Newtonian potential and the Einstein equivalence principle

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Abstract

An object in a gravitational field has an associated velocity from the Einstein equivalence principle. Do these velocities add relativistically when objects merge? This possibility leads to a reconsideration of the nature of gravitational potential energy.

1 Introduction

This essay points out a possible inconsistency in accepted physics which is evident from the Einstein equivalence principle. Einstein's prediction of a gravitational redshift in the frequency of light approximately equal to the difference in Newtonian gravitational potential divided by the square of the speed of light [1] has been verified experimentally [2]. The derivation supposes equivalence between a system of measurement undergoing uniform acceleration and one in a gravitational field.

Let us follow the steps of the derivation in the accelerated system which is undergoing uniform acceleration g in the upward direction to mimic gravity. The time required for radiation emitted downward to travel a distance h to meet the rising measurement system is h/c, where c is the speed of light. In that time, the measurement system has acquired a velocity $v = gh/c$ upward. From the special theory of relativity, the radiated energy is then detected by the measurement system and found to be increased approximately by a factor of $1 + gh/c^2$ corresponding to a redshift $z = -gh/c^2$ (a blue shift). For the system in the gravitational field, the change in radiated energy is then attributed to the change in gravitational potential given by gh.

The derivation, consisting of finding a velocity to infer a redshift, ends at this point but consider this fictional velocity, v, and its relation to special relativity. If every material body has an associated velocity then one might wonder whether the velocities, and hence the gravitational redshifts, of two bodies combine relativistically on merging to form a single body with a new velocity. This new velocity would obey the composition law from special relativity, not the

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Galilean law of addition. For velocities v_1 and v_2 associated with the gravitational fields of two bodies, the relativistic redshift from the merged body would be given by the product

$$1 + z \ = \ \sqrt{\frac{1 - v_1/c}{1 + v_1/c}} \sqrt{\frac{1 - v_2/c}{1 + v_2/c}} \tag{1}$$

The situation is illustrated clearly in the relationship between redshift and gravitational potential energy. Merging two masses with redshifts, z_1 and z_2, would result in a redshifted photon wavelength close to $(1 + z_1)(1 + z_2)$ times the original, not simply $1 + z_1 + z_2$ which might be seen as approximating this product of redshifts, which is itself a first-order approximation to (1) taken term by term. Calculating the first-order approximation separately for each term retains the product form, and this concept is essential. The nonlinearity of combining redshifts by a process of multiplication is fundamental but has not been consistently applied at the very foundation of gravitation. In Newton's derivation of gravitational potential, contributions to the potential, and thus to the redshift, from concentric shells of matter are added together. It has been proposed [3, 4] that the calculation of gravitational potential energy ought to be based on multiplying fractional energies due to shells of matter, corresponding to the more accurate product form of the approximation for multiple redshifts.

Since the mass of each Newtonian shell is infinitesimal, this relativistic approximation will be used in the reformulation of gravitational potential energy which follows. It will become evident that the exact relativistic form based on (1) will also be satisfied. While a shell of matter influences the mass-energy of a test particle, the test particle will be presumed to have no effect on the shell. The resulting expression for potential energy is an exponential transformation of Newtonian potential requiring an arbitrary factor for normalization. This factor is unknown *a priori*, but must be equal to the rest energy of the test particle being influenced by the shells in order to correspond to modern physics. Given this equivalence, a radial escape velocity can be calculated from the modified potential and used to determine Gullstrand-Painlevé coordinates.

2 Reformulation of classical potential

2.1 Calculation of gravitational potential energy

Given a force, $F(x)$, as a function of distance, x, the change in Newtonian potential energy after moving from a to b is defined generally as

$$U = -\int_a^b F(x)\mathrm{d}x \tag{2}$$

but only gravity with radial displacement will be considered. Proceeding with the implementation of the compounding calculation which replaces the operation of addition with multiplication, the measure of gravitational potential energy, $\widehat{U}$, derived from the rule of redshifts can be determined as

$$\widehat{U} = \widehat{U}_{\text{o}} \exp\left(U/\widehat{U}_{\text{o}}\right) \tag{3}$$

The initial energy, $\widehat{U}_{\text{o}}$, will be taken to be a constant evaluated at the point of reference, a. According to an early interpretation of Mach's principle [5], the rest energy of an object is gravitational potential energy due to the separation of the object from distant matter. This is in line with the supposition that rest energy is the appropriate scale for these calculations, and $\widehat{U}$ might also be called Machian potential energy.

