SUBTITUTING MATHEMATICS FOR NATURE

(Wandering towards a goal)

FQXI ESSAY

P.W.WANJOHI

Abstract

The axiom of extensionality states that two sets which contain the same elements are the same set; therefore the converse of the statement "substituting mathematics for nature" is also true. This essay, as the title suggests, is an attempt to show that the study of mathematics and that of nature, especially its physical descriptions, are intertwined.

Introduction

How can mindless mathematical laws give rise to aims and intentions? Obviously, mathematics is too rigorous to be mindless. But does it have any aims or intentions? It is hard for man to ignore the usefulness of mathematics in his constant search for solutions to not only practical, existential problems and dilemmas but also fundamental and lofty questions of pure science far removed from engineering and technological development. But in so far as mindless mathematical laws imply pure, abstract reasoning pursued with no apparent practical consideration, both the practically useful pursuit of mathematics as well as the apparently egotistic one lead one to the other at some point if not now.

Mathematical reasoning fits the universe so remarkably well, but does it lead to infallible truths or can it wander too far away from reality? Why should the physical world conform to patterns of man's reasoning which mathematics surely is?

Goal of Mathematics

Many mathematicians pursue the subject simply because they like it. To them the subject offers intellectual challenge and values that draw them to it far more strongly than money or power attracts people generally. They enjoy the excitement of the quest for new results, the thrill of discovery, the satisfaction of mastering difficulties, and the pride in achievement. There are moreover delights and aesthetic values to be derived from surveying orderly chains of reasoning, such as occur in most proofs, from the contemplation of the results themselves, and from grasping the ideas that make the proofs work. Those portions of mathematics which prove valuable in the study of nature offer additional satisfaction of unifying a multitude of seemingly disorganized facts and of comprehending nature's ways.

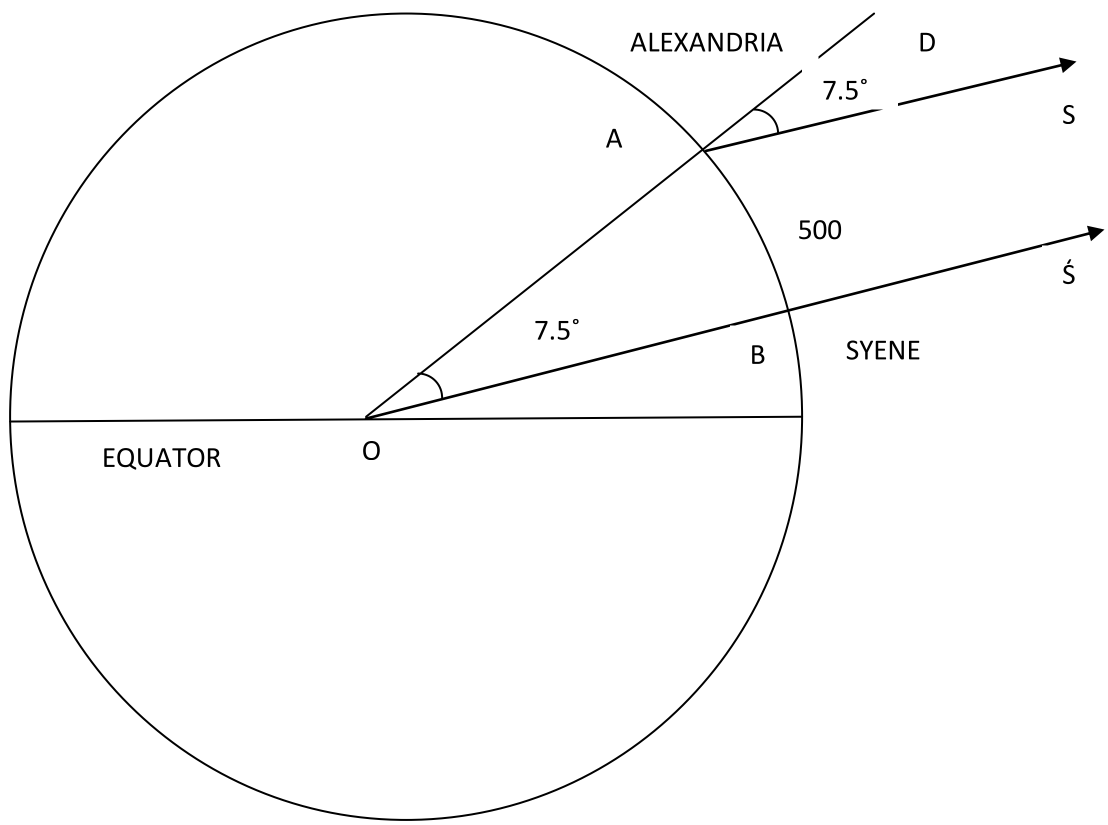
Nevertheless, great mathematicians throughout history could discern directions of prevailing scientific investigations and therefore developed mathematical concepts and techniques that were clearly instrumental in the investigation of nature and the entire scientific enterprise. Such was the usefulness of their axioms that, today, mathematics is the heart of our best scientific theories be it mechanics, relativity and quantum theory.

