Infinite Resolution

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Abstract. It was reported that General Relativity predicts its own breakdown, because of singularities. We will see that the mathematics of General Relativity can be naturally extended to work fine when the metric becomes degenerate. Then is proposed an extended version of Einstein's equation which remains valid at singularities. The time evolution is expressed by equations which allow passing beyond the singularities. Consequently, the problems of singularities, including Hawking's information paradox, vanish.

The core principle used to extend the mathematics and physics of General Relativity beyond the singularities provides a surprising answer to the question:

"Is there a deep, foundational reason why reality must be purely analog, or why it must be digital?"

Prologue: Discrete *vs.* continuous

Is reality discrete or continuous? Is it possible to find the answer by experiment? Or at least from theoretical arguments?

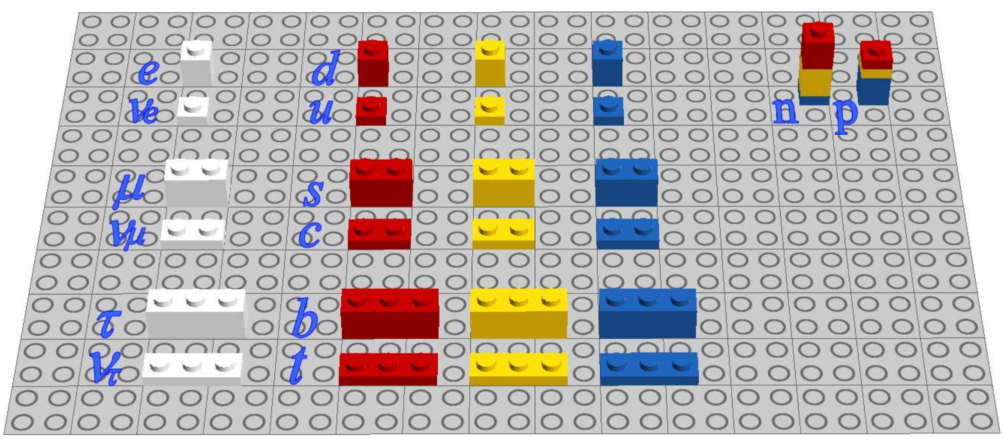

Figure 1. Is reality discrete or continuous?

Most successful theories in fundamental physics are based on continuous spaces, and on continuous sets of numbers. Should we conclude from this that Nature is fundamentally continuous? Couldn't this be just a historical accident, as many physicists believe? It could, indeed, and at very small scales, the spacetime itself may very well be discrete. Therefore, FQXi asks the following question:

Question #1: Is there a deep, foundational reason why reality must be purely analog, or why it must be digital?

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Could we let experiment decide between continuous and discrete? I find it hard to believe, because the reality we are trying to observe is necessarily finer than our tools. What we can do is to let the experiment decide between two theoretical views, by rejecting one of them and confirming the other.

In principle, it seems that we can discretize a continuous by sampling. For example the discrete versions of Gauge Theories and of General Relativity (GR) can be made to reproduce at any degree of accuracy the original, continuous versions. Reciprocally, there are ways to make a discrete theory continuous, by "interpolating" it, or by "embedding" it in a continuous background. Or it is possible that a continuous theory is mathematically equivalent with a discrete one, and yet none of them being an approximation or an extension of the other.

So it seems hard to give an answer to Question #1. The reason is that the theories we know so far don't seem to make use in an essential, irreducible way of the discrete or continuous nature of the reality they propose. It follows that probably

The best evidence for the continuity of reality would be provided by a theory which is based in an essential, irreducible way, on the necessity that spacetime and the values of the fields are divisible *ad infinitum*.

In addition, this theory should be mathematically and logically consistent, very well corroborated by experiments, and as simple as possible. Please note that I do not ask this theory to be the theory of everything, just to be a good theory.

My main claim is that such a theory is the version of General Relativity proposed below.

General Relativity is arguably the most successful theory in Physics. It is based on a small number of hypotheses, but it makes a wide range of accurate predictions. The way to obtain so much from so little is by developing mathematically and logically the consequences. But the mathematics of GR is difficult, although the main idea, "gravity is due to the curvature of spacetime", is very simple.

General Relativity was considered mathematically consistent, until Penrose and Hawking proved that it predicts in an unavoidable way the existence of singularities [9, 10, 11, 12]. Their *singularity theorems* are usually viewed as the irrefutable proof that *General Relativity predicts its own breakdown*.

The singularities are the second reason why physicists are trying discrete approaches – the first reason, which I will not discuss here, being the infinities which seem to occur when we try to quantize gravity. Therefore, another question is justified:

Question #2: Can the problem of singularities in General Relativity be resolved by keeping the continuous spacetime?

Or does it lead us to the necessity of a discrete approach? My second claim is that the problem of singularities and the *information loss paradox* can be answered by Singular General Relativity – which is just General Relativity with a "bug fix". In addition, this solution provides an unexpected answer to Question #1.

The answer to Question #2 involves some new Differential Geometry, and here I will try to describe it by requiring from the reader as little mathematical background as possible. I moved the more "technical details" to the endnotes, and the complete details can be find in the references.

1. From fantasy to mathematical proof

"I was borne violently into the channel of the Ström, and in a few minutes, was hurried down the coast into the 'grounds' of the fishermen."

Edgar Allan Poe, A Descent into the Maelström, 1841

I was trying to see with my mind's eyes what is happening inside a black hole, when approaching the singularity. My meditation led me to an unexpected vision. In my vision, the bodies that fell into the black hole were violently stretched and destroyed, as expected. After a short time, as they reached the singularity, there was no distance to separate them. But the particles continued to occupy distinct positions. *Although there was no distance between them, the points of space maintained their identities.* This way, if the black hole evaporates, the information is completely retrieved.

