The illusion of Hilbert space

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Abstract

I argue that Hilbert space, one of the foundational elements of quantum physics, is unphysical in the context of quantum many-body systems. Physical states reside in a tiny corner of Hilbert and are not best thought of as exponentially long vectors. The important question is how to characterize the space of physical states, and I suggest that it may be useful to take a quantum computer's view of the world. Finally, I apply this reasoning in a specific case to obtain a description of the universal aspects of quantum ground states in terms of an emergent entanglement geometry.

INTRODUCTION

I think Hilbert space is an illusion. There is no doubt that it is a sometimes convenient description of the physical states of a quantum system, but I will convince you that it is ultimately a false ideal that leads to unphysical thinking. The illusion is very subtle because it is totally harmless when we consider systems of few components. Indeed, the illusion is only revealed when we turn to larger systems, to quantum many-body systems, such as superconductors or the universe itself. The fundamental problem is that the many-body Hilbert space is just too big.

Before proceeding, let me immediately say that this is not an essay about the foundations of quantum mechanics, at least not in the sense that I am proposing some alternative theory that makes no mention of Hilbert space. Hilbert space continues to play a background role in the argument, but it becomes a secondary structure. Instead, this is an essay about the foundations of quantum matter in which we argue that Hilbert space should be given up as a false foundation and replaced with something more physical.

Quantum matter is general term for any kind of large system composed of many small pieces subject to the rules of quantum physics. We have plenty of examples of effectively classical matter, but there are a growing number of experimental situations in which truly quantum many-body phenomena are apparent. The foundational issue at hand is emergence. We simply have new phenomena and new questions to ask when the system is a large piece of quantum matter. We are especially struck by the feature of universality, where many microscopic systems give rise the same effective physics at low energies and long distances. At the same time, the old questions, e.g. what is the exact spectrum of energy levels, become less and less interesting. Quantum matter cannot be understand as the sum (really tensor product) of its parts, so the fundamental question is how do we understand the universal physics of quantum matter?

The essay is divided into roughly three parts. First, I will argue that the many-body Hilbert space is a superfluous and illusory concept. The material in this section is part of the lore of quantum matter, although it is difficult to find it written down in one place. Second, I will sketch a formal replacement for Hilbert space that focuses more on process than state. This section draws on ideas from quantum computation and boils down to the idea that physical states are those preparable by a quantum computer. Finally, I will specialize to a particular problem and show how to visualize the quantum state as a dynamical process using an emergent geometrical description. This descriptions draws on ideas from entanglement based simulation methods [1, 2, 3, 4] and holographic duality [5, 6, 7].

THE ILLUSION OF HILBERT SPACE

Imagine we have a single electron trapped in a potential well somewhere in space. Since the electron cannot move, its only degree of freedom is its spin. The electron spin can be either up, represented by $|\uparrow\rangle$, or down, represented by $|\downarrow\rangle$, or indeed any combination of up and down, represented by $c_\uparrow|\uparrow\rangle + c_\downarrow|\downarrow\rangle$. The space of possible electronic spin states forms the Hilbert space $\mathcal{H}$ of the system. In this case, it is a two complex dimensional vector space (the amplitudes $c_\uparrow$ and $c_\downarrow$ are complex) which we may call $\mathcal{H}_1 = \mathbb{C}^2$. A quantum state $|\psi\rangle$ is then any vector in the Hilbert space $\mathcal{H}_1$. Given the state we can compute physically measurable correlations. For example, the expectation value of an observable $\hat{O}$ is