Indeed, physical science has reached a level of dalliance with mathematics that they are now essentially indistinguishable, one from the other. Science has become a collection of mathematical theories adorned with only a few physical facts. Further, the ultimate goal of modern scientific theory is to subsume all its results under one mathematical principle whose implications would describe the entire spectrum of nature's operations.

The precise manner in which mathematics produces answers to physical problems may be the answer. For example, simple arithmetic proves very effective in handling personal day to day financial transactions to arrive at the best propositions and avoid imprudent loss of money. With some geometry, it is possible to see clearly which plot of land among several of different spatial shapes is larger or smaller than the other; and what design of a fence or building wall encompasses more area, given an equal amount of fencing or walling material.

These are just examples of the many problems in which intuition can only be of incidental use and mathematics must carry the entire burden. One of the simplest and yet most impressive is Eratosthenes' calculation of the circumference of the earth. Eratosthenes (275-194 BC) knew the earth is spherical. He also knew that the city of Alexandria was due north of the city of Syene by a distance of 500miles along the surface of the earth (Fig. 1).

.



It was summer time and the noon sun shone directly down into a well at Syene. This means that the sun was directly overhead at that time; i.e. the direction of the sun was OBŚ. At Alexandria at the same instant, the direction of the sun was AS whereas the overhead direction is OAD. The sun being far away, the direction AS is the same as BŚ or AS and BŚ are parallel lines. Hence, it follows from an axiom of geometry that angles DAS and AOB are equal. Eratosthenes measured angle DAS and found 7.5˚.This then is the size of AOB. But this angle is 7.5/360 or 1/48 of the entire angle at O. It follows that arc AB is 1/48 of the entire circumference and since AB is 500 miles, the entire circumference is 48x500 or 24000 miles.

It is clear from such a simple example that mathematical reasoning can produce knowledge which guesswork, intuition and experience can only produce inaccurately or not at all.

Mathematical concepts and mathematical methods of obtaining knowledge have been most effective in representing and investigating the motions of heavenly bodies and the motions of objects on or near the surface of the earth, the phenomena of sound, light, heat, electricity, and electromagnetic waves, the structure of matter, the chemical reactions of various substances, the structure of the eye, ear and other organs of the body, and dozens of other major scientific phenomena.

The seemingly unprofitable amusements of speculative brains have produced rewards over and above the satisfaction of curiosity and the improvement in man's physical comforts be they better crops and animal husbandry, prediction of weather, better marine and aerial navigation, more efficient industries, modern transport and infrastrure, trains, airplanes, automobiles, ships, movies, radio, television, telephony, highly useful home appliances, electricity, medical treatments, computerization, internet, social media platforms among others.

Many fears and superstitions about the heavens have also been eliminated by just those people who studied the skies to satisfy their intellectual curiosity. Modern astronomical doctrines reveal an invariable order and a mathematical pattern to which heavenly bodies adhere. Thus such events as eclipses can be predicted precisely.

