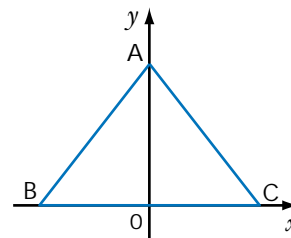


### 3.4 Reflections of Functions

A coordinate grid is superimposed on a cross section of the Great Pyramid, so that the  $y$ -axis passes through the vertex of the pyramid. The  $x$ -axis bisects two opposite sides of the square base. The two sloping lines in the cross section,  $AB$  and  $AC$ , are altitudes of two triangular faces of the pyramid.



#### Web Connection

[www.school.mcgrawhill.ca/resources/](http://www.school.mcgrawhill.ca/resources/)  
To find more information about the Great Pyramid and other Egyptian pyramids, visit the above web site. Go to **Math Resources**, then to *MATHEMATICS 11*, to find out where to go next. Write a brief report about the construction techniques used.

#### INVESTIGATE & INQUIRE

1. Copy and complete each table of values and sketch each pair of graphs on the same set of axes.

a)  $y = \sqrt{x}$

x	y
0	
1	
4	
9	
16	

$y = -\sqrt{x}$

x	y
0	
1	
4	
9	
16	

b)  $y = x^2$

x	y
-2	
-1	
0	
1	
2	

$y = -x^2$

x	y
-2	
-1	
0	
1	
2	

2. In question 1a), for equal values of the  $x$ -coordinates, how do the  $y$ -coordinates of  $y = \sqrt{x}$  compare with the  $y$ -coordinates of  $y = -\sqrt{x}$ ? How are the graphs the same? How are they different?

3. In question 1b), for equal values of the  $x$ -coordinates, how do the  $y$ -coordinates of  $y = x^2$  compare with the  $y$ -coordinates of  $y = -x^2$ ? How are the graphs the same? How are they different?

4. Make a conjecture about the relationship between the graphs of  $y = f(x)$  and  $y = -f(x)$ .

5. Test your conjecture by graphing each of the following pairs of graphs on a graphing calculator.

a)  $y = \sqrt{x-1}$  and  $y = -\sqrt{x-1}$

b)  $y = (x+3)^2$  and  $y = -(x+3)^2$

6. Copy and complete each table of values and sketch each pair of graphs on the same set of axes.

a)  $y = x + 2$

$y = (-x) + 2$

b)  $y = 2x - 4$

$y = 2(-x) - 4$

x	y
-2	
-1	
0	
1	
2	

x	y
-2	
-1	
0	
1	
2	

x	y
-2	
-1	
0	
1	
2	

x	y
-2	
-1	
0	
1	
2	

7. In question 6a), for equal values of the  $y$ -coordinates, how do the  $x$ -coordinates of  $y = x + 2$  compare with the  $x$ -coordinates of  $y = (-x) + 2$ ? How are the graphs the same? How are they different?

8. In question 6b), for equal values of the  $y$ -coordinates, how do the  $x$ -coordinates of  $y = 2x - 4$  compare with the  $x$ -coordinates of  $y = 2(-x) - 4$ ? How are the graphs the same? How are they different?

9. Make a conjecture about the relationship between the graphs of  $y = f(x)$  and  $y = f(-x)$ .

10. Test your conjecture from question 9 by graphing each of the following pairs of graphs on a graphing calculator.

a)  $y = \sqrt{x}$  and  $y = \sqrt{-x}$

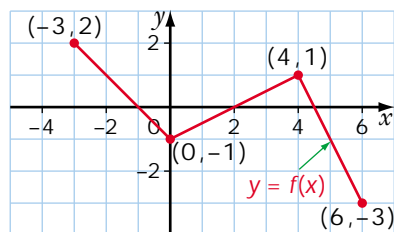
b)  $y = (x-4)^2$  and  $y = (-x-4)^2$

11. In the cross section of the Great Pyramid, one altitude can be modelled by the equation  $y = 1.27x + 146$ . Which altitude, AB or AC, can be modelled by this equation? Explain.