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle. \tag{1}$$

Finally, there is a special operator called the Hamiltonian H, which is physically the energy of the system, that generates dynamical evolution for the quantum state. Schrodinger's equation for the time-dependent state $|\psi(t)\rangle$ (using units where $\hbar = 1$) is

$$i\partial_t |\psi(t)\rangle = H|\psi(t)\rangle. \tag{2}$$

Returning to our electron pinned in a potential well, we can ask what kinds of Hamiltonians can be realized physically. We can certainly apply a magnetic field $\vec{B}$. The electron spin $\vec{S}$ is associated with a magnetic dipole moment $\vec{\mu}$ so classical electromagnetism gives an energy $H = -\vec{\mu} \cdot \vec{B}$. We will dispense with the numerical factors and simply observe that $\vec{\mu} \propto \vec{S}$ and hence the Hamiltonian may be taken to be

$$H_1 = -\Delta \sigma^z \tag{3}$$

where Δ is an energy proportional to $|\vec{B}|$ and $\vec{B}$ points in the z direction. The operator σ^z appearing in Eq. 3 is the usual Pauli matrix

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis. With this Hamiltonian the electron spin wants to align with the field because the lowest energy state is $|\uparrow\rangle$ ($E_\uparrow = -\Delta$ and $E_\downarrow = \Delta$).

An important question is what states in Hilbert space can be realistically achieved? We could solve Eq. 2 for various choices of H and initial states $|\psi(0)\rangle$ to get some intuition, but there is simple and general argument that provides an answer to our question. The trick is to imagine letting the direction of the magnetic field vary slowly in time. Our intuition is that if we start in the ground state and we change the Hamiltonian slowly enough (adiabatically), then the quantum state will always be close to the instantaneous ground state of the Hamiltonian. How slowly we need to go depends on the size of the energy splitting Δ. Roughly speaking, if we change the field over a time T satisfying $T \gg 1/\Delta$ (remember $\hbar = 1$) then the spin will not be excited out of the ground state. Thus since every state in the Hilbert space $\mathcal{H}_1$ is the ground state of the magnetic field Hamiltonian for some direction of the field, every state can be reached from any other state by the quantum dynamics in Eq. 2 in a time of order $1/\Delta$. Hence apply a big enough field and you can explore the Hilbert space of the spin as fast as you want. What could be simpler? Clearly no illusions here.

Does anything change for two two spins? Let's bring up another electron next to our original ($N = 2$ electrons). A basis of states for the composite system can be obtained by tensor product, e.g. $|\uparrow\uparrow\rangle \equiv |\uparrow\rangle \otimes |\uparrow\rangle$, from the states of the component systems. The full Hilbert space is now $\mathcal{H}_2 = \mathbb{C}^4$ since there are four combinations of up and down between the two spins. We can still reach any state from any other by slowly changing the Hamiltonian over a time of order $1/\Delta$ where Δ is the smallest energy splitting. The Hilbert space $\mathcal{H}_2$ still looks like a sensible place, but remember that you were warned that the illusion was subtle for small systems.

There is one important new feature of the two particle system. This feature is called entanglement, and it occurs when the state of the composite system doesn't factorized over the subsystems. For example, the state $|\downarrow\downarrow\rangle$ factorizes and is called a product state, but the state

$$|S = 0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \tag{5}$$

is entangled because it cannot be written as a product, $|S = 0\rangle \neq |\phi_1\rangle|\phi_2\rangle$. $|S = 0\rangle$ is nothing but the spin singlet state of the two spins, but more importantly, it can be regarded as the elementary example of entanglement. Entanglement between two systems can be quantified by counting how many effective singlets they share. Entanglement will appear again later, but for now note that it is an essential ingredient in truly many-body states (otherwise they would break up into pieces that don't interact).

Now we make the leap to a number N of electron spins equal to, say, Avogadro's number $N = 10^{23}$. No problem, you say, I can easily list a basis of states for the system of N electron spins. The basis is

$$|m_1\rangle \otimes ... \otimes |m_N\rangle \tag{6}$$

where $m_i = \uparrow$ or $m_i = \downarrow$ depending on whether the i-th spin is up or down. There are 2^N such states and each can be the ground state of a simple Hamiltonian, indeed

$$H_N = \sum_{i=1}^{N} -\Delta_i \sigma_i^z \tag{7}$$

has just such a ground state where $m_i = \uparrow$ if $\Delta_i > 0$ and $m_i = \downarrow$ if $\Delta_i < 0$. But of course, this isn't the whole story since we must also permit superpositions. Indeed, the formal Hilbert space of the system is a 2^N dimensional complex vector space $\mathcal{H}_N = \mathbb{C}^{2^N}$.