These advances and applications surely gives man a good reason to reason about nature; and in this regard mathematics was ,is and will be very helpful. Yet it is still not clear how a purely human creation, which mathematics is, can give us such insight into and power over nature.

The idea of the Greeks of 6^{th}, 5^{th}, and 4^{th} centuries BC was basically simple. They observed nature and found that certain simple forms such as lines, triangles, and circles occur repeatedly. The heavenly bodies are spheres; light seems to travel in straight lines; the surfaces of lakes are flat; the sides of buildings are rectangles. Number or quantity was also suggested repeatedly by collections and sizes of objects. These concepts of number and geometrical forms, in view of their very prevalence, were deemed worthy of study.

The Greeks also noted that certain facts about these concepts are obvious and seemingly basic. Circles are determined by choosing a centre, and a radius; any two right angles are always equal; equal numbers added to equal numbers or equal lengths added to equal lengths yield equals; etc. So, they selected the most obvious of these facts to see what reasoning could deduce from them. Surely if some new facts could be derived, these facts would apply to all those physical objects that possessed the basic properties in the first place. If the area of a circle could be shown by reasoning to be π times the square of the radius, then the area of any circular piece of land should also be π times the square of its radius. Further reasoning unearthed new facts, which observation alone could not suggest. Such advantages and many more accrued from reasoning about common concepts on the basis of clearly evident facts. Reasoning can therefore produce knowledge that not only covers a multitude of cases in one swoop but may produce physically meaningful information that is entirely unforeseen.

The intimate connection between mathematics and events in the physical world is reassuring, for it means that we not only can hope to understand the mathematics proper but also expect physically meaningful conclusions.

Methodology

Many methods of reasoning or drawing conclusions are available to man. Of these, the most widely used are reasoning by analogy, induction, and deduction. The essence of analogical reasoning is to find a similar situation or circumstance and to argue that what is true for the similar case applies also to the one in question. Obviously one must be able to find a similar situation and one must take the chance that the differences don't matter. Despite the uncertainties attached to this method of reasoning, it is constantly used because similar situations are usually readily available. However, though one starts with some facts that are reliable, he obtains a conclusion that is not necessarily reliable. But it is possible to reason in such a manner that starting with reliable facts; one is able to draw unquestionable conclusions.

An experimenter usually reasons by induction. The essence of this method is that one observes repeated instances of the same phenomena and concludes that the phenomenon will always occur. Conclusions obtained by induction are evidence – based, especially when the instances observed is large. Nevertheless, there is much room for error.

Finally there are ways of combining facts so as to obtain a new but equally sound one. This is deductive reasoning. A deductive argument consists in combining accepted facts in ways that compel acceptance of the conclusion. This characterization of deductive reasoning does not specify exactly what kinds of combinations of accepted facts yield inescapable conclusions. Despite these drawbacks, mathematicians ever since Greek times have limited themselves to conclusions which can be established deductively on the basis of a thoroughly reliable premise. Of course the conclusions deduced from such premises are themselves reliable and hence maybe used in turn as premises for further deductive reasoning. This means that theorems already established may be used as evidence for new proofs. No matter how many successive deductive arguments are involved, each yields certain conclusions. We see therefore why it is that mathematics has attained its reputation for the certainty of its results.

This contrasts with the methodology of physical and social sciences which includes induction and analogy in addition to deductive proof. Thus, where no facts are available, direct trial, experimentation and even sheer guess work are employed in order to supply likely conclusions more rapidly than would be the case if deductive proof is needed. The scientist feels free to draw conclusions based on observation, experimentation and experience. For example he may reason that sound waves behave like water waves or test a possible cure for human disease through animal trials. At some stages of his work, the scientist may reason deductively, employing the concepts and methodology of mathematics proper, but he certainly does not confine himself to deductive arguments only. A famous example of this contrast is the Goldbach's hypothesis which conjectures that every even number can be expressed as the sum of two prime numbers. Thousands of even numbers have been tested over a period of three centuries and the conjecture has been verified in each case. Inductive reasoning could conclude that every even number is the sum of two prime numbers. But deductive reasoning requires that all possible even numbers must be tested first before the conjecture is accepted as a theorem of mathematics.