12. What is the equation of the other altitude? Explain.

### EXAMPLE 1 Comparing $y = f(x)$ and $y = -f(x)$

- a) Given the graph of  $y = f(x)$ , as shown, graph  $y = -f(x)$  on the same axes.
- b) Describe how the graph of  $y = -f(x)$  is related to the graph of  $y = f(x)$ .



### SOLUTION

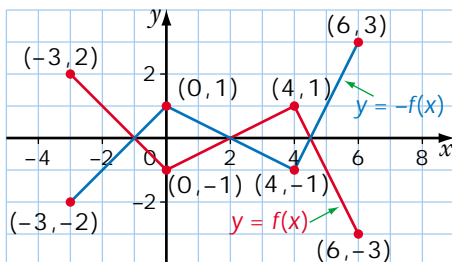
- a) Use the given graph to complete a table of values for the function  $y = f(x)$ . Then, complete a table of values for the function  $y = -f(x)$ , and draw the graph.

$$y = f(x)$$

x	y
-3	$f(-3) = 2$
0	$f(0) = -1$
4	$f(4) = 1$
6	$f(6) = -3$

$$y = -f(x)$$

x	y
-3	$-f(-3) = -2$
0	$-f(0) = 1$
4	$-f(4) = -1$
6	$-f(6) = 3$



- b) The graphs of  $y = f(x)$  and  $y = -f(x)$  are congruent.

If  $y = -f(x)$ , then  $y = -1f(x)$ , so each  $y$ -value on the graph of  $y = -f(x)$  is the corresponding  $y$ -value on the graph of  $y = f(x)$  multiplied by  $-1$ .

The point  $(x, y)$  on the graph of the function  $y = f(x)$  becomes the point  $(x, -y)$  on the graph of  $y = -f(x)$ . For example,  $(-3, 2)$  becomes  $(-3, -2)$ , and  $(6, -3)$  becomes  $(6, 3)$ .

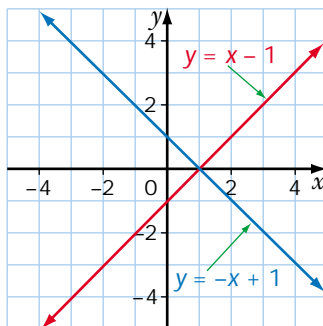
The graphs have the same  $x$ -intercepts.

The graph of  $y = -f(x)$  is a reflection of the graph of  $y = f(x)$  in the  $x$ -axis.

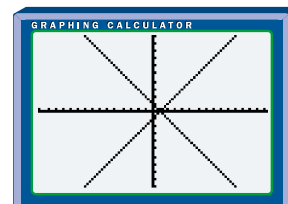
As noted in Example 1, the two graphs have the same  $x$ -intercepts. Points that lie on the  $x$ -axis have a  $y$ -coordinate of 0, so they are unaltered by the transformation of  $y = f(x)$  to  $y = -f(x)$ . Points that are unaltered by a transformation are said to be **invariant**.

In general, the point  $(x, y)$  on the graph of the function  $y = f(x)$  becomes the point  $(x, -y)$  on the graph of  $y = -f(x)$ . The graph of  $y = -f(x)$  is a reflection of the graph of  $y = f(x)$  in the  $x$ -axis. Points that lie on the  $x$ -axis are invariant, because their  $y$ -coordinate is 0.

If, in Example 1,  $y = f(x)$  were defined by an equation, such as  $f(x) = x - 1$ , then  $y = -f(x)$  would be defined by the equation  $y = -(x - 1)$  or  $y = -x + 1$ . The graph of  $y = -f(x)$  would be a reflection of the graph of  $y = f(x)$  in the  $x$ -axis. The  $x$ -intercepts of the two graphs would be the same. There would be one invariant point,  $(1, 0)$ .



