Seeking the Analytic Quaternion

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Abstract

By combining the complex analytic Cauchy-Riemann derivative with the Cayley-Dickson construction of a quaternion, possible formulations of a quaternion derivative are explored with the goal of finding an analytic quaternion derivative having conjugate symmetry. Two such analytic derivatives can be found. This unanticipated finding may have significance in areas of quantum mechanics where quaternions are fundamental, especially regarding the enigmatic phenomenon of complementarity, where a quantum process seems to present two essential aspects.

Introduction

Early progress in complex analysis was due to the realization, by Cauchy and Riemann in the nineteenth century, that a function of a complex variable has two complex derivatives. One derivative is called analytic, the other non-analytic. The analytic derivative is taken with respect to the complex variable, and the non-analytic derivative is taken with respect to its conjugate, corresponding to opposite directions of rotation of the variable. A quaternion variable can be formed from two complex variables by the Cayley-Dickson construction. It is proposed that quaternionic and octonionic analysis ought to be based on these two foundations, and constructed to satisfy a presumed symmetry shared by analycity and conjugation. Under this hypothesis there are four branches of a quaternion derivative, two of which are analytic.

A function of a real or complex variable is analytic if its Taylor series converges to the function. Proving analycity entails a suite of techniques involving derivatives and limits of power series, but a simpler meaning will suffice for this essay - an analytic function is one that is capable of being analyzed. The issue of analycity gained importance with the development of complex analysis and the difficulties encountered due to different directions of rotation of a complex variable. These difficulties were overcome when it was found that there are two branches of a function of a complex variable. Each branch has a complex derivative associated with it according to the variable's direction of rotation. These are the analytic and non-analytic branches. Recognition of this separation allows a complex function to be analyzed in two parts, each complex variable having opposite rotation.

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An analytic function can then be defined as one whose complex variables rotate in only one direction. Non-analycity is related to conjugation which reverses the rotational direction of a complex variable. The non-analytic derivative is the derivative of a function with respect to the conjugate of a complex variable. The non-analytic derivative of an analytic function is zero. For example, consider a complex variable, z, and its conjugate, z^*, and their derivatives (analytic derivatives on the left),

$$\mathrm{d}z/\mathrm{d}z = 1 \qquad \mathrm{d}z/\mathrm{d}z^* = 0 \tag{1}$$
$$\mathrm{d}z^*/\mathrm{d}z = 0 \qquad \mathrm{d}z^*/\mathrm{d}z^* = 1 \tag{2}$$

The variable, z, is an analytic function, so the non-analytic derivative, $\mathrm{dz}/\mathrm{dz}^*$, is zero. On the other hand, z^* is a non-analytic function whose analytic derivative is zero. Thus, the conjugate of a variable can be treated as a constant when taking a complex derivative. Quaternion and octonion analogs for (1)-(2) will be presented, but corresponding analycity is not investigated beyond that.

It is the symmetry shared by analycity and conjugation shown in the above equations that is the focus of this essay. In the conventional approach to quaternionic analysis due to Fueter, this correspondence is lost. Fueter considers the asymmetric representation of a quaternion consisting a real variable and three imaginary variables with independent Cauchy-Riemann equations. The Cauchy-Riemann equation produces analytic or non-analytic complex derivatives from real derivatives of a complex function. See [1] for an exposition on quaternionic analysis and Fueter's work.

The Cayley-Dickson construction forms a quaternion number from two complex numbers. This symmetric view of a quaternion (two complex numbers *vs* one real and three imaginary numbers) leads to an extension of the Cauchy-Riemann equation to quaternions via the Cayley-Dickson construction resulting in four branches of quaternion analycity which can be arranged to have the symmetry required for a correspondence between analycity and conjugation.

Quaternion analycity then depends on complex analycity and requires that complex derivatives are either both analytic or both non-analytic. This property can be extended to octonions where an analytic octonion derivative would require that quaternion derivatives are both analytic or both non-analytic.