Physically, what kinds of superpositions can we achieve? Clearly something like like

$$|\psi_x\rangle = \otimes_i (|\uparrow_i\rangle + |\downarrow_i\rangle) \tag{8}$$

is an allowed state (it simply represents all the spins pointing in the x-direction). On the other hand, the "cat state"

$$|\psi_{cat}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_1 \dots \uparrow_N\rangle + |\downarrow_1 \dots \downarrow_N\rangle\right) \tag{9}$$

is less obviously sensible since it represents a macroscopic superposition, but even this state is perhaps not too crazy for moderate N. Note that we are able to give a succinct classical description of the 2^N amplitudes for both these states.

The simplest way to see that there must be states that are qualitatively unlike the relatively simple states above is to count possible states. Imagine that our spins are arranged in a one dimensional chain so that each spin can only talk to a few of its neighbors. The Hamiltonian of this system will consists of many local terms describing the interactions of neighboring spins. Locality means that only neighboring spins interact and that interactions involve only a few spins at a time. Now suppose we can describe all these interactions with a few parameters, call them $g_1, ..., g_k$. An example of such a Hamiltonian, called the quantum Ising model, is

$$H_{Ising} = -g_1 \sum_i \sigma_i^z \sigma_{i+1}^z - g_2 \sum_i \sigma_i^x. \tag{10}$$

The g_1 term favors neighboring spins to align along the z-axis while the g_2 term favors each spin to point in the x-direction. When $g_2 \gg g_1$ the ground state of H_{Ising} looks like $|\psi_x\rangle$ while when $g_1 \gg g_2$ the ground state looks like $|\psi_{cat}\rangle$.

More generally, the ground state of the Hamiltonian $H_N(g_1, ..., g_k)$ depends on these k parameters, but the state is a 2^N component vector and hence all 2^N components are determined by just a few numbers! Even if N is merely 100 and $k = 2$ we are talking about a two dimensional subset of a 2^{100} dimensional space! Furthermore, if we change one of the parameters by a small amount, the quantum state changes drastically

$$\langle\psi(g)|\psi(g + \delta g)\rangle \sim e^{-N\delta g}. \tag{11}$$

In other words, the overlap between nearby states, even within the same phase of matter, is nearly zero. The quantum state does not appear to be capturing the universal physics of quantum matter in that two almost orthogonal states can describe nearly identical physics.

Perhaps we just don't have enough parameters? But even if we allow an independent magnetic field for every spin and a different two spin interaction between every spin, we would still only have roughly $N + N^2$ parameters, far lass than 2^N. So what, after all, are all those complex numbers really telling us? Do we need them to predict the results of physical measurements? If so, we're in trouble. Ignoring causality and the lack of materials, even if we filled up our entire Hubble volume, the whole visible universe, with our best classical storage device, we could only store the quantum state of a few hundred spins using this huge classical memory. Suddenly the illusory nature of Hilbert space is brought into focus.

So far we have considered only equilibrium states, but to drive home the message, let us return again to quantum dynamics. I argued above that one could easily get around the whole Hilbert space of a single spin in a short time. What about moving around in the many-body Hilbert space? Another simple counting argument will reveal the futility of this idea. The Hamiltonian dynamics in Eq. 2 may be formally solved to yield

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \equiv U(t)|\psi(0)\rangle. \tag{12}$$

The operator $U(t)$ is unitary, so a reasonable simplified model of quantum dynamics arises by imagining we can apply an arbitrary two spin unitary transformation to any two spins we like at every time step. Formally, this can be achieved by considering a very special time dependent Hamiltonian. If each two spin unitary has a few parameters, then applying M unitaries over M time steps gives us a state with roughly M parameters. Thus again we see that in order to have a state where we can really control all the amplitudes, we must evolve for a time that is exponentially large in the number of spins so that $M \sim 2^N$. The expanding universe will have long since gone cold and empty before we can achieve such control even for a few hundred spins.