The window can be adjusted using the ZSquare instruction in the ZOOM menu.



### EXAMPLE 2 Comparing $y = f(x)$ and $y = f(-x)$

Let  $f(x) = 2x + 1$ .

- Write an equation for  $f(-x)$ .
- Graph  $y = f(x)$  and  $y = f(-x)$  on the same axes or in the same viewing window of a graphing calculator.
- Describe how the graph of  $y = f(-x)$  is related to the graph of  $y = f(x)$ .

#### SOLUTION

- Substitute  $-x$  for  $x$  in  $2x + 1$ .

$$2(-x) + 1 = -2x + 1$$

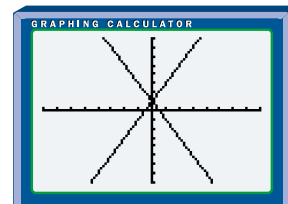
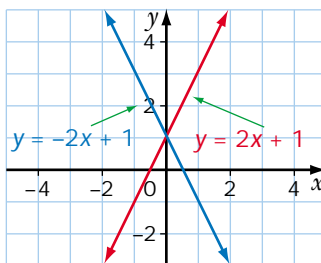
$$\text{so } f(-x) = -2x + 1$$

- Graph both functions using paper and pencil or a graphing calculator.

$$y = 2x + 1$$

$$y = -2x + 1$$

x	y	x	y
-2	-3	2	-3
-1	-1	1	-1
0	1	0	1
1	3	-1	3
2	5	-2	5



c) The graphs of  $y = f(x)$  and  $y = f(-x)$  are congruent.

The point  $(x, y)$  on the graph of  $y = 2x + 1$  becomes the point  $(-x, y)$  on the graph of  $y = -2x + 1$ . For example, the point  $(2, 5)$  becomes the point  $(-2, 5)$ . The point  $(-1, -1)$  becomes the point  $(1, -1)$ .

The graphs have the same  $y$ -intercept.

The graph of  $y = -2x + 1$  is the graph of  $y = 2x + 1$  reflected in the  $y$ -axis.

Note that there is one invariant point,  $(0, 1)$ . For both linear functions, the domain and range are the set of real numbers.

---

In general, the point  $(x, y)$  on the graph of the function  $y = f(x)$  becomes the point  $(-x, y)$  on the graph of  $y = f(-x)$ . The graph  $y = f(-x)$  is a reflection of the graph of  $y = f(x)$  in the  $y$ -axis. Points that lie on the  $y$ -axis are invariant, because their  $x$ -coordinate is 0.

The results for reflecting graphs of functions in the  $x$ -axis and the  $y$ -axis can be summarized as follows.

#### Reflection in the $x$ -axis

The graph of  $y = -f(x)$  is the graph of  $y = f(x)$  reflected in the  $x$ -axis.

#### Reflection in the $y$ -axis

The graph of  $y = f(-x)$  is the graph of  $y = f(x)$  reflected in the  $y$ -axis.

---

### EXAMPLE 3 Reflecting a Radical Function

If  $f(x) = \sqrt{x-2}$ , write an equation to represent each of the following functions, describe how the graph of each function is related to the graph of  $y = f(x)$ , sketch each graph, state the domain and range, and identify any invariant points.

a)  $y = -f(x)$       b)  $y = f(-x)$

#### SOLUTION

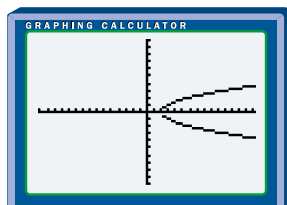
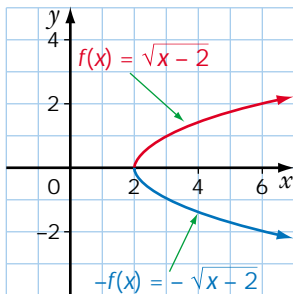
a) For  $y = f(x)$       For  $y = -f(x)$   
 $y = \sqrt{x-2}$        $y = -\sqrt{x-2}$

The graph of  $y = -f(x)$  is the graph of  $y = f(x)$  reflected in the  $x$ -axis. Both graphs have the same  $x$ -intercept, 2.

