Quaternion Spectra

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Abstract

Quaternions arise from the solution of an elementary two-dimensional polynomial, and can be visualized in terms of amplitude and phase spectra by means of a multidimensional Fourier transform. Three-dimensional quaternion configurations can be found in the structure of an octonion, including one that lies outside a plane, a tetrahedron. These configurations are formulated as polynomials and spectra of their quaternion and nonquaternion solutions are discussed in relation to the Higgs field.

1 Introduction

With respect to *its* or *bits*, I expect most of us would agree with Wheeler's "supreme goal" [1], "*Deduce the quantum* from an understanding of *existence*." If we take existence to mean physical existence, that means comparing models of our understanding to experiment and observation. We must also necessarily endevour to comprehend the models and their origins. Examination of quaternion spectra falls into the latter category.

The underlying theme of this essay is resonance, a process involving transfer and storage of energy. The line of inquiry will follow a first-order polynomial. In the one-dimensional case, where such representations are easily realizable, polynomials are used to filter out resonance. The inverse of the polynomial corresponds to the resonant process. When viewing the spectra, it should be kept in mind that the search for zeroes of a polynomial is really the search for resonant frequencies which determine characteristics of the process.

The spectrum [2] is calculated using a Fourier transform which can be expressed concisely as a polynomial in a complex variable $z = \exp(-i2\pi f)$ called the unit delay operator where f is a real-valued frequency in the range $-1/2 \leq f < 1/2$. The delay operator z lies on the unit circle $|z| = 1$ and is strictly a function of frequency. The one-dimensional technique extends to higher dimensions.

Mathematical constructions called division algebras feature prominently in modeling the physical world from quantum mechanics to the space-time algebra. There are only four possible division algebras. They are the real numbers, complex numbers, quaternions and octonions, composed of 1, 2, 4 or 8 elements

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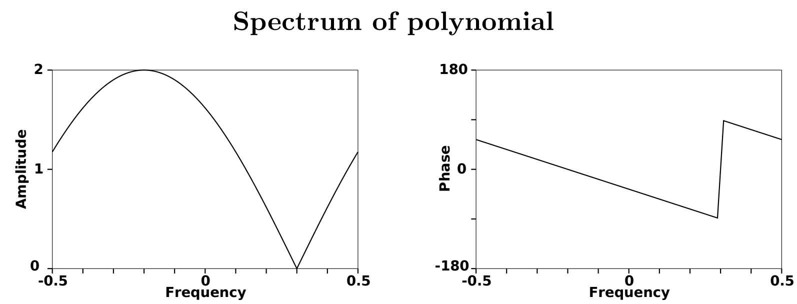


(a) (b)

Figure 1: Evaluation of polynomial $p(z) = 1 - az$ with $z = e^{-i2\pi f}$ gives the spectrum. The frequency at the point of zero amplitude can be set by solving $1 - az_o = 0$ to get $a = z_o^*$ with $z_o = e^{-i2\pi f_o}$ and $f_o = 0.3$ as displayed. Having obtained a value for the coefficient a, the spectrum is given by the amplitude and phase of $p(z)$ taken over the full range of frequencies. (a) Amplitude of polynomial, $|1-az|$, is maximum at antipode located at $-z_o$ or $f_o - 1/2 = -0.2$ (b) Phase of polynomial.

respectively. The elements map nicely onto a point, line, area or volume. Taking a Fourier transform on a mapping allows quaternions and octonions to be displayed as amplitude and phase spectra. Configurations of quaternions embedded in the vector space of octonions appear in three forms and each of these forms has three orientations. Two forms correspond to slices through the octonion vector space and are essentially two-dimensional. The third form is a tetrahedron in three dimensions.

To complicate matters, a two-dimensional polynomial can be found with quaternion and nonquaternion solutions. Its quaternion solutions correspond to quantum states in equal superposition only. Quaternions whose existence is derived from the elementary polynomial would implicitly have this limitation. A nonquaternion solution to this polynomial is included because it has the ability to provide a unique spectral representation, whereas quaternions must exhibit two spectral copies.