I have shown that Hilbert space is a very big place. Most of the states in a many-body Hilbert space never arise as ground states of local Hamiltonians. Similarly, most of the states in Hilbert space can never be reached by time evolution with a local Hamiltonian. Such evolutions would take much longer than the age of the universe, not to mention the lifetime of the experimenter or of the human species. Thus since most of the states in Hilbert space are unphysical, Hilbert space is a kind of illusion. We will never be able to verify the existence or properties of most of the states in Hilbert space and hence, for all practical purposes, Hilbert space is nothing more than a background construct.

THINKING LIKE A QUANTUM COMPUTER

The next logical step is then to ask what does the space of physical states look like? As I mentioned at the beginning, we will keep Hilbert space around as a kind of kinematic stage on which quantum dynamics play out, but we don't want to characterize physical states in terms of exponentially long vectors. We've already seen a clue to the answer when thinking about the Hilbert space of a single spin: we were quite content if we could dynamically reach the state of interest in a reasonable time. Our definition of the space of physical states is then very simple. A state is physical if it can be reached from a product state like $| \uparrow_1 ... \uparrow_N \rangle$ after time evolution, Eq. 2, with a local Hamiltonian for a reasonable time. The notion of a reasonable time is obviously imprecise and depends on the experimental circumstances, but certainly we can't consider time evolution taking longer than the age of the universe.

Locality is very important in this definition since any two states in Hilbert space can be connected by time evolution with a time of order one if we allow completely arbitrary Hamiltonians. Suppose you want to go from state $|\psi(0)\rangle$ to state $|\psi(1)\rangle$. Pick any unitary U with $|\psi(1)\rangle = U|\psi(0)\rangle$. Finally, define the Hamiltonian H_{bad} by $U = e^{-iH_{bad}}$ (any H_{bad} satisfying the equation will work and there are many solutions). We call this Hamiltonian H_{bad} bad because although it does what we want, it is typically extremely non-local and hence could never be engineered in a lab. Here non-locality means not only that all spins talk to each other but also that H_{bad} involves two, three, ... even N spin interactions.

A formalization of the above definition of physical states is provided by the notions of quantum computation and quantum complexity. Simply put, a quantum computer is a (quantum) information processing device that takes in an input state $|\psi_i\rangle$, performs some kind of time evolution described by a unitary operator U, and then measures the final state $|\psi_f\rangle = U|\psi_i\rangle$ to obtain the answer to a question (see [8] for an introduction). A quantum algorithm is simply a choice of initial state, unitary evolution, and measurement scheme that enables you to answer a question in a reliable way. For example, we might want to answer the question, "What is the ground state energy of H_{Ising} as a function of g_1 (to some accuracy)?" Here is a possible quantum algorithm: find a time evolution U that prepares the ground state of H_{Ising} from the initial state $| \uparrow_1 \dots \uparrow_N \rangle$ and then measure the energy of the resulting state.

Would this algorithm work? In fact, it will because we actually know an efficient classical algorithm to do the same thing (called quantum Monte Carlo), and quantum computers are at least as powerful as classical computers. More generally, quantum complexity theory is tasked with determining what a quantum computer can achieve given reasonable resources, including energy, time, and space. In the context of complexity theory, "reasonable resources" means resources that scale polynomially with the number of elements in the computer, be they spins, mobile electrons, or whatever (in general the elements are called qubits). So on a quantum computer composed of N spins, an algorithm that requires a running time of order N^3, say, is considered reasonable while an algorithm that has a running time of order 2^N is considered unphysical. Nevertheless, an important point to understand is that these are asymptotic statements. Certainly we are not maintaining that an algorithm taking time 2^N when $N = 2$ is somehow impossible to run, but we are claiming that the algorithm becomes rapidly infeasible to implement as N gets large.