For  $y = f(x)$ , the domain is  $x \geq 2$  and the range is  $y \geq 0$ .

For  $y = -f(x)$ , the domain is  $x \geq 2$  and the range is  $y \leq 0$ .

There is one invariant point,  $(2, 0)$ .



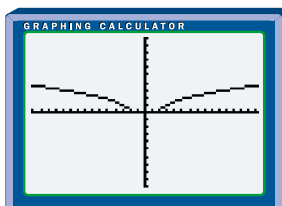
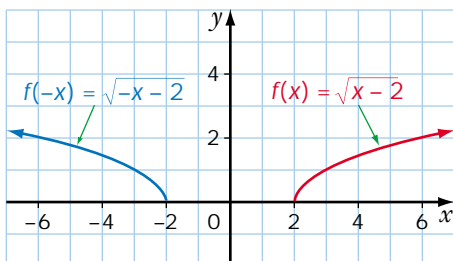
b) For  $y = f(x)$       For  $y = f(-x)$   
 $y = \sqrt{x-2}$        $y = \sqrt{-x-2}$

The graph of  $y = f(-x)$  is the graph of  $y = f(x)$  reflected in the  $y$ -axis.

For  $y = f(x)$ , the domain is  $x \geq 2$  and the range is  $y \geq 0$ .

For  $y = f(-x)$ , the domain is  $x \leq -2$  and the range is  $y \geq 0$ .

There are no invariant points.



#### EXAMPLE 4 Reflecting a Quadratic Function

If  $f(x) = x^2 + 1$ ,

a) write an equation for  $y = -f(x)$  and  $y = f(-x)$

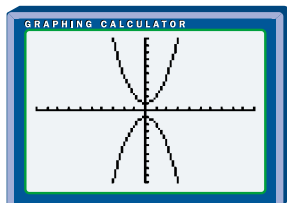
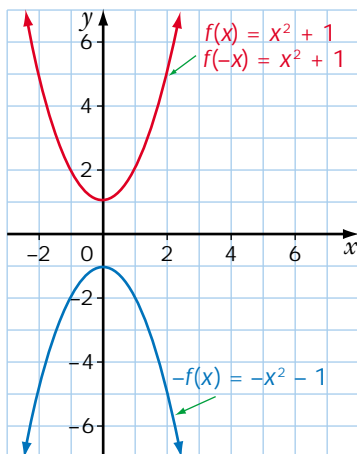
b) describe how the graph of each equation from part a) is related to the graph of  $y = f(x)$ , and sketch the three graphs on the same axes

#### SOLUTION

a) For  $y = f(x)$       For  $y = -f(x)$   
 $y = x^2 + 1$        $y = -x^2 - 1$

For  $y = f(x)$       For  $y = f(-x)$   
 $y = x^2 + 1$        $y = (-x)^2 + 1$   
 $y = x^2 + 1$        $y = x^2 + 1$

b) The graph of  $y = -f(x)$  is the graph of  $y = f(x)$  reflected in the  $x$ -axis. Since the graph of  $y = x^2 + 1$  is symmetrical about the  $y$ -axis, replacing  $x$  with  $-x$  has no effect on the graph. So, the graphs of  $y = f(x)$  and  $y = f(-x)$  are the same.



### EXAMPLE 5 Reflecting a Quadratic Function

If  $f(x) = x^2 + 6x$ , write an equation for each of the following functions, describe how the graph of each function is related to the graph of  $y = f(x)$ , sketch each graph, and identify any invariant points.