Mathematics therefore achieves a reputation for precise reasoning at the expense of limiting its results to those that can be established deductively. But far from being shortsighted, mathematics is infinitely wise. By relying upon and exploiting deductive reasoning, it has routinely obtained results that would have been very difficult or even impossible to obtain by other methods. Its objectivity has accomplished far more than we would hope to obtain by observation, experience and experimentations. This is because our senses are limited. The eye sees only a small range of light waves and is easily deceived as to sizes and locations of objects. The ear hears only a limited range of sound waves. Touch, smell and taste are even more imprecise. On the other hand, man's deductive reasoning can encompass distances, sizes, sounds, and temperatures beyond the range of senses and even imagination. Mathematics has been able to create spaces of arbitrary dimensions and predict the existence of imperceptible radio waves and such uncharacteristic phenomena as black holes.

But mathematics is not entirely a deductive science. Though it derives conclusions that follow from axioms, these initial premises cannot be obtained deductively. They are derived from observation and experience. Deciding what to prove and how to go about making the proofs is also not deductive.

The most fertile source of mathematical ideas is nature herself. The clearest of such a process is elementary geometry. Precise statements of theorems could be proved from direct experience with physical objects. Thus theorems on area, perimeter and angle sum of common figures were developed at the earliest by mathematicians such as Pythagoreans and Euclid. To determine the area of a complex figure, Galileo used to make another out of cardboard and compared its weight with a cardboard model of a figure whose area is known. The relative weights of course corresponded with relative area.

After some theorems have been proved suggestions for others are obtained by the process of generalization. In arithmetic and algebra, direct calculations with numbers, which is analogous to measurement in geometry, suggests possible theorems. For example, simple calculations show that the sum of the first n odd numbers where n is any integer, is the square of n. Of course no mortal man could make the infinite set of calculations required to establish the conclusion *for every n.* However, the endless calculations give the mathematician something to work on. However, one cannot account for the discovery of what to prove or how to make deductive proof entirely by sheer observation, measurement or calculations. Therefore some methods of proof seem so artificially ingenious that they are devious, mean and underhand. Yet on the basis of such abstractions mathematics are created others that are even more remote from anything real. Negative numbers, irrational numbers, complex numbers, equations involving unknowns, formulas, derivatives, integrals and other such concepts are abstractions built upon abstractions. Surprisingly, the process of abstraction is far more natural than it appears at first. Such abstract concepts are actually derived from properties of physical objects. Physical straight lines have thickness, color, molecular structure and rigidity. Mathematical straight lines have none of these, and so do other geometrical forms, concepts of numbers, arithmetic, algebra etc.

Arithmetic

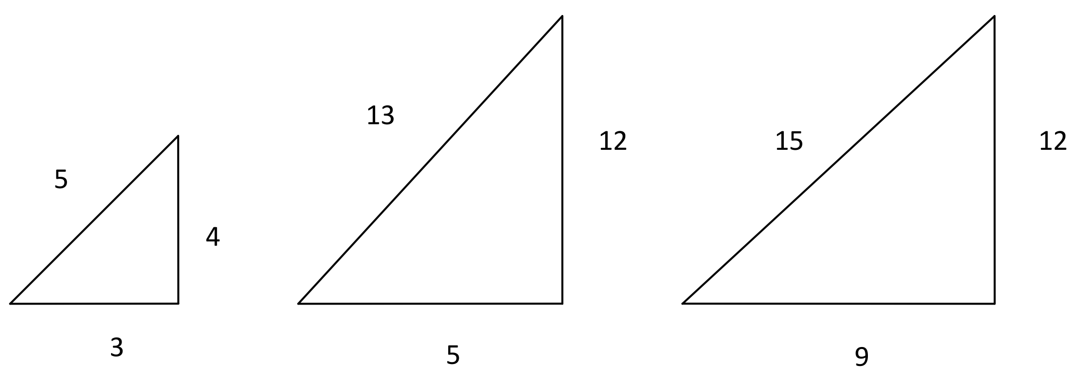
The Pythagorean school of mathematics inaugurated arithmetic by use of pebbles or dots, there being no number symbols at that ancient time. They classified these numbers according to shapes produced by the arrangement of the dots or pebbles. 3,6,10 and so on were called triangular numbers because they could be arranged to form triangles. Numbers 4,9,16 and so on were called square numbers because they could be arranged in perfect square shapes. From these simple geometric arrangements, some properties of whole numbers became evident e.g. the sum of two triangular numbers is always a square number. Or that if n^2 is any square number, then
$n^2+2n+1=(n+1)^2$