With the modern terminology of quantum computation in hand, we can rephrase our original definition of physical states in the following precise way. A state is physical if it can be prepared using a quantum computer from a product state of N spins in a time that scales polynomially with N. Unfortunately, while this is a precise and physical definition, it doesn't seem terribly useful until quantum computers hit the shelves. Can we put ourselves in the shoes of a quantum computer now, to see the world as a quantum computer sees it? What does that even mean?

If you ever played with blocks as a kid, you probably tried to build various structures with blocks. Sometimes they were stable and sometimes they fell to pieces. As you built more structures you probably slowly got a better sense of what could and couldn't be achieved with the blocks. Now I want you to imagine that you're a baby quantum computer (or perhaps a baby with access to a quantum computer). Your blocks are small unitary transformations called quantum gates. They have names like the Hadamard gate,

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{13}$$

the $\pi/8$ gate,

$$U_{\pi/8} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}, \tag{14}$$

and the CNOT gate,

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{15}$$

You start at the ground (a product state) and build upwards by applying your blocks however you like. At first you probably don't get anywhere interesting, but after a time you might start to get the hang of things. You learn how to make all kinds of interesting states, things like metals and insulators, magnets and superfluids. You might do puzzles of a sort, where you're given a state and your job is to break it up into unentangled pieces as fast as possible. I think a quantum computer growing up like this is bound to have a rather different perspective on quantum matter than the average person.

How does such a computer (or a person who grew up with such a computer) understand quantum matter? One thing is immediately clear, the computer doesn't know anything about 2^N complex coefficients in some abstract space. It understands process and history, since for it quantum states and even measurements are all a matter of choosing the right dynamics. The exponential audacity of Hilbert space would be the furthest thing from its mind. Of course, one day our quantum computer may go to graduate school and then it will probably be forced to learn about Hilbert space. But I have to imagine that it will always regard exponentially big vectors with suspicion. So to ask our question again, can we get a glimpse of how a quantum computer would think about quantum states?

THE GEOMETRY OF ENTANGLEMENT

It will help to focus on a very particular question about quantum matter. In the context of our spin model, we can ask whether the system magnetizes at zero temperature or whether quantum fluctuations "melt" the magnetic order and give rise to a liquid-like state [9]. In other words, we ask what the ground state of the system looks like. To answer this question, our considerations above suggest that we abandon, at least partially, the framework of Hilbert space and think about process and dynamics. This is true even though we eventually want to describe equilibrium states. We are obviously not going to have the same richness of intuition as the quantum computer, but we can ask if there is a way to visualize at least the most basic aspects of quantum ground states as a quantum computer would understand them.

Remember that we argued that entanglement was a fundamental component many-body states. We might even call it the fabric of quantum matter. We set for ourselves the task of understanding the structure of entanglement in quantum ground states, specifically the universal aspects of entanglement. Let's begin with an unentangled state given by $|\uparrow \ldots \uparrow\rangle$. A quantum computer would have quickly learned how to generate entanglement from this state (a combination of Hadamard and CNOT will work). So perhaps we should imagine building up a quantum state by systematically creating the necessary entanglement. In fact, its more convenient to imagine starting with the state of interest and systematically disassembling it by removing all entanglement.

Since we're interested in universal physics, the key point is that the quantum computer acts locally while universal physics emerges at long distances. Many different choices of spin Hamiltonian will give the same physics at long wavelengths and low energy, so what we're really talking about is coarse-graining. When we measure thermodynamic quantities or electrical conductivity or even spectral information we are still ultimately probing the collective dynamics of many degrees of freedom. Coarse-graining or renormalization is the process whereby we focus on the universal aspects of this collective dynamics which emerges at low energies (low temperatures) and long distances.