- a)  $y = -f(x)$       b)  $y = f(-x)$

### SOLUTION

a) For  $y = f(x)$ ,  $y = x^2 + 6x$ .

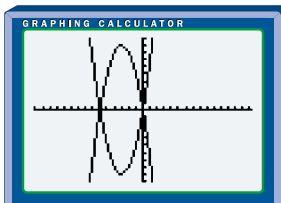
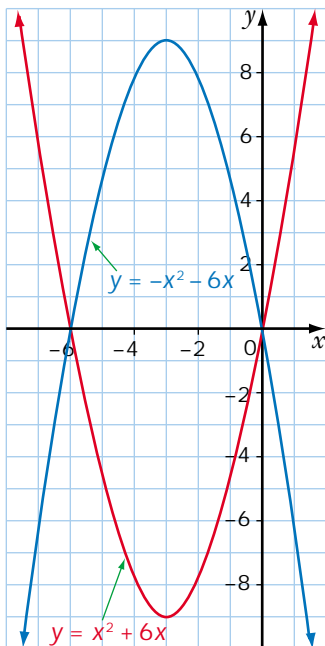
The graph of  $y = x^2 + 6x$  is a parabola that opens up.

Since  $x^2 + 6x = 0$  has roots  $x = 0$  and  $x = -6$ , the  $x$ -intercepts of  $y = x^2 + 6x$  are 0 and  $-6$ .

The vertex lies on the axis of symmetry, so the  $x$ -coordinate of the vertex is  $-3$ .

$$\begin{aligned} \text{When } x = -3, y &= (-3)^2 + 6(-3) \\ &= 9 - 18 \\ &= -9 \end{aligned}$$

So, the coordinates of the vertex are  $(-3, -9)$ .



Sketch the graph of  $y = f(x)$  using the points  $(0, 0)$ ,  $(-6, 0)$ , and  $(-3, -9)$ .

$$\begin{aligned} \text{For } y &= -f(x) \\ y &= -(x^2 + 6x) \\ &= -x^2 - 6x \end{aligned}$$

The graph of  $y = -f(x)$  is the graph of  $y = f(x)$  reflected in the  $x$ -axis.

For  $y = f(x)$ , the domain is all real numbers and the range is  $y \geq -9$ .

For  $y = -f(x)$ , the domain is all real numbers and the range is  $y \leq 9$ .

Both graphs have the same  $x$ -intercepts,  $0$  and  $-6$ .

There are two invariant points,  $(0, 0)$  and  $(-6, 0)$ .

b) For  $y = f(-x)$   

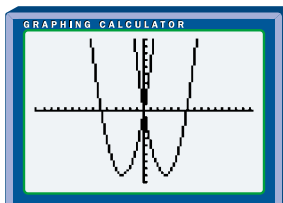
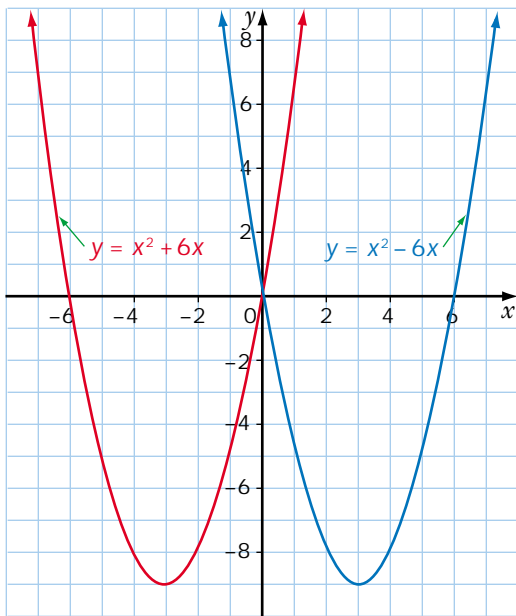
$$y = (-x)^2 + 6(-x)$$

$$= x^2 - 6x$$

The graph of  $y = f(-x)$  is the graph of  $y = f(x)$  reflected in the  $y$ -axis. For both  $y = f(x)$  and  $y = f(-x)$ , the domain is all real numbers and the range is  $y \geq -9$ .