2 Polynomial spectra

A polynomial in one dimension can be constructed from a sequence of complex coefficients $(1, -a)$ mapped to the time domain $(1, z)$ taken from the present and past of the time line $(\ldots, z^{-1}, 1, z, z^2, \ldots)$ comprised of unit delays. The inner product gives $(1, -a) \cdot (1, z) = p(z) = 1 - az$. An example of its spectrum is shown in Fig. 1. The frequency domain is circular and wraps around. Finding the coefficient a for $p(z) = 0$ at a given frequency is straightforward.

A polynomial in two dimensions requires the introduction of a second delay operator $w = \exp(-i2\pi g)$ with frequency g in the added dimension. But the polynomial must not have one-dimensional factors. For example, the polynomial $1 \pm z \pm w \pm zw$ has eight possible realizations. Four can be seen to be $(1 \pm z)(1 \pm w)$. The remaining four are unfactorable and correspond to the set of 2×2 Hadamard matrices. Following these clues, one of the unfactorable elementary polynomials is constructed from mapping the coefficients $(1, -a, -b, -ab)$, where b is an added complex coefficient, onto the 2D time domain $(1, z, w, zw)$ to get

$$p(z, w) = 1 - az - bw - abzw \tag{1}$$

whose solution involves a quadratic equation. Among its solutions are the set of Hadamards. The coefficients must satisfy $|a| = |b| = 1$ for the solution to be quaternion.

By comparison the quaternion as a polynomial

$$q(z, w) = u + vz - v^* w + u^* zw \tag{2}$$

is mapped from a matrix form of the quaternion,

$$Q = \begin{bmatrix} u & v \\ -v^* & u^* \end{bmatrix}$$

with u, v complex. The condition $|u| = |v|$ yields an equal superposition of Pauli basis states, and thus also includes the set of Hadamards. To distinguish these states, which are common to both polynomials (1) and (2), any quaternion in a state of equal superposition will be called a Hadamard quaternion (or Hadamard solution). Furthermore, to distinguish the other extreme, any quaternion with $|u| = 0$ or $|v| = 0$ will be called a Pauli quaternion. The polynomial (1) can have no Pauli quaternions as solutions. Two-dimensional spectra of the original Hadamard and Pauli sets are shown in Fig. 2. The frequency domain in two dimensions is toroidal. The naming convention used here extends the meaning to all translations of the amplitude spectra on the torus. In other words, the spectra will appear similar for members in the same class, except shifted in frequency.

Although the elementary polynomial (1) cannot produce Pauli quaternions, its solutions include a subset of the biquaternions which I have been calling nonquaternion, but "almost quaternion" might be a better description because both quaternions and "nonquaternions" have four degrees of freedom compared to eight for biquaternions. The biquaternions include all complex 2×2 matrices. The nonquaternion polynomials are mapped from

$$A = \begin{bmatrix} u & v \\ -v^* & u^{-1} \end{bmatrix}, \qquad B = \begin{bmatrix} u & v \\ -v^{-1} & u^* \end{bmatrix}. \tag{3}$$

The magnitude of v in A, or u in B will determine whether the nonquaternion polynomial is classified as Hadamard or Pauli.