Consider the process, illustrated in Fig. 1, which we call entanglement renormalization [10, 11]. We start with some complex many-body state. The first thing we do is remove a little bit of local entanglement between nearby spins (blue squares in Fig. 1). As a result some of the degrees of freedom become unentangled with the rest and we can coarse-grain the system by removing the unentangled states (red triangles in Fig. 1) since they no longer contribute to physics at longer length scales. Notice how the size of the original lattice has been shrunk by a factor of two in this process, so we really are coarse-graining the system. Now we have a new renormalized state in terms of new coarse-grained degrees of freedom. Thus we can repeat our procedure at this longer

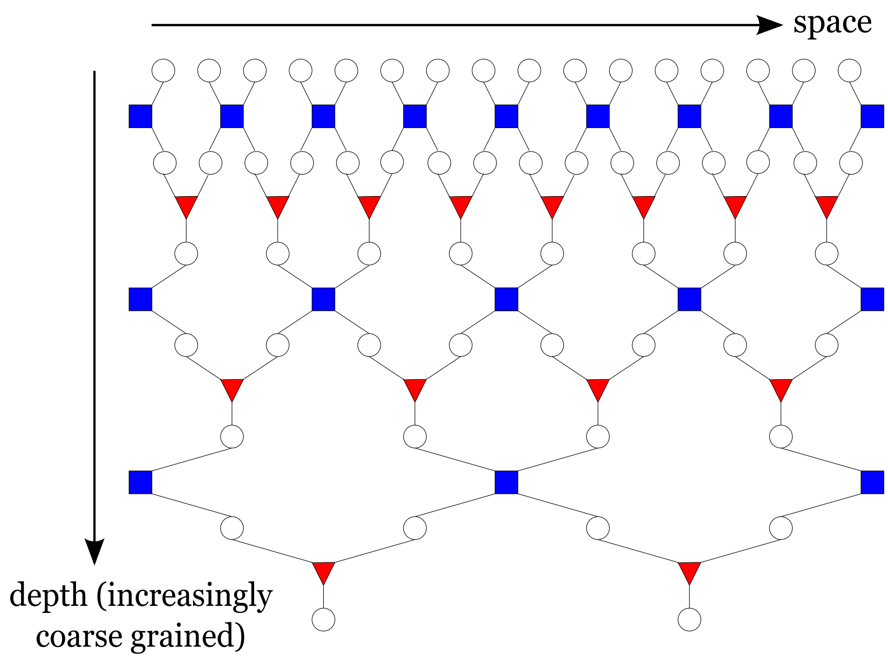

Figure 1: Spins are open circles, blue squares denote unitaries removing entanglement, while red triangles denote transformations that coarse-grain the system. We have shown the original spatial dimension together with a new emergent dimension which corresponds to length scale in the system.

length scale. We again remove local entanglement, producing unentangled degrees of freedom, and then coarse-grain the system. Indeed, the whole procedure can be repeated until we have removed all the entanglement in the quantum state or until there are no degrees of freedom left. Reversing the procedure, we can imagine building up the whole complex many-body state from unentangled degrees of freedom at all different length scales.

Entanglement renormalization is thus a renormalization process, a picture of the whole history of the quantum state with entanglement displayed scale by scale. We have the geometry of the original collection of spins as well as the geometry of this renormalization process which exists in an emergent dimension. Specifically, we have the network geometry displayed in Fig. 1, a kind of discrete geometry controlled by the connectivity of the network. It is as if we have taken a system living on a one dimensional line and expanded it into a new dimension by breaking it down scale by scale. Where the one dimensional description is complex, an exponentially long vector, the higher dimensional description is simple, describes the structure of entanglement in the quantum state visually, and gives a simple process by which the state can be constructed.