When used in reference to quaternions or octonions in this essay, it must be understood that the terms 'analycity' and 'analytic' will refer to a mathematical property which has not been demonstrated. At this point, it is not even clear how to apply the various branches of a derivative in order to do so. It is the prerequisite development of symmetry in the quaternion derivative through a hypothetical relationship between conjugation and analycity that will be the concern here. Hence formulation of quaternion and octonion derivatives will be guided by consideration of the conjugate derivative, and resulting forms will be presumed to exhibit analycity inferred from its value just as in equations (1)–(2) for complex variables.

Cauchy-Riemann equation for functions of a complex variable

The Cauchy-Riemann equation originated from the effort to define a complex derivative. Consider a function $f = f(x)$ of a complex variable $x = a + b\,i$ where a and b are real variables. The function is analytic if it satisfies the Cauchy-Riemann equation which equates appropriately rotated[1] real derivatives,

$$\frac{\mathrm{d}f}{\mathrm{d}a} = -\frac{\mathrm{d}f}{\mathrm{d}b}\,i \tag{3}$$

Adding these expressions gives the analytic derivative with respect to the complex variable, x, in terms of real derivatives as

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{2}\left(\frac{\mathrm{d}f}{\mathrm{d}a} - \frac{\mathrm{d}f}{\mathrm{d}b}\,i\right) \tag{4}$$

Taking their difference gives the non-analytic derivative (the derivative with respect to the conjugate variable, x^*) as

$$\frac{\mathrm{d}f}{\mathrm{d}x^*} = \frac{1}{2}\left(\frac{\mathrm{d}f}{\mathrm{d}a} + \frac{\mathrm{d}f}{\mathrm{d}b}\,i\right) \tag{5}$$

which will be zero if the function satisfies the Cauchy-Riemann equation. The complex derivative is a unique concept and not the same as the gradient, which would have the form of (5) except for the factor of one half.

Cayley-Dickson construction of the quaternion

Consider another complex number $y = c + d\,i$ with real c and d. The Cayley-Dickson construction forms a quaternion, q, from complex numbers x and y using a new imaginary number, j, as

$$q = x + y\,j \tag{6}$$

Defining a third imaginary number, $k = ij$, gives the quaternion as

$$q = a + b\,i + c\,j + d\,k \tag{7}$$

The Cauchy-Riemann-Fueter equations for functions of a quaternion

The Cauchy-Riemann-Fueter equations come from applying (3) to each real derivative of a quaternion function $f = f(q)$ so that

$$\frac{\mathrm{d}f}{\mathrm{d}a} = -\frac{\mathrm{d}f}{\mathrm{d}b}\,i = -\frac{\mathrm{d}f}{\mathrm{d}c}\,j = -\frac{\mathrm{d}f}{\mathrm{d}d}\,k \tag{8}$$

Analogous to the complex case, a quaternion derivative can be formed by adding these real derivatives to get

$$\frac{\mathrm{d}f}{\mathrm{d}q} = \frac{1}{4}\left(\frac{\mathrm{d}f}{\mathrm{d}a} - \frac{\mathrm{d}f}{\mathrm{d}b}\,i - \frac{\mathrm{d}f}{\mathrm{d}c}\,j - \frac{\mathrm{d}f}{\mathrm{d}d}\,k\right) \tag{9}$$

[1] The rotation comes from equating $\mathrm{d}f/\mathrm{d}a = \mathrm{d}f/\mathrm{d}(bi)$ which becomes (3). Carrying a 90° rotation (the imaginary number, i) with the real variable, b, makes the complex derivative different from a gradient.

with a condition for regularity (not analycity) given by

$$\frac{\mathrm{d}f}{\mathrm{d}a} + \frac{\mathrm{d}f}{\mathrm{d}b}\, i + \frac{\mathrm{d}f}{\mathrm{d}c}\, j + \frac{\mathrm{d}f}{\mathrm{d}d}\, k = 0 \tag{10}$$

Analycity has been shown to be limited to constant and some linear functions. Fueter proposes no correspondence between conjugate derivatives and analycity.