The graphs intersect at  $(0, 0)$ .

There is one invariant point,  $(0, 0)$ .



## Key Concepts

- A function and its reflection image in the  $x$ -axis or  $y$ -axis are congruent.
- The graph of  $y = -f(x)$  is the graph of  $y = f(x)$  reflected in the  $x$ -axis.
- The graph of  $y = f(-x)$  is the graph of  $y = f(x)$  reflected in the  $y$ -axis.
- Points that are unaltered by a transformation are known as invariant points.

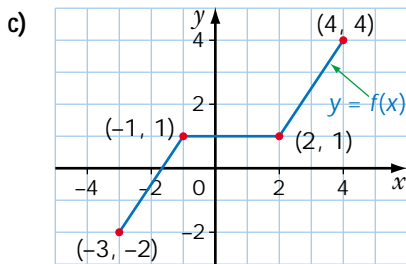
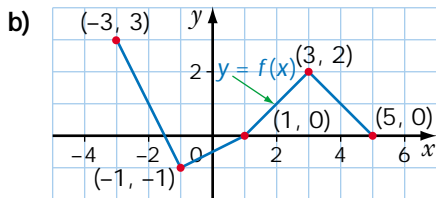
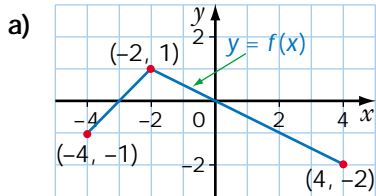
## Communicate Your Understanding

1. If  $f(x) = \sqrt{x-3}$ , describe how you would find the equations of  $y = -f(x)$  and  $y = f(-x)$ .
2. If  $f(x) = x^2 + 8x$ , describe how you would find the equations of  $y = -f(x)$  and  $y = f(-x)$ .
3. If  $f(x) = x^2 - 4x$ , describe how you would find any points that are invariant under a reflection in
  - a) the  $x$ -axis
  - b) the  $y$ -axis

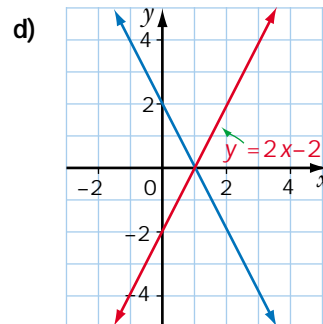
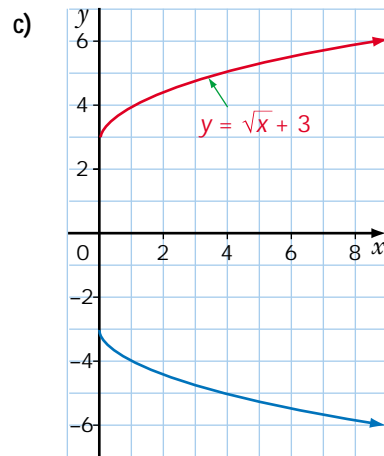
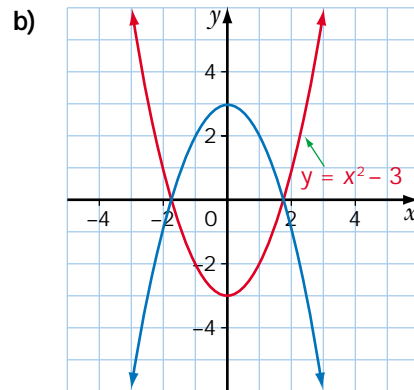
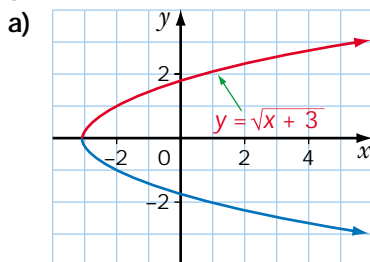
## Practise