Thus, to the rudimentary Pythagoreans' concept of whole numbers was later added the concept of zero which enabled the representation of large quantities in tens of tens of tens of tens etc. Other improvements were the now familiar methods of operating with fractions and decimal representation of fractions. Arithmetic enables the usual operations of, addition, subtraction, multiplication and division. This is in addition to such practical applications such as ratios and rates which enabled the study of relationships such as air/water pressure; blood pressure/heart beat etc. Relative distances of the then known planets from the sun enabled the discovery of other by- then- unknown planets such as Neptune. The asteroids orbiting the sun between Mars and Jupiter are now known to be remnants of an exploded planet that lay in that position.

About 1860, Dmitri Ivanovich Mendeleev, by use of simple numbers and fractions, framed one of the leading ideas of modern chemistry: the Periodic Table. By arranging elements according to their atomic weights, he found that every eighth element among the first sixteen had similar chemical properties to the first one of that set of eight. Such periodicity has been used to discover new elements, their atomic weights and properties such as ability to combine with other elements to form new molecules.
The theory of number sets that has been investigated for many years solely because of its intrinsic interest has proved immensely useful today. For curiosity's sake, mathematicians have considered writing quantities in bases other than ten. Thus in base 6 , the symbol 10 can mean 6.To write 7 in base 6, we would write 11 i.e. 1x6+1,just as 11 in base 10 means 1x10+1. Moreover we can perform the usual arithmetic operations in this base, only we would have to learn new addition and multiplication tables such that if, in base ten 4+5=9, in base six 9 would be written 13. Base two especially impressed seventeenth century mathematician Gottfried Wilhelm Leibniz. In that base, all numbers are written in terms of 0 and 1 only. Thus eleven, which equals $1x2^3+0x2^2+1x2+1$, would be written 1011.
The subject of bases other than ten was regarded until recently as an intellectual amusement but with advent of computing machines, it is no longer purely an academic, aesthetic or amusing idea. All electronic computers perform their various operations in base two. The results are then converted to base 10 on which our mathematics and life experience is based. As a biological computing machine, the brain also most likely operates in this base.

Irrational Numbers

The Pythagorean Theorem states the general fact that the square of the hypotenuse of any right angled triangle equals the sum of the squares of the arms (Fig, 2).



This result was very pleasing to the Pythagoreans. But ,as mathematicians, they liked combinations so much that they engaged in an endless search for all possible triples of the form $a^2+b^2=c^2$. One Pythagorean decided one day to examine the simplest case of the geometric theorem: Suppose a right-angled triangle has arms each 1 unit long. This curiosity caused untold woes to later generations of mathematicians.
It turned out that $1^2+1^2=2$ therefore the hypotenuse is √2. There was no simple fraction whose square is 2. The nearest was 49/25.This meant that √2 could not equal a fraction and therefore no precise decimal number. In geometrical terms, it meant that the hypotenuse of a right angled triangle whose arms are 1 respectively cannot be ascertained to whole numbers or fractions of whole numbers by any physical measurement no matter how precise. Therefore was born the irrational number, a number not expressible as a ratio of whole numbers. Therefore the square

root of any whole number that is not a perfect square e.g. $\sqrt{3}$,$\sqrt{5}\sqrt{6}$,the cube of any whole number that is not a perfect cube e.g. $\sqrt[3]{2}$,$\sqrt[3]{3}$ etc are irrational. Irrational numbers, including π, though seemingly meaningless in geometrical terms are increasingly useful and convenient in the fields of algebra and calculus and perhaps proves the reality of the Planck length, the smallest unit or subdivision of length below which precise measurement becomes impossible.