Extending Cauchy-Riemann via the Cayley-Dickson construction

An overlooked possibility for extending the Cauchy-Riemann equation to a function $f = f(q)$ of a quaternion variable, q, is to procede from the Cayley-Dickson construction (6) and form the "Cayley-Dickson-Cauchy-Riemann" equation using j instead of i, and complex derivatives in place of real derivatives in (3),

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{\mathrm{d}f}{\mathrm{d}y}\, j \tag{11}$$

The derivative of a function of a complex variable has two branches, one for analytic functions (4) and one for non-analytic (5). Being composed of two complex variables, the resulting expression for a quaternion derivative will have four branches, and each can be clearly identified as analytic or non-analytic. If complex derivatives come from branches with similar analycity, the quaternion derivative will be analytic. Otherwise, the quaternion derivative will be non-analytic. Hence, a quaternion derivative formed from two complex non-analytic derivatives is analytic. Unlike the conventional approach, the conjugate of a quaternion variable is a non-analytic function whose analytic derivative is zero.

When a quaternion derivative is evaluated, two of the four real derivatives will cancel.[2] Thus the following expressions omit the expected factor of 1/2 in anticipation of the cancellation. Also, reversal of the sign in the second branch of the quaternion derivative (13) is necessary to have the branches balance analycity with non-analycity. The quaternion derivative is then composed from complex derivatives as

$$f'_{\mathrm{aa}} = \frac{\mathrm{d}f}{\mathrm{d}q} = \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}f}{\mathrm{d}y}\, j \qquad \text{(analytic)} \tag{12}$$

$$f'_{\mathrm{an}} = \frac{\mathrm{d}f}{\mathrm{d}q^*} = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}f}{\mathrm{d}y^*}\, j \qquad \text{(non analytic)} \tag{13}$$

$$f'_{\mathrm{na}} = \frac{\mathrm{d}f}{\mathrm{d}q^*} = \frac{\mathrm{d}f}{\mathrm{d}x^*} - \frac{\mathrm{d}f}{\mathrm{d}y}\, j \qquad \text{(non analytic)} \tag{14}$$

$$f'_{\mathrm{nn}} = \frac{\mathrm{d}f}{\mathrm{d}q} = \frac{\mathrm{d}f}{\mathrm{d}x^*} - \frac{\mathrm{d}f}{\mathrm{d}y^*}\, j \qquad \text{(analytic)} \tag{15}$$

where the four branches of a quaternion derivative are denoted by f'_{aa}, f'_{an}, f'_{na} and f'_{nn} with "a" for analytic and "n" for non-analytic complex derivative, and where the first subscript indicates the x complex derivative and the second is

[2] The initial part of the Cauchy-Riemann-Fueter equations, *i.e.* (8) equating appropriately rotated real derivatives, is implicitly assumed.

for y. For example, the analytic branch composed from two analytic complex derivatives (12) is given by

$$f'_{\mathrm{aa}} = \frac{1}{2}\Big(\frac{\mathrm{d}f}{\mathrm{d}a} - \frac{\mathrm{d}f}{\mathrm{d}b}\,i - \frac{\mathrm{d}f}{\mathrm{d}c}\,j + \frac{\mathrm{d}f}{\mathrm{d}d}\,k\Big) \tag{16}$$

The derivative of a quaternion variable and its conjugate are of interest. For $f = q$ the real derivatives are

$$\frac{\mathrm{d}q}{\mathrm{d}a} = 1 \qquad \frac{\mathrm{d}q}{\mathrm{d}b} = i \qquad \frac{\mathrm{d}q}{\mathrm{d}c} = j \qquad \frac{\mathrm{d}q}{\mathrm{d}d} = k \tag{17}$$

and, noting that $i^2 = j^2 = k^2 = -1$, thus $q'_{\mathrm{aa}} = dq/dq = (1+1+1-1)/2 = 1$. The sign of the three derivatives associated with the imaginary numbers changes for the conjugate, so that $q^{*\prime}_{\mathrm{aa}} = dq^*/dq = (1-1-1+1)/2 = 0$, allowing the same connection between conjugation and analycity found in the complex case.