**A**

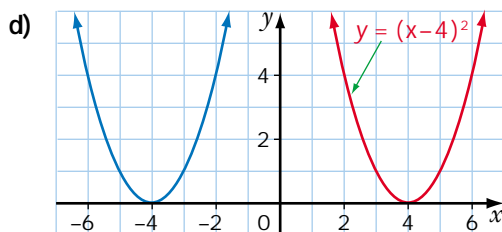
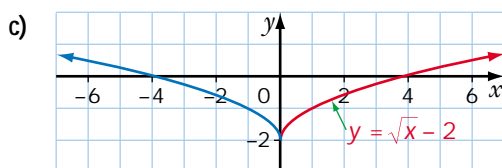
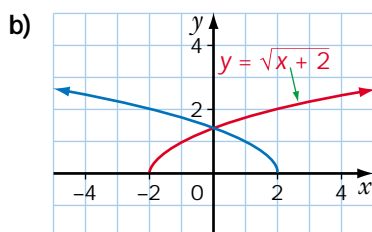
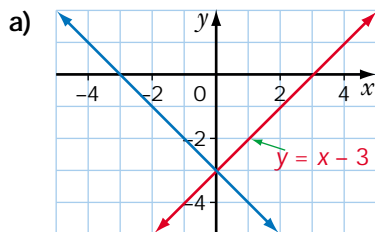
1. Copy the graph of  $y = f(x)$  and draw the graphs of  $y = -f(x)$  and  $y = f(-x)$  on the same axes.



2. The blue graph is a reflection of the red graph in the  $x$ -axis. The equation of the red graph is given. Write the equation of the blue graph.



3. The blue graph is a reflection of the red graph in the  $y$ -axis. The equation of the red graph is given. Write the equation of the blue graph.



4. Graph  $f(x)$  and sketch the specified reflection image.

- $g(x)$ , the reflection of  $f(x) = x + 1$  in the  $x$ -axis
- $h(x)$ , the reflection of  $f(x) = \sqrt{x} + 5$  in the  $y$ -axis
- $g(x)$ , the reflection of  $f(x) = x^2 - 4$  in the  $x$ -axis
- $h(x)$ , the reflection of  $f(x) = \sqrt{x + 5}$  in the  $y$ -axis

5. a) Given  $f(x) = 2x - 4$ , write equations for  $-f(x)$  and  $f(-x)$ .

b) Sketch the three graphs on the same set of axes.

c) Determine any points that are invariant for each reflection.

d) State the domain and range of each function.

6. a) Given  $f(x) = -3x + 2$ , write equations for  $-f(x)$  and  $f(-x)$ .

b) Sketch the three graphs on the same set of axes.

c) Determine any points that are invariant for each reflection.

7. a) Given  $f(x) = x^2 - 4x$ , write equations for  $-f(x)$  and  $f(-x)$ .

b) Sketch the three graphs on the same set of axes.

c) Determine any points that are invariant for each reflection.

8. a) Given  $f(x) = x^2 - 9$ , write equations for  $-f(x)$  and  $f(-x)$ .

b) Sketch the three graphs on the same set of axes.

c) Determine any points that are invariant for each reflection.

9. a) Given  $f(x) = (x + 4)(x - 2)$ , write equations for  $-f(x)$  and  $f(-x)$ .

b) Sketch the three graphs on the same set of axes.

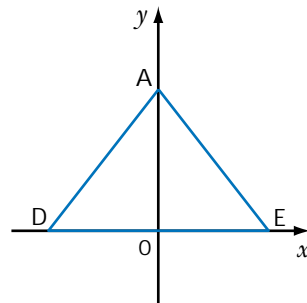
c) State the domain and range of each function.

