SUPPLEMENTARY MATERIAL
DEA Concepts and Returns to Scale Classification

Data Envelopment Analysis (DEA) is a nonparametric method used to evaluate the efficiency of decision-making units (DMUs) by comparing their input-output ratios. Within this framework, the concepts of pure efficiency, technical efficiency, and returns to scale are crucial for understanding productivity dynamics, as highlighted in studies such as Banker et al. (1984), Bogetoft & Otto (2011), Charnes et al. (1978), Cooper et al. (2007), Cooper et al. (2011), and Ferreira & Gomes (2009)which provide foundational insights for the development of this document.

1. Pure Efficiency and Technical Efficiency

Pure efficiency in DEA refers to the optimal use of inputs to produce outputs without considering the scale of production. It is measured by the distance of a DMU from the efficiency frontier, indicating how well resources are utilized. A DMU is considered fully efficient if it lies on the frontier, while those below it is deemed inefficient. The efficiency measured by the DEA BCC model captures this concept and is known as **pure technical efficiency**.

Technical efficiency, on the other hand, encompasses both pure efficiency and the scale of production. It assesses whether a DMU can produce the maximum output from a given set of inputs. The DEA CCR model measures **overall technical efficiency**, which includes both pure efficiency and scale efficiency components. Thus, the efficiency score of a DMU under the CCR model (assuming constant returns to scale) accounts for both management practices and the appropriateness of its operational scale.

2. Returns to Scale (RTS) and Scale Efficiency (SE)

Returns to scale (RTS) describe how output changes in response to proportional changes in inputs. They can be classified as **increasing, constant, or decreasing**:

- **Increasing returns to scale (IRS):** A proportional increase in inputs results in a more than proportional increase in output. This indicates that the DMU is operating below its optimal scale and should expand production.
- **Constant returns to scale (CRS):** A proportional increase in inputs leads to a proportional increase in output, indicating that the DMU is operating at its optimal scale.
- **Decreasing returns to scale (DRS):** A proportional increase in inputs results in a less than proportional increase in output, meaning that the DMU is operating above its optimal scale and may need to downsize.

Scale efficiency (SE) is given by the ratio between technical efficiency from the DEA CCR model and the DEA BCC model:

$$SE = \frac{TE_{CCR}}{TE\ _{BCC}}$$

where:

- **SE = 1**: The DMU operates at the optimal scale (CRS).
- **SE < 1**: The DMU is scale inefficient, meaning that RTS is either increasing (IRS) or decreasing (DRS).

A DMU's return to scale classification can be determined by analyzing its reference set in DEA models:

- If the sum of lambda values equals 1, the DMU exhibits **constant returns to scale (CRS)**.
- If the sum is less than 1, the DMU operates under **increasing returns to scale (IRS)**.
- If the sum is greater than 1, the DMU operates under **decreasing returns to scale (DRS)**.

3. Classification of DMUs Based on Technical Efficiency and RTS

Once DEA models are applied, DMUs can be classified based on their efficiency and scale properties. While DEA provides valuable insights into efficiency and productivity, it is essential to recognize its limitations. For instance, the classical RTS concept may misrepresent scaling activities, potentially obscuring productivity improvement opportunities. Therefore, six distinct situations may occur, as shown in Table 1.

Table 1 - Classification of DMUs Based on Technical Efficiency and RTS

Returns to Scale	**Technical Efficiency**	
	Efficiency	**Inefficiency**
Constant	This is the best performance. The DMU uses resources without waste and operates at an optimal scale. Output increases should maintain the same proportion of input usage. Cost increases are proportional to output increases	Despite operating at the optimal scale, there is technical inefficiency. This means that input usage can be reduced while maintaining the same output (input-oriented). Conversely, output can increase while using the same inputs (output-oriented). By eliminating technical inefficiencies, the DMU becomes efficient with constant returns
Increasing	Despite being technically efficient, no inputs are used in excess, and the output volume is below the optimal scale. This means that the DMU can increase production at decreasing costs. Thus, output growth should occur by incorporating inputs while maintaining the proportional relationships between input and output quantities.	In this situation, the DMU exhibits technical inefficiency due to excessive input usage and scale inefficiency. The latter occurs because the DMU is operating below the optimal scale. To improve technical efficiency, excess input usage must be eliminated. To operate at the optimal scale, production must increase. In summary, the DMU should increase output, ensuring that the ratio between input usage and output volume decreases.
Decreasing	In this case, the DMU is technically efficient but operating above the optimal scale. One alternative is to reduce the DMU's output while maintaining the same input-output ratio. Since there is no technical inefficiency, overutilization of the facility may be advantageous. Another option to increase output is adopting quantitative policies, meaning that improving factor productivity would allow production growth without requiring additional inputs. However, in this situation, output expansion will occur at increasing costs.	In this case, the DMU is operating above the optimal scale and exhibits technical inefficiency. Both issues need to be addressed. To improve technical efficiency, excessive input usage must be eliminated, which means producing more with the same inputs. Regarding scale efficiency, production can be reduced at each DMU, or a larger number of smaller DMUs can be used to produce the same total output. This decision depends on market conditions, competitiveness, and industry structure. Additionally, technology improvements can enhance factor productivity, optimizing input utilization.

Source: Adapted from Ferreira & Gomes (2009)

References

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