The roles of the variable and its conjugate are reversed for the two non-analytic branches of the quaternion derivative, f'_{an} and f'_{na}, again like the complex derivative. The two non-analytic branches correspond to the derivative with respect to the conjugate quaternion variable, and are zero for analytic functions. One of the non-analytic quaternion derivatives formed from complex derivatives with differing analycity (13) is given by

$$f'_{\mathrm{an}} = \frac{1}{2}\Big(\frac{\mathrm{d}f}{\mathrm{d}a} - \frac{\mathrm{d}f}{\mathrm{d}b}\,i + \frac{\mathrm{d}f}{\mathrm{d}c}\,j + \frac{\mathrm{d}f}{\mathrm{d}d}\,k\Big) \tag{18}$$

In this case $q'_{\mathrm{an}} = dq/dq^* = (1+1-1-1)/2 = 0$, and the derivative of the conjugate is $q^{*\prime}_{\mathrm{an}} = dq^*/dq^* = (1-1+1+1)/2 = 1$ as expected for a non-analytic derivative.

The other non-analytic quaternion derivative (14) is given by

$$f'_{\mathrm{na}} = \frac{1}{2}\Big(\frac{\mathrm{d}f}{\mathrm{d}a} + \frac{\mathrm{d}f}{\mathrm{d}b}\,i - \frac{\mathrm{d}f}{\mathrm{d}c}\,j + \frac{\mathrm{d}f}{\mathrm{d}d}\,k\Big) \tag{19}$$

In this case $q'_{\mathrm{na}} = dq/dq^* = (1-1+1-1)/2 = 0$, and the derivative of the conjugate is $q^{*\prime}_{\mathrm{na}} = dq^*/dq^* = (1+1-1+1)/2 = 1$.

The analytic quaternion derivative formed from two non-analytic complex derivatives (15) is given by

$$f'_{\mathrm{nn}} = \frac{1}{2}\Big(\frac{\mathrm{d}f}{\mathrm{d}a} + \frac{\mathrm{d}f}{\mathrm{d}b}\,i - \frac{\mathrm{d}f}{\mathrm{d}c}\,j - \frac{\mathrm{d}f}{\mathrm{d}d}\,k\Big) \tag{20}$$

In this case $q'_{\mathrm{nn}} = dq/dq = (1-1+1+1)/2 = 1$, and the derivative of the conjugate is $q^{*\prime}_{\mathrm{nn}} = dq^*/dq = (1+1-1-1)/2 = 0$ as required.

Non-associativity of quaternion derivative

Consider the following analytic quaternion derivatives of basic linear functions formulated using two analytic complex variables (16):

$$\frac{\mathrm{d}q}{\mathrm{d}q} = 1 \qquad \frac{\mathrm{d}(i\,q)}{\mathrm{d}q} = i \qquad \frac{\mathrm{d}(j\,q)}{\mathrm{d}q} = j \qquad \frac{\mathrm{d}(k\,q)}{\mathrm{d}q} = k \tag{21}$$

$$\frac{\mathrm{d}q}{\mathrm{d}q} = 1 \qquad \frac{\mathrm{d}(q\,i)}{\mathrm{d}q} = i \qquad \frac{\mathrm{d}(q\,j)}{\mathrm{d}q} = j \qquad \frac{\mathrm{d}(q\,k)}{\mathrm{d}q} = -k \tag{22}$$

The last of the above equations stands out because of the negative sign on the k basis. The other analytic quaternion branch has a similar reversal affecting the i basis for the function (qi), and the non-analytic branches also have one basis[3] that does not conform, leading to possible restrictions on linear analytic forms, or at least complicating their development.

Now consider a constant quaternion, u, and encapsulation of the above two sets of equations (21)-(22) by the derivatives

$$\frac{\mathrm{d}(u\,q)}{\mathrm{d}q} = u \qquad\qquad \frac{\mathrm{d}(q\,u)}{\mathrm{d}q} = u^{\mathrm{K}} \tag{23}$$

where the superscript $^{\mathrm{K}}$ in u^{K} operates to reverse the k basis in u.

