = 1
\]

This means all the states are aperiodic. Consequently aMGM chain is aperiodic.

**Theorem B.6:** aMGM chain is ergodic.

**Proof:** Based on Lemmas B.2, B.3, B.5 and ergodicity definition.

**REFERENCES**