Algebra

A good example of the role of algebra in investigating quantitative physical problems is furnished by the exploits of Archimedes, the greatest mathematician of antiquity. Wishing to complete an assignment given to him of determining how much silver and how much gold was in the King's crown, he supposed that the crown which, for example, weighed 10 pounds was made up of W_1 pounds of silver and W_2 pounds of gold. He found that 10 pounds of pure silver displaced 30 cubic inches of water. Hence W_1 pounds of silver would displace $(W_1/10)$ x30 $=3W_1$ cubic inches of water. Since ten pounds of pure gold displaced 15 cubic inches of water,W_2 pounds of gold would displace $(W_2 /10)$x15 = $2/3W_2$ cubic inches of water.
On measuring the volume of water displaced he found it to be, say, 20 cubic inches of water, hence,

1) $3W_2 +3/2W_2=20$
2) $W_{1+}W_2 =10$
3) $3W_1+3W_2 =30$
4) $3/2W_2=10$
5) $W_2=6\frac{2}{3}$
6) $W_{1=}3\frac{1}{3}$

Therefore the crown contained $3\frac{1}{3}$ pounds of silver and $6\frac{2}{3}$ pounds gold.
Many such real world problems are now routinely resolved through algebra. It is a machine that mechanizes thinking.
Algebra's series of powerful techniques apply to large classes of numbers and hence are useful for thousands of applications where arithmetic would deal with only one at a time. Apart from such first degree equations, similar techniques can solve more complex second and third, fourth, fifth and even higher degree equations involving much more complex problems. But the Frenchman Evariste Galois (1811-1832) showed that general equations of degrees higher than fourth cannot be solved by algebraic operations. Thus he created the theory of groups a subject now at the base of modern abstract algebra. The mathematician may also tackle general problems out of curiosity and therefore inadvertedly develop formulae for solving many similar and intractable special cases. The theory investigated for its own sake finds new applications never intended and certainly not foreseen.

Study of Light

To the mathematically minded scientist, light offers another domain of inquiry in his search for laws of nature. Though progress in the discovery of the mathematical laws of light has been slow, some remarkable laws have nevertheless been obtained and continue to be obtained. The depths of these laws continue to keep pace with mathematics proper. These include the straight line motion of light, its angles of reflection and refraction, the wave function of light, radio waves and other electromagnetic waves.

Astronomy

A combination of simple algebra, simple geometry and trigonometry has enabled man to master astronomy, so much so that, though it is the most physically unreachable domain of nature, our understanding of it is quite precise.

Trigonometric ratios based on the concept of similar right triangles are used to triangulate the heavens, determine the velocity of light and express the precise law of refraction for light when it passes from one medium to another. Copernicus' heliocentric theory and Kepler's laws on gravitation and motions of heavenly bodies, which opened heavenly secrets to the human mind, were all based on sound mathematical reasoning. Because we today accept the heliocentric theory and Kepler's laws, we no longer appreciate the full significance of Copernicus' and Kepler's achievements and what their mathematics really accomplished.

Curve and Equation

Curves represent geometrically the quantitative and numerical information needed for many practical applications. Fermat and Descartes developed algebraic expressions to ease the obtaining of information contained in curves such the path followed by a projectile, area of curved surfaces, the elliptical orbits of the planets, the paths followed by a bent or refracted light beam among others. This branch of geometry came to be known as analytical geometry. Thus was created the X and Y coordinates, being distances in numbers of each point on the curve, from two arbitrarily chosen but fixed axes, to be used as means of deducing facts about curves. Hence, for example, $x^2+y^2=r^2$ is the equation for a circle, a circle being no more than a collection of points that are at the same distance from the centre *0*, and *r* being the radius.

When idealized, all physical objects are curves and surfaces. Therefore they can be represented by equations and the shapes and motions studied by applying algebra to these equations. This method is very basic to science and engineering.