Consider another constant quaternion, v, and the product, uqv. Quaternions are associative so $u(qv)$ and $(uq)v$ are equal. However the derivative discriminates between the two formulas so that, assuming a right-associated chain rule for functions of functions,

$$\frac{\mathrm{d}[u(qv)]}{\mathrm{d}q} = \frac{\mathrm{d}[u(qv)]}{\mathrm{d}(qv)}\,\frac{\mathrm{d}(qv)}{\mathrm{d}q} = uv^{\mathrm{K}} \qquad\qquad \frac{\mathrm{d}[(uq)v]}{\mathrm{d}q} \neq v^{\mathrm{K}}u \tag{24}$$

It can be shown that for the elementary quadratic function, q^2,

$$\frac{\mathrm{d}q^2}{\mathrm{d}q} = q^{\mathrm{K}} + q \tag{25}$$

It can also be shown that

$$\frac{\mathrm{d}(q\,q^*)}{\mathrm{d}q} = q^{*\mathrm{K}} \qquad\qquad \frac{\mathrm{d}(q^*q)}{\mathrm{d}q} \neq q^* \tag{26}$$

indicating another selection since $q\,q^* = q^*q$. Judging from these examples for one branch, the price of symmetry is structural complexity.

There is one linear form that is well-behaved. Functions of the form uq with constant u always have the same derivative (u) for the analytic branches, and zero for the non-analytic branches. It is only derivatives of the commuted form qu that have a nonconforming basis.

[3] The non-analytic quaternion derivative f'_{an} (13) is zero for the functions in question except $\frac{\mathrm{d}(q\,i)}{\mathrm{d}q^*} = 2i$. Likewise, the non-analytic quaternion derivative f'_{na} (14) is zero except $\frac{\mathrm{d}(q\,j)}{\mathrm{d}q^*} = 2j$. An analytic derivative takes the nonconforming basis one step backward instead of forward – a non-analytic derivative takes the nonconforming basis two steps forward instead of not moving at all.

Complex matrix form of quaternion derivative

Recall that two complex variables

$$x = a + b\,i \qquad y = c + d\,i$$

were used to form a quaternion, q, via the Cayley-Dickson construction

$$q \;=\; x + y\,j \;=\; a + b\,i + c\,j + d\,k$$

The real quaternion basis in complex matrix form is

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{27}$$

where a quaternion is formed from real coefficients (a, b, c, d) as

$$\mathbf{Q} = \begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix} = a\,\mathbf{1} + b\,\mathbf{I} + c\,\mathbf{J} + d\,\mathbf{K} \tag{28}$$

and its conjugate is given by

$$\mathbf{Q}^{\mathbf{H}} = \begin{bmatrix} x^* & -y \\ y^* & x \end{bmatrix} = a\,\mathbf{1} - b\,\mathbf{I} - c\,\mathbf{J} - d\,\mathbf{K} \tag{29}$$

Note that the conjugate of the quaternion is the Hermitian transpose of the matrix. All of the formulas in the preceding sections can use the matrix basis by substituting $(\mathbf{1}, \mathbf{I}, \mathbf{J}, \mathbf{K})$ for $(1, i, j, k)$.

The Cayley-Dickson construction can be put in matrix form. Instead of complex variables x and y, start with complex diagonal matrices

$$\mathbf{X} = \begin{bmatrix} x & 0 \\ 0 & x^* \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} y & 0 \\ 0 & y^* \end{bmatrix} \tag{30}$$

A quaternion is then constructed as

$$\mathbf{Q} = \mathbf{X} \;+\; \mathbf{Y}\,\mathbf{J} \tag{31}$$

Treatment of the quaternion derivative can procede as before, ultimately taking derivatives with respect to the real variables (a, b, c, d). The matrix form is complicated by the presence of conjugate variables which are redundant, but in some sense informative. In the initial development, the matrix form was found to have the advantage that (31), for instance, can be distinguished immediately from its commuted alternative as the obvious place to start. This is not so clear otherwise using imaginary numbers where a bit of a search would be involved to see which possibility could be eliminated.

Summary

This work was motivated in part by a perceived lack of symmetry between analytic and non-analytic derivatives in conventional approaches to quaternion analycity, in comparison to the complex derivative. The picture of an analytic quaternion that emerges is one of some complexity with four branches of the derivative. While these branches are complicated by non-associative exceptions, their relatively simple form may provide an avenue for their analysis. Notably, none of the four branches of quaternion derivative correspond to the possibility considered by Fueter. There is one analytic derivative for complex variables, but there are two analytic derivatives for quaternions in the Cauchy-Riemann-Cayley-Dickson scheme.