2.2 Free fall and the potential energy function

Consider the energy of an object of mass, m, falling from infinity toward a much larger body of mass, M. The Newtonian potential energy of the object at a distance, R, from the body is

$$U(R) = -\int_{\infty}^{R} F(r)\mathrm{d}r = \frac{-GMm}{R}. \tag{4}$$

This classical gravitational potential energy function is limitlessly negative, having a maximum, $U(\infty) = 0$. The Machian gravitational potential energy of the object calculated by the redshift rule (3) is

$$\widehat{U}(R) = \widehat{U}(\infty) \exp\left(\frac{-GM}{Rc^2}\right) \tag{5}$$

which includes the rest energy, $\widehat{U}(\infty) = mc^2$ at the point of reference $r = \infty$. It can be seen that potential energy calculated by the rule of redshifts is a finite positive quantity and confirms the postulate, a corollary of Mach's principle, that the kinetic energy gained in free fall can be no greater than the rest energy.

3 Reformulated potential in modern gravitation

3.1 Representation in the Schwarzschild metric

Lacking a theory which incorporates the nonlinear concept discussed in the Introduction, the following demonstration is suggestive of what might be expected. The Machian gravitational potential will be used to form Gullstrand-Painlevé coordinates which are completely determined by escape velocity. Since the metric in general relativity can be determined from Newtonian potential by way of classical escape velocity and Gullstrand-Painlevé coordinates, it is interesting to see what happens on substituting Machian escape velocity for Newtonian, the difference being a relativistic accounting of motion and a weak-field accounting of the relativistic effect of gravity. Note that the weak-field assumption arises from considering shells of infinitesimal mass and does not necessarily preclude application of the resulting metric to strong fields.

According to (5), with a scale factor given by

$$\widetilde{\sigma}(r) = \exp\left(\frac{-GM}{rc^2}\right) \tag{6}$$

the rest energy, E_o, of an object in a gravitational field falls to $E_o\widetilde{\sigma}$. The diminished rest energy can be increased kinetically by the Lorentz factor to give $E = E_o\widetilde{\sigma}\gamma$ with $\gamma = (1 - v^2/c^2)^{-1/2}$. The condition for radial escape is $E = E_o$ or $\widetilde{\sigma}\gamma = 1$. This gives an escape velocity $c(1 - \widetilde{\sigma}^2)^{1/2}$ which is limited to c. The conventional Schwarzschild scale factor from general relativity

$$\sigma(r) = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \tag{7}$$

is associated with the escape velocity $(2GM/r)^{1/2}$, which is the same as Newtonian escape speed.

For the exponentially transformed field of gravitational potential, the coordinate system of Gullstrand and Painlevé [7] gives the Schwarzschild metric in geometrized units ($c = G = 1$) as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + (\mathrm{d}r + \beta\mathrm{d}\tau)^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{8}$$

where τ is proper time in the frame of an object in free fall, initially at rest at infinity, with speed $\beta = (1 - \widetilde{\sigma}^2)^{1/2}$. Since β is less than the speed of light for any non-zero radius, there is no absolute event horizon. In static Schwarzschild coordinates, instead of $g_{tt} = -\sigma^2$ that term becomes $g_{tt} = -\widetilde{\sigma}^2$, and the resulting metric equation is then given by

$$\mathrm{d}s^2 = -\widetilde{\sigma}^2\mathrm{d}t^2 + \widetilde{\sigma}^{-2}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{9}$$

where t is proper time in the frame of a motionless object.

3.2 Prospects for experimental verification

No test of general relativity has ruled out the exponential metric. A second-order test of local position invariance could determine which version of the gravitational redshift is supported experimentally. Hatch [6] noted that the sign of the second-order term in the power series expansion of the Schwarzschild scale factor (7) is opposite that of the exponential scale factor (6). The frequency of a clock in a gravitational field varies directly as the scale factor. Expanding the Schwarzschild scale factor (7) gives

$$\sigma = 1 - \frac{GM}{rc^2} - \frac{1}{2}\left(\frac{GM}{rc^2}\right)^2 - \ldots \tag{10}$$

while the power series expansion of the exponential scale factor (6) is

$$\widetilde{\sigma} = 1 - \frac{GM}{rc^2} + \frac{1}{2}\left(\frac{GM}{rc^2}\right)^2 - \ldots \tag{11}$$

The difference, occurring in the sign of the second-order term, is detectable in principle using current technology. A clock test of relativity at four solar radii was proposed by L. Maleki and J.D. Prestage [8]. High temperature would make a direct test like this extremely challenging, but advances in clock accuracy could alleviate that problem by allowing a greater distance. The Gravitational Time Delay Mission [9] would measure the Shapiro time delay of a laser beam passing the Sun. A reference spacecraft at the Earth-Sun L1 Lagrange point in conjunction with a solar satellite could perform the measurement with a predicted accuracy sufficient to indicate the second-order term. The proposed LATOR mission [10] would use the international space station and two satellites in solar orbit employing laser interferometry to determine the second-order term by measuring the deflection of light. LATOR is projected to achieve the highest accuracy, but any of these experiments could decide between σ and $\widetilde{\sigma}$. Ben-Amots [11] has shown how an exponential potential might apply to electrostatics as well as gravitation, and that may provide another line of investigation. There is also the possibility that observations of Sgr-A* [12] near the center of the galaxy will reveal some clue. After nearly a century of technological progress since Eddington's confirmation of a first-order deflection of light, we are approaching an era of significant new information which will start with the measurement of a second-order effect.