2D Spectra of Hadamard and Pauli quaternions

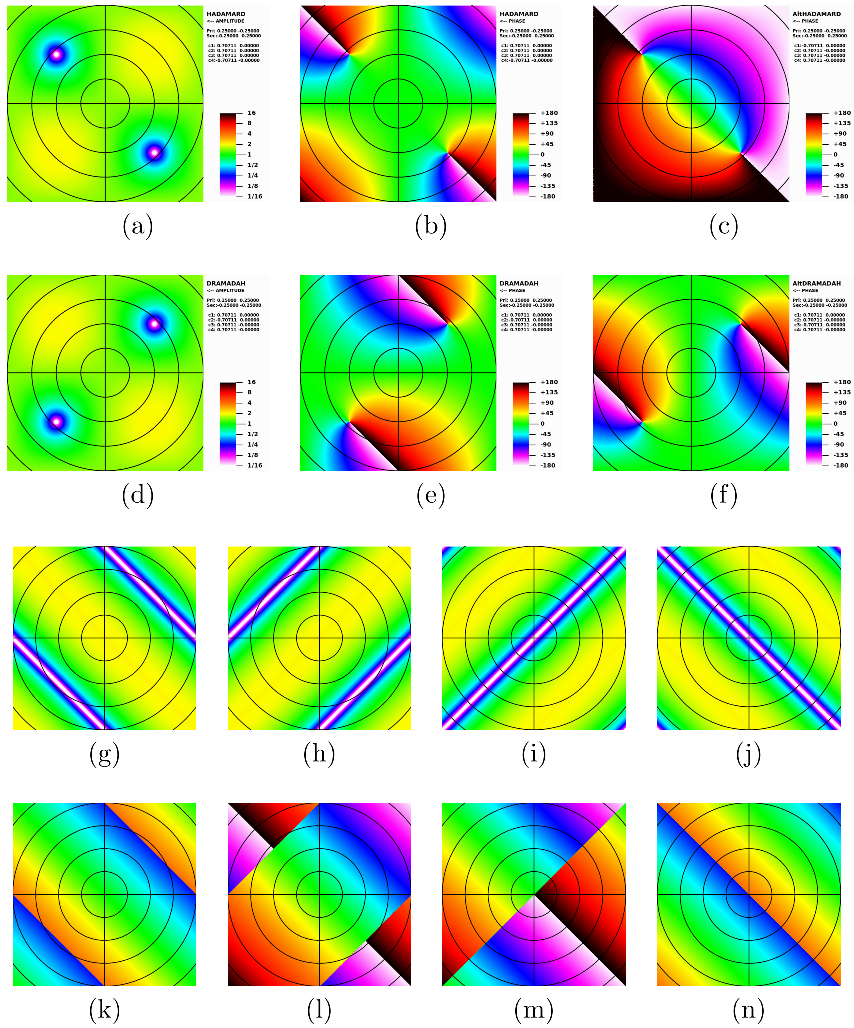
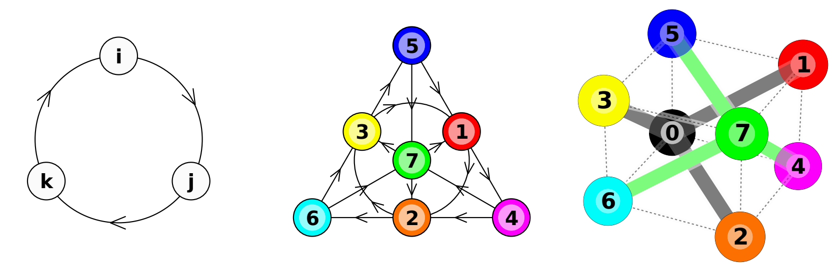


Figure 2: Hadamard in top two rows, Pauli in bottom two rows. The frequencies are in the range $[-1/2, 1/2]$ with f horizontal and g vertical. (a) Amplitudes of Hadamard polynomials H_0 and H_1 are identical. (b) Phase of $H_0 = 1 + z + w - zw$, a Hadamard matrix. (c) Phase of alternate solution, $H_1 = 1 - z - w - zw$, another Hadamard matrix. (d) Amplitude of polynomials H_2 and H_3. (e) Phase of $H_2 = 1 + z - w + zw$. (f) Phase of alternate solution, $H_3 = 1 - z + w + zw$. (g) Amplitude spectrum of identity matrix: $1 + zw$, (h) Pauli-X: $z + w$, (i) Pauli-Y: $z - w$, (j) Pauli-Z: $1 - zw$. (k-n) Phase of identity and Pauli matrices.

Fano plane



(a) (b) (c)

Figure 3: (a) A quaternion's imaginary components i, j and k multiply pairwise according to a cyclic ordering around the figure. For instance, $ij = k$ but $ji = -k$. (b) The Fano plane shows the seven imaginary elements of an octonion having seven sets of quaternion cyclic ordering given by three sides of the triangle (426, 635, 514), the three altitudes (473, 671, 572), and the circle of midpoints (123). (c) In the vector space of the octonions which includes the real part, each element is at a corner of a cube. This is the "time domain" representation of an octonion. The quaternion configuration formed from the circle of midpoints is shown schematically linked together by dark lines. The green lines represent its complement in the structure of the octonion.