10. a) Given  $f(x) = \sqrt{x+4}$ , write equations for  $-f(x)$  and  $f(-x)$ .  
 b) Sketch the three graphs on the same set of axes.  
 c) State the domain and range of each function.

11. a) Given  $f(x) = \sqrt{x+4}$ , write equations for  $-f(x)$  and  $f(-x)$ .  
 b) Sketch the three graphs on the same set of axes.  
 c) State the domain and range of each function.

## Apply, Solve, Communicate

12. **Great Pyramid** A coordinate grid is superimposed on a cross section of the Great Pyramid, so that the  $y$ -axis passes through the vertex of the pyramid. Two opposite vertices of the square base are on the  $x$ -axis. The two sloping lines in the cross section, AD and AE, are opposite edges of the pyramid.



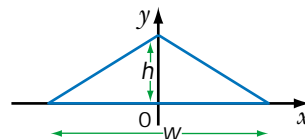
- a) In the cross section, one edge can be modelled by the equation  $y = 0.9x + 146$ . Which edge, AD or AE, can be modelled by this equation? Explain.  
 b) What is the equation of the other edge? Explain.

### B

13. **Communication** How are the graphs of  $y = x^2 - 2x$  and  $y = 2x - x^2$  related. Explain.

14. a) Given  $f(x) = x^2 - x - 6$ , determine the coordinates of the points where the graph crosses the  $y$ -axis and the  $x$ -axis.  
 b) Use the points you found in part a) to sketch the graphs of  $y = -f(x)$  and  $y = f(-x)$ .

15. **Sloping roof** The diagram shows a set of coordinate axes superimposed on the cross section of a sloping roof of height  $h$  metres and width  $w$  metres. The equation of one half of the cross section is  $y = -0.7x + 1.9$ .



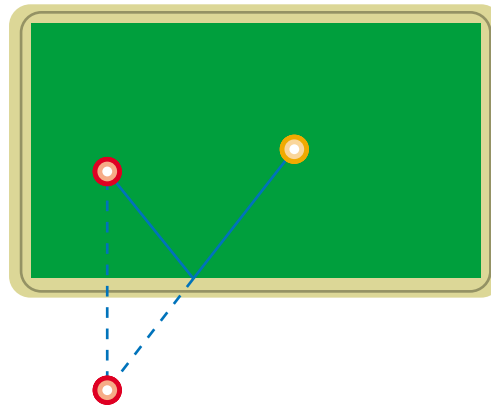
- a) What is the equation of the other half of the cross section?  
 b) What is the height of the roof,  $h$ ?  
 c) What is the width of the roof,  $w$ , to the nearest tenth of a metre?  
 d) State the domain and range of the function that models each half of the roof.



## ACHIEVEMENT Check

Knowledge/Understanding Thinking/Inquiry/Problem Solving Communication Application

A property of a billiard table is that the ball bounces off the side at the same angle that it strikes the side. To find where to hit the ball on the side of the table, you aim for the reflection of the ball you want to hit, as shown.



a) Show on graph paper where ball 1 should be hit so that it bounces off the bottom side and hits ball 2.

b) Show on graph paper where ball 1 should be hit so that it bounces off the left side and hits ball 2.

c) Show on graph paper where ball 1 should be hit so that it bounces once off the top or bottom side and then once off the left or right side and hits ball 2.

d) If the table is placed on a grid with the origin at the bottom left corner, then ball 1 is at the point  $(5, 6)$  and ball 2 is at the point  $(11, 3)$ . Find the coordinates of the point where ball 1 bounces off the bottom side in part a). Label it point H.

e) Find the equation of the path ball 1 takes to the bottom side at H.

f) Find the equation of the path ball 1 takes from point H to ball 2.

g) What is the relationship between the slope of the line from part e) and the slope of the line from part f)? Explain your answer.