4 Conclusion

This essay presents a relativistic reformulation of Newtonian gravitational potential energy as a consequence of the Einstein equivalence principle. The essential difference is that a relativistic treatment involves multiplication of quantities where they would be added classically. It is likely that a relic of Galilean relativity remains in the modern theory of gravitation.

The reformulated potential can be expressed tentatively in a modern context. With its associated horizon-free metric, this reformulation of gravitational potential exhibits properties such as finite normalization and flat geometry which are desirable for any quantum theory of gravity, but unattainable from conventional theory. A reconsideration of the role of gravitational energy is in order.

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6 Technical notes

6.1 Note 1

The implementation of the rule of multiple redshifts is similar to the calculation of compound interest. After one year of simple interest at a fixed rate of u per year, a unit investment increases u in value to $1+u$. Had the interest been compounded n times, the value of the investment would have become $(1+u/n)^n$. If the interest had been compounded continuously, then $n \to \infty$, and the unit investment would have become $\widehat{u} = \mathrm{e}^u$.

A continuously variable interest rate is handled in the same way. Integration of a function is a continuous summation defined as a limit. Given $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$ with the limits a and b in years,

$$u = \int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{12}$$

where u represents the accrued profit on a unit investment without compounding and f is the instantaneous profit – i.e., the product of the initial investment and an instantaneous interest rate. A function, $\widehat{u}$, can be defined to represent the total accrued value of an initial investment $\widehat{u}_\mathrm{o}$ with continuously compounded variable interest (at an effective rate given by $f/\widehat{u}_\mathrm{o}$):

$$\widehat{u} = \widehat{u}_\mathrm{o} \lim_{n\to\infty} \prod_{i=1}^{n} [1 + f(x_i)\Delta x/\widehat{u}_\mathrm{o}] \tag{13}$$

This can be rewritten using natural logarithms as

$$\widehat{u} = \widehat{u}_\mathrm{o} \lim_{n\to\infty} \exp\left(\sum_{i=1}^{n} \ln[1 + f(x_i)\Delta x/\widehat{u}_\mathrm{o}] \right) \tag{14}$$

Since $\ln[1 + f(x_i)\Delta x/\widehat{u}_\mathrm{o}] \to f(x_i)\Delta x/\widehat{u}_\mathrm{o}$ as $\Delta x \to 0$,

$$\widehat{u} = \widehat{u}_\mathrm{o} \exp\left(\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x/\widehat{u}_\mathrm{o} \right) \tag{15}$$

which simplifies to

$$\widehat{u} = \widehat{u}_\mathrm{o} \exp(u/\widehat{u}_\mathrm{o}) \tag{16}$$

In order to implement the compounding scheme, it is necessary to consider only non-dimensional functions in the exponent in (16) such as a ratio like an interest rate, or a redshift for which an initial wavelength is known. There is a related complication. The initial value, $\widehat{u}_\mathrm{o}$, may be unknown but can be inferred independently. The calculation of potential energy is such a problem, with indications that rest energy is the appropriate normalization.

6.2 Note 2

The exact relativistic form based on (1) will also be satisfied. Because the numerator and denominator can be separated into terms like (13), the exact relativistic form becomes something like

$$\sqrt{\frac{\widehat{u}_\mathrm{o}\exp(-u/\widehat{u}_\mathrm{o})}{\widehat{u}_\mathrm{o}\exp(u/\widehat{u}_\mathrm{o})}} = \exp(-u/\widehat{u}_\mathrm{o}) \tag{17}$$

The requirement $\widehat{u}_\mathrm{o} = 1$ would be appropriate for calculating redshift z, for example, involving dimensionless quantities like the scale factor.

6.3 Note 3

While I am satisfied with the derivation of gravitational potential energy in the first part of the essay, the second part seems rather problematic. It may be necessary to finagle the Einstein equations into yielding an exponential solution (presumably) instead of Schwarzschild. Without doing something to incorporate the nonlinearity inherent in the redshift due to its multiplicative nature into the field equations, it is difficult to trust that the deflection of light, for instance, will be calculated with sufficient accuracy for a second-order test. The expected exponential time dilation, for the static case at least, may be less suspect.