Conclusion

What can be said to address the theme of this competition, *Wandering Towards a Goal: How can mindless mathematical laws give rise to aims and intention?*

One definition of mathematics is the study of structure. Understanding structure can be taken as an overarching goal, if only to allow structure in physics, for example, to be expressed quantitatively. Technical aspects of analycity which were neglected in the above presentation require more study. If this is indeed useful new mathematics, that in itself would point to the rather unpredictable nature of such developments.

Clearly, a re-examination of the role of a quaternion derivative would be required in the context of the rules of quantum mechanics, with the aim of discovering some correspondence.

An interesting hypothesis is that the presence of two analytic derivatives could be linked to complementarity, the property that a quantum process can be described in two mutually exclusive classical ways.

Ultimately, it is mathematics that allows us to entertain the notion of understanding quantum mechanics.

References

[1] Sudbery A., Quaternionic Analysis, Math. Proc. Camb. Phil. Soc., 85, 199-225 (1979)
`http://dougsweetser.github.io/Q/Stuff/pdfs/Quaternionic-analysis.pdf`

[2] Baez J.C., The octonions, Bull. Amer. Math. Soc. 39, 145-205 (2002)
`http://math.ucr.edu/home/baez/octonions/`

Extension to derivative of octonion function

Octonion analycity requires the combination of two quaternions of similar analycity. An octonion [2] variable $\phi = p + q\,l$ is created from two quaternion variables p and q and a new imaginary number, l, using the Cayley-Dickson construction. Consider an octonion function $f = f(\phi)$. Extending the Cayley-Dickson-Cauchy-Riemann equations for the quaternion derivative (12)-(15) gives this generic expression for the octonion derivative in terms of quaternion derivatives,

$$f'(\phi) \;=\; \frac{\mathrm{d}f}{\mathrm{d}\phi} \;=\; \frac{\mathrm{d}f}{\mathrm{d}p} - \frac{\mathrm{d}f}{\mathrm{d}q}\,l \;=\; f'(p) - f'(q)\,l \tag{32}$$

where $f'(p)$ stands for the derivative of the octonion function, f, with respect to a quaternion part (p) of the octonion variable, ϕ, which produces the following sixteen possible branches of an octonion derivative,