The appearance of an emergent dimension, where a d dimensional system can be related to a $d + 1$ dimensional description, is strongly reminiscent of the idea holography in black hole physics. Recall that a black hole has the peculiar feature that while taking up a fixed volume its entropy is proportional to its surface area. This realization has profound consequences for gravity, which should apparently be understood as a holographic system in one less dimension. I have argued [11] that in essence entanglement renormalization is a version of holographic duality (AdS/CFT) [5, 6, 7] that applies to general local quantum states. Here we have seen the beginnings of this argument by asking how a quantum computer might understand entanglement in a complex quantum state.

We have been led to a description of entanglement in terms of an emergent dimension associated with length scale in the original system. I believe this point of view teaches us a number things. First, it gives us a visual representation of quantum states in terms of entanglement and renormalization. For example, phases with an energy gap to excitations and short-range entanglement quickly run out of entanglement and the entanglement geometry ends. Additionally, phases with gapless excitations or long-range entanglement are described by an entanglement geometry that goes on forever. They are entangled at all length scales. Furthermore, the connection between entanglement renormalization and holography suggests that holography is also giving us a picture of entanglement in certain quantum many-body systems [12]. Finally, the point of view we have taken does not merely lead to a new physical picture, it also gives powerful new computational tools [1, 2, 4, 3]. The entanglement geometry represented by the network in Fig. 1 can be represented efficiently on a classical computer [10]. This enables us to compute correlations, determine ground states, and generally elucidate the physics of quantum matter in a way that simply isn't possible when thinking about exponentially long vectors in Hilbert space.

You should be convinced that exponentially long vectors in Hilbert space isn't the right way to think about quantum matter. The idea of thinking like a quantum computer, that is thinking in terms of quantum processes and dynamics, is a powerful way to give a physically meaningful notion of quantum states. Hilbert space thus survives but only as a stage which is anyway mostly unused. By trying to build a quantum state out of entanglement, the way a quantum computer might do it, we are led to a geometric representation of quantum states in terms of an emergent dimension, a possible connection between quantum matter and holography and string theory, and many new and powerful computational tools for understanding the physics of quantum matter. All we have to do is give up the illusion of Hilbert space.

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REFERENCES

[1] U. Schollwöck. The density-matrix renormalization group. *Rev. Mod. Phys.*, 77:259–315, Apr 2005.

[2] G. Vidal. Class of quantum many-body states that can be efficiently simulated. 101:110501, 2008.

[3] Z.-C. Gu, M. Levin, and X.-G. Wen. Tensor entanglement renormalization. *Phys. Rev. B*, 78:205116, 2008.

[4] F. Verstraete, J. Cirac, and V. Murg. Peps. *Adv. Phys.*, 57:143, 2008.

[5] Juan M. Maldacena. The large n limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231, 1998.

[6] E. Witten. Anti de sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253, 1998.

[7] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov. Guage theory correlators from non-critical string theory. *Phys. Lett. B*, 428:105, 1998.

[8] Phillip Kaye, Raymond Laflamme, and Michele Mosca. *An Introduction to Quantum Computing*. Oxford University Press, Inc., New York, NY, USA, 2007.

[9] Minoru Yamashita, Norihito Nakata, Yoshinori Senshu, Masaki Nagata, Hiroshi M. Yamamoto, Reizo Kato, Takasada Shibauchi, and Yuji Matsuda. Highly mobile gapless excitations in a two-dimensional candidate quantum spin liquid. *Science*, 328(5983):1246–1248, 2010.

[10] G. Vidal. Entanglement renormalization. *Phys. Rev. Lett.*, 99:220405, Nov 2007.

[11] B. Swingle. Entanglement Renormalization and Holography. *ArXiv: 0905.1317 (accepted Phys. Rev. D)*, May 2009.

[12] S. Ryu and T. Takayanagi. Holographic derivation of entanglement entropy from ads/cft correspondence. *Phys. Rev. Lett.*, 96:181602, 2006.