Fundamental Principles

The way to obtain correct and basic principles, Galileo indicated, was to pay attention to what nature says rather than what the mind believes. To Leonardo da Vinci, sciences that arise and end in thought do not give truths because in these mental considerations, no experience enters, and without this nothing is sure. Newton emphasized his reliance upon mathematics and said he used experiments largely to make his results physically intelligible and to convince the "vulgar". Thus a few fundamental principles derived from nature and much mathematical reasoning constituted Galileo's and Newton's scientific method. Galileo idealized just as the mathematician does and thereby penetrated the phenomena to obtain the basic principles. Thus one can use the machinery of mathematics to deduce new truths, for example by combining several formulas by use of proper algebraic steps to obtain a totally new fact. Such is the power of mathematics that science absorbs so much mathematics that for the first time, the danger is evident that it is going to contain little else.

Very often a mathematical formula that fits some phenomena of nature will apply far beyond the range of conditions under which it was originally derived. If it does apply, there is some heuristic evidence in favour of the original theorem. For example there is no reason why the distance *d* that a body falls due to gravity in *t* seconds $d=16t^2$ cannot represent the number of apples a man eats each month. This is an example of taking advantage of the abstractness of mathematics.

Take the mathematical science of sounds. The mathematical tool that proves to be the key to the analysis of sound is the very same function which represents the motion of masses on springs and of pendulums: i.e. oscillations or sinusoidal motion of the form y=D sin Ft =D sin 2π ft where y is air displacement, D is the amplitude or maximum displacement, F is the number of oscillations in 2π units of t, and t is the time. F=$\frac{F}{2π}$ frequency in one second.
In so far as mathematics is concerned, a periodic function is no more than a periodic relation between two variables. Hence for every periodic function, no matter how complex, the shape of the graph within the period is a sum of sine functions (Ohm's Law / Fourier's theorem).
Thus based on such exploits, mathematics is on the right track to seek basic quantitative laws. However, to describe mathematically what must happen and to make objects actually behave in that way are totally different matters. Many decades of ingenious experimental work had to pass before electrical phenomena predicted earlier through mathematical work by Michael Faraday could be physically realized. James Clerk Maxwell expanded the mathematical foundations of electricity into the four keys laws of electromagnetism. But Maxwell's labor in formulating the mathematical laws of electromagnetism paid unexpected dividend: the reciprocal relation between a magnetic field and an electrical field, so that the law of conservation of energy could hold. This is the electromagnetic field, a combination of changing electric and magnetic fields that can travel far out into space. These electromagnetic waves are none other than radio waves, now used by modern communication gadgets.
Differential calculus is another example of fruitful mathematical imagination based on the fundamental idea of *limits* and is used to calculate instantaneous rates of change such as velocity and acceleration. For acceleration due to gravity, the derivative of $y=16x^2$ holds regardless of the physical meaning of y or x. Hence we can apply the derivative to thousands of physical situations in which $y=16x^2$ applies.
In science's aim of deducing all phenomena of nature by application of mathematical reasoning, calculus is a major tool. This is through the inverse process of finding original formulas from known rates. Some of the greatest developments in mathematics, differential equations, the calculus of variations, differential geometry, potential theory and a host of other subjects collectively known as analysis were developed and explored by means of calculus.
The same concept helped solve another class of problems: areas bounded by curves, and the volumes of figures bounded by surfaces. The essential mathematical point in this process is that a *limit* of a sum can be obtained by reversing differentiation of the limit of a sum of the form

$$\text{Lim } h \to 0 \sum_{i=1}^{n} y_1 h_1$$

Gradually however, mathematicians, scientists and philosophers are increasingly aware of the complexity of nature's simplicity. Thus they persist in the goal of deriving as many facts and phenomena from as few principles as possible. And they are willing to pay the mathematical price, such as the mastery of differential equations and differential geometry.
But since reasoning on the basis of axioms contradicting Euclid's yielded theorems that applied to the same physical world where Euclid's also operates, it is no longer possible to assert that mathematics yields truths, for truth is unique. Rather, mathematics produces a representation of physical reality.