$$f'_{\mathrm{aa\,aa}}(\phi) = f'_{\mathrm{aa}}(p) - f'_{\mathrm{aa}}(q)\,l \qquad \text{(analytic)} \tag{33}$$
$$f'_{\mathrm{aa\,an}}(\phi) = f'_{\mathrm{aa}}(p) + f'_{\mathrm{an}}(q)\,l \qquad \text{(non analytic)} \quad * \tag{34}$$
$$f'_{\mathrm{aa\,na}}(\phi) = f'_{\mathrm{aa}}(p) + f'_{\mathrm{na}}(q)\,l \qquad \text{(non analytic)} \quad * \tag{35}$$
$$f'_{\mathrm{aa\,nn}}(\phi) = f'_{\mathrm{aa}}(p) - f'_{\mathrm{nn}}(q)\,l \qquad \text{(analytic)} \tag{36}$$
$$f'_{\mathrm{an\,aa}}(\phi) = f'_{\mathrm{an}}(p) - f'_{\mathrm{aa}}(q)\,l \qquad \text{(non analytic)} \tag{37}$$
$$f'_{\mathrm{an\,an}}(\phi) = f'_{\mathrm{an}}(p) - f'_{\mathrm{an}}(q)\,l \qquad \text{(analytic)} \tag{38}$$
$$f'_{\mathrm{an\,na}}(\phi) = f'_{\mathrm{an}}(p) - f'_{\mathrm{na}}(q)\,l \qquad \text{(analytic)} \tag{39}$$
$$f'_{\mathrm{an\,nn}}(\phi) = f'_{\mathrm{an}}(p) - f'_{\mathrm{nn}}(q)\,l \qquad \text{(non analytic)} \tag{40}$$
$$f'_{\mathrm{na\,aa}}(\phi) = f'_{\mathrm{na}}(p) - f'_{\mathrm{aa}}(q)\,l \qquad \text{(non analytic)} \tag{41}$$
$$f'_{\mathrm{na\,an}}(\phi) = f'_{\mathrm{na}}(p) - f'_{\mathrm{an}}(q)\,l \qquad \text{(analytic)} \tag{42}$$
$$f'_{\mathrm{na\,na}}(\phi) = f'_{\mathrm{na}}(p) - f'_{\mathrm{na}}(q)\,l \qquad \text{(analytic)} \tag{43}$$
$$f'_{\mathrm{na\,nn}}(\phi) = f'_{\mathrm{na}}(p) - f'_{\mathrm{nn}}(q)\,l \qquad \text{(non analytic)} \tag{44}$$
$$f'_{\mathrm{nn\,aa}}(\phi) = f'_{\mathrm{nn}}(p) - f'_{\mathrm{aa}}(q)\,l \qquad \text{(analytic)} \tag{45}$$
$$f'_{\mathrm{nn\,an}}(\phi) = f'_{\mathrm{nn}}(p) + f'_{\mathrm{an}}(q)\,l \qquad \text{(non analytic)} \quad * \tag{46}$$
$$f'_{\mathrm{nn\,na}}(\phi) = f'_{\mathrm{nn}}(p) + f'_{\mathrm{na}}(q)\,l \qquad \text{(non analytic)} \quad * \tag{47}$$
$$f'_{\mathrm{nn\,nn}}(\phi) = f'_{\mathrm{nn}}(p) - f'_{\mathrm{nn}}(q)\,l \qquad \text{(analytic)} \tag{48}$$

Note: * indicates a sign change like that required for the quaternion derivative.

As with complex and quaternion derivatives, the derivative of an octonion variable with respect to the conjugate octonion variable is zero. Cancellation among components again leads to a missing factor of two compared to Cauchy-Riemann. The octonion derivative has non-associative exceptions for linear functions similar to the quaternion.

Quaternion Exceptions

Consider functions of the form qe_ν, where $e_\nu = 1, i, j \,\text{or}\, k$. Each of the four branches of quaternion derivative with respect to q (or q^* for non-analytic derivatives) will break with associativity for one of the functions. These exceptions are given by the value 2 or -1 in the following table. For example, the fourth branch (15) of the derivative of qi can be found from the fourth row in the table in the column under i which shows -1 as the entry, so that the derivative of qi is $-i$ for that branch.

		1	i	j	k
aa	a	1	1	1	-1
an	n	0	2	0	0
na	n	0	0	2	0
nn	a	1	-1	1	1

Octonion Exceptions

Octonions have three exceptions for each branch instead of just one. Here e_ν represents the octonion basis. As an example, for the function ϕe_6 of an octonion, ϕ, the third branch of the derivative (35), which is a non-analytic exception, is found from the third row of the seventh column of the table (under e_6) to be $2e_6$.

			e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
aaaa	aa	a	1	1	1	-1	1	-1	-1	1
aaan	an	n	0	2	2	0	0	2	0	0
aana	an	n	0	2	2	0	0	0	2	0
aann	aa	a	1	1	1	-1	1	1	-1	-1
anaa	na	n	0	2	0	0	2	0	0	2
anan	nn	a	1	1	-1	-1	1	-1	1	1
anna	nn	a	1	1	-1	-1	1	1	-1	1
annn	na	n	0	2	0	0	2	2	0	0
naaa	na	n	0	0	2	0	2	0	0	2
naan	nn	a	1	-1	1	-1	1	-1	1	1
nana	nn	a	1	-1	1	-1	1	1	-1	1
nann	na	n	0	0	2	0	2	2	0	0
nnaa	aa	a	1	-1	1	1	1	-1	-1	1
nnan	an	n	0	0	2	2	0	2	0	0
nnna	an	n	0	0	2	2	0	0	2	0
nnnn	aa	a	1	-1	1	1	1	1	-1	-1