3 Quaternion spectra in three dimensions

The Fano plane provides a means of identifying quaternion configurations in the three-dimensional structure of an octonion [3] which has its elements, a real number or one of seven imaginary numbers, at the corners of a cube as shown in Fig. 3. Taken with the common real element, the three sides of the triangle correspond to the three sides of the cube. The altitudes of the triangle correspond to three diagonal cleaves through the cube. The circle of midpoints of the triangle is unique since, when combined with the real part, it alone corresponds to a figure existing outside a plane – a tetrahedron with vertices at alternate corners of the cube. Three-dimensional spectra shown in Fig. 5 can be obtained by incorporating a third delay operator and forming a polynomial. Quaternion polynomials will then be of the form

$$q(z_\mathrm{x}, z_\mathrm{y}, z_\mathrm{z}) = u + vI - v^*J + u^*K \tag{4}$$

where I, J and K are delays composed from the set $z_\mathrm{x} = e^{-i2\pi f}$, $z_\mathrm{y} = e^{-i2\pi g}$ and $z_\mathrm{z} = e^{-i2\pi h}$ and there are now three frequencies for the three directions. A tetrahedron might have $I = z_\mathrm{x}z_\mathrm{y}$, $J = z_\mathrm{y}z_\mathrm{z}$ and $K = z_\mathrm{z}z_\mathrm{x}$.

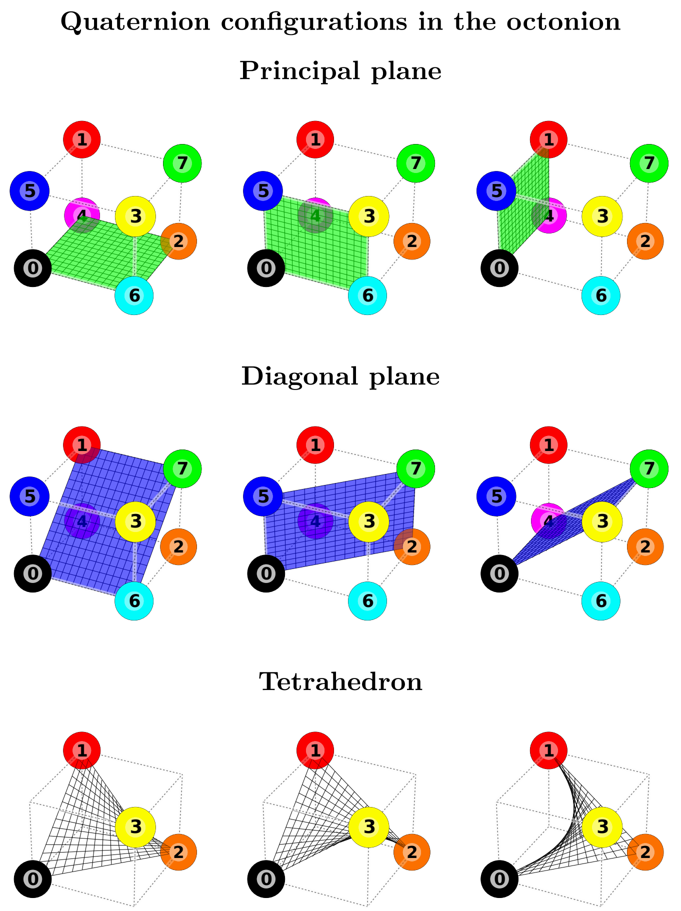


Figure 4: The three tetrahedral configurations can be viewed as permutations of arrangements of the corners of a square sheet, so in a sense even this configuration is two-dimensional. Taking corner 0 (the real part) as fixed, the opposite corner on the sheet is at point 3, 2, or 1 for the tetrahedron from left to right. What distinguishes tetrahedrons is the 90° twist giving the sheet a saddle shape. The corners of principal or diagonal plane configurations are coplanar. Permuting the corners of the sheet for these configurations, if allowed, would result in a 180° twist (not shown).

Typical Hadamard quaternion and nonquaternion spectra

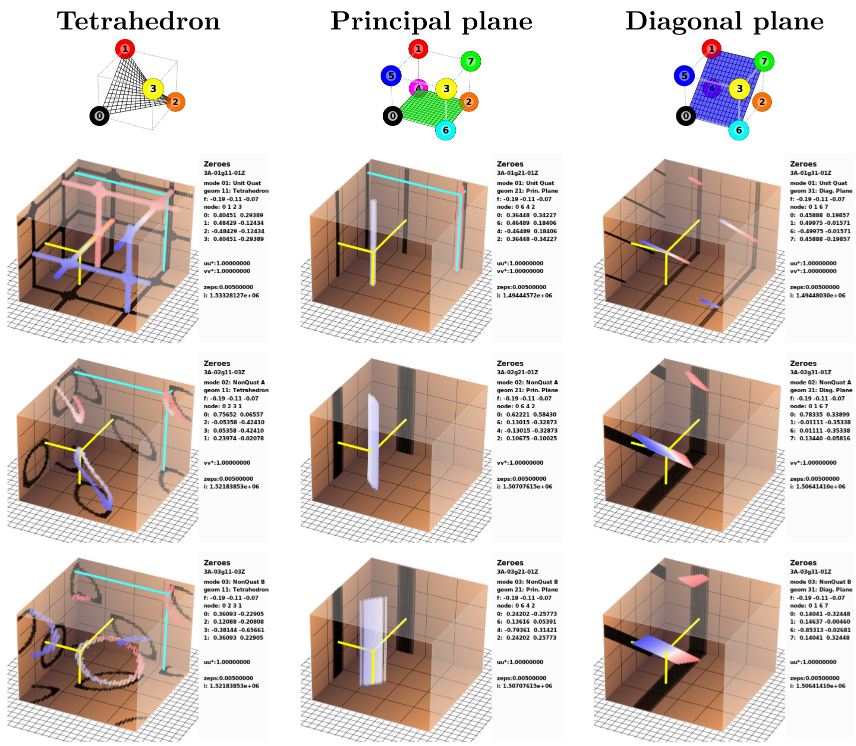


Figure 5: **Top row:** Spectra of quaternion solutions involve pairs of straight lines of zeroes for the condition $|u| = |v| = 1$ in Eq. (2). The tetrahedron and principal plane geometries have antipodal zeroes. Quaternion solutions for the diagonal plane geometry have a shifted antipodal zero. **Second row:** Nonquaternion solutions are lines when A has $|v| = 1$ or B has $|u| = 1$ in Eq. (3). For the tetrahedral geometry, solutions are primarily a pair of antipodal loops. Nonquaternion solutions show a ribbon-like quality for the plane geometries which is a side-effect of displaying grid points whose amplitude is within a small range of zero. **Bottom row:** The difference between the two types of nonquaternions is seen in the orientation of the ribbon for plane geometries. *Yellow struts indicate frequency intended for root. Blue struts indicate 3D antipode. Locations of zeroes are tinted red in the background, blue in the foreground. Black areas are 'shadows' indicating location of zeroes.*

All tetrahedrons and all quaternions have two copies.

Dimensional frames of reference

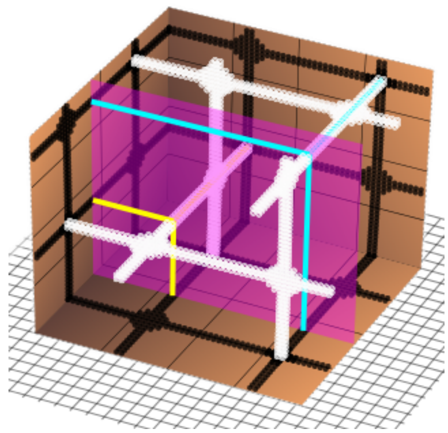


Figure 6: A 2D quaternion solution to the tetrahedral polynomial is used to place antipodal zeroes in the indicated x-y frequency plane. The solution lies along two lines of zeroes from a set of six lines that look like interconnected piping. This 2D solution is the same as the 3D quaternion solution shown in Fig. 5. The 3D nonquaternion solutions in Fig. 5 usually give rise to a pair of loops, but the extent of the loops varies with frequency. The extent is greatest whenever one of the frequencies approaches $\pm 1/4$ resulting in a pipe-like structure shown here. Note the shadows parallel to the colored x-y plane indicating those frequencies in the z direction.

The example in Fig. 6 could serve as a demonstration of different dimensional frames of reference, a phrase used by Armin Nikkah Shirazi to help convey the concepts of 'actual' and 'actualizable' in his dimensional theory of quantum mechanics [4]. To illustrate an essential point, he notes that the way to extend the existence of a point in a plane from 2-space to 3-space is to consider a line through the point parallel to the extra dimension. He then goes on to apply this concept to spacetime worldlines and arrives at the Feynman path integral from elementary principles. In the present case, placing antipodal zeroes in a principal plane (a plane orthogonal to a coordinate axis) produces a quaternion solution consisting of lines of zeroes passing through isolated points in the plane into the third dimension. In Fig. 2, the Hadamard polynomials are zero only at two points in the frequency domain, while Pauli polynomials show a line of zeroes. Because of this difference in dimensionality between their polynomial zeroes, the *actualizable* state would correspond to the Hadamard condition which is an equal superposition of *actual* Pauli states. Whether there is something more substantial to this interesting connection is a matter for further study.

4 Discussion

Quaternions show up in a variety of contexts, but perhaps most topical is in relation to the Higgs field, given last year's possible discovery of its boson. The field is thought to exist throughout space, even in the absence of gravitational or electromagnetic fields. The mechanism involves the theory of superconductivity and a Higgs condensate. The primary structure of the field is that of the unit quaternion.

Do the three quaternion configurations in the octonion play a role in the Higgs mechanism? The tetrahedron stands out as three-dimensional. The extent of spectral zeroes for the diagonal plane configuration is twice that of the principal plane. From these basic considerations, it is tempting to associate the tetrahedron with spin 1/2 (matter), the principle plane configuration with spin 1 (electromagnetism), and the diagonal plane with spin 2 (gravitation). Perhaps the observation of three generations of fermions has something to do with the three configurations of quaternions.

In both two and three dimensions, the frequencies $\pm 1/4$ are associated with the Hadamard condition which implies equal superposition of Pauli states. These central frequencies would seem to be a significant point of reference. It is notable that a condensate composed of octonion cells half-occupied by tetrahedral quaternions could match Vladimir Tamari's conclusions [5] about structure of the vacuum involving Kepler packing. In that case a zero-phase condition ought to be attainable through symmetric configurations in a $3 \times 3 \times 3$ time domain corresponding to the face-centered cube of Kepler packing.

Finally, this essay raises a point concerning the origin and use of quaternions in quantum mechanics. The elementary polynomial (1) has both quaternion and nonquaternion solutions. If it should happen that the elementary polynomial is somehow at the root of the theory, then quaternions may not be the only forms required to express the Higgs mechanism. Nonquaternion forms could also be involved. The tetrahedral configuration is well matched to a quaternion solution because they both necessarily result in a spectrum consisting of two copies. Quaternions might be unsuitable for the planar configurations for that very reason, since electromagnetism and gravitation are directional. The unambiguous single copy provided by nonquaternions might be an advantage in that case. More examples of spectra with solutions can be found in [6].

I can imagine the picture of existence that emerges could be analogous to a quantum computer operating coherently in the Higgs condensate. The operational units might involve "mass qubits" and computational gates having two functions: determining the state of actualization based on Nikkah Shirazi's dimensional theory, while performing the required operation by superposition or mixture of alternative states. In this view, *its* are *actual* states and *bits* are *actualizable* alternative states which influence the outcome. Even with these assumptions, the relation between *its* and *bits* remains unknown.

References

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5 Notes

Constant factors have been omitted from polynomials and matrices. Note that the usual convention for normalizing the determinant to form a unit quaternion by scaling u and v will not work for nonquaternions of the form (3). The quaternion determinant is $|u|^2 + |v|^2$ while the nonquaternion determinants are $1 + |v|^2$ or $1 + |u|^2$. Scaling u and v by the square root of the determinant normalizes the quaternion, but would not normalize the nonquaternions.

A point in the frequency domain represents a complex-valued plane wave heading towards the origin in the time domain. Its frequency f obeys $f^2 = f_{\mathrm{x}}^2 + f_{\mathrm{y}}^2 + f_{\mathrm{z}}^2$. As a plane wave passes an array of sensors, it is the geometric *(spatial)* arrangement of the sensors that determines the frequency sensed in each direction. This simple interpretation holds as long as points in the time domain constructed from delay operators correspond to places in Euclidean space.

None of the spectra in Fig. 5 can be localized to a single zero in the 3D frequency domain. Only the nonquaternion planar configurations allow that possibility. Adding nonquaternions from two different orientations of diagonal plane configurations, for instance, can produce a polynomial with a single spectral zero.