Can two points in space be separated by no distance, and yet be distinct?

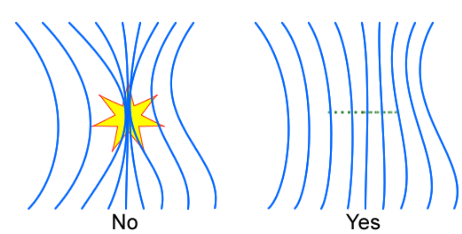
Initially, this possibility seemed absurd to me, but I could not get rid of the strong feeling that this is what really happens inside the singularity. In my vision, this separation saved the information to be destroyed, and when the black hole evaporated, it ensured the retrieval of all information. Moreover, the curvature and other geometric properties of spacetime survived miraculously to the singularity, although we know that they appear to be infinite at that place. And all these happened without violating the fundamental laws of General Relativity, but rather because of them.

Of course, this product of my imagination is not an evidence. I had to prove it somehow. But how? An experiment was out of our reach, so I thought maybe I can derive these results mathematically.

A mathematical proof starts with a *hypothesis*, and leads to a *conclusion*. The hypothesis is in this case that some principles assumed by General Relativity are valid. I needed a mathematical proof joining them with the conclusion that the information falling into a black hole, although violently mixed up, can survive passing through the singularity.

But wouldn't such a proof contradict other mathematical proofs based on GR – the singularity theorems of Penrose and Hawking? The *singularity theorems* show that under some conditions, which are general enough to be fulfilled in our Universe, if GR is correct, singularities occur. The singularities are points where it is believed that the time evolution gets stuck. If we know how matter is distributed in space, and how it moves, we can in principle deduce how it was arranged in the past. But if for some reason the space becomes stretched so that more points which were supposed to be distinct become identical, then a kind of short-circuit happens, and the information is "overwritten". As a consequence, the Universe "forgets" its prior configuration, this causing the "memory leak" [1]. This not only violates the classical time evolution laws, but also the quantum ones.

But what the singularity theorems predict is that the distance between some points becomes zero, not that they are "overwritten" or destroyed. Apparently, if the distance between two points is zero, then they are one and the same point. But this is not necessarily true. In fact, there are two ways in which two points can be separated. They can be separated *topologically*, if they are actually distinct, and *metrically*, if the distance between them is 0.



The topology of spacetime doesn't care about distance between points, it only cares if one of them has a neighborhood which doesn't contain the other. Topological spaces don't need the notion of distance for telling whether two points are separated. Of course, our intuition keeps telling us that zero distance means coincidence. This is because we think so much in terms of the Euclidean distance, which indeed has the property that the only situation when the distance between two points is 0 is when they coincide.

In order to avoid the information "overwriting", we need to allow the distances in space to become zero even if the points may be distinct. This means that we need to allow metric to become degenerate (we will see soon what this means). But this is anyway what the singularity theorems predict, that the metric becomes degenerate. What they do not predict, and it is assumed implicitly and unjustified, is that we should identify the points just because the distance between them is 0.

Unfortunately, allowing the points to remain distinct is not enough, because when the metric becomes degenerate, important geometric objects associated to the metric have infinite values. This is really a problem, because these objects appear in the equation governing GR – Einstein's equation. I needed to show that the information contained in those objects can be written in a form which does not become infinite, and that we can replace Einstein's equation with a more general version, which is valid even inside the singularities.

2. Distance and metric tensor in General Relativity

The notion of distance in GR is contained in the *metric tensor* g_{ab}, which is the fundamental entity. Please see the end note [2] for a brief introduction. The mathematics of the metric tensor, of curved spaces and of General Relativity is named *semi-Riemannian geometry*. It uses extensively both the metric and its inverse, therefore it assumes that the metric has an inverse – *i.e.* it is *non-degenerate*. Some important objects in semi-Riemannian geometry are *tensors*, briefly reminded in the end note [3], with the fundamental operations addition, multiplication and *contraction*.

The *metric tensor* g_{ab} is a tensor field of type $(0, 2)$. It defines on the tangent space T_pM at each point an inner product. Its components form a matrix, whose inverse gives a tensor field of type $(0, 2)$ – an inner product on the cotangent space T_p^*M. The metric g_{ab} determines the *Riemann curvature* $R^a{}_{bcd}$. From the Riemann curvature we can extract by contraction a "summary curvature" $R_{bd} := R^c{}_{bcd}$, which is the *Ricci curvature*. By "summarizing" again we obtain the *scalar curvature* $R = g^{ab}R_{ab}$. The *Einstein tensor* is $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$, and is related in General Relativity to the flows of energy and momentum (named the *stress-energy tensor*) of the matter by *Einstein's equation*:

EINSTEIN TENSOR = STRESS-ENERGY TENSOR

At singularities the metric becomes degenerate – its determinant becomes 0. Unfortunately, many obstacles make impossible to write Einstein's equation for degenerate metric.

I soon realized that what I needed was a new branch of Differential Geometry, one which deals with metrics which can be degenerate like it does with the non-degenerate metric of Lorentz. A new geometry – *Singular Semi-Riemannian Geometry* [4] – was needed.

3. Exploring a space with degenerate metric

I was excited and worried at the same time, because I needed some geometric operations and objects which apparently are impossible to be defined at the singularities. Practically, I needed a series of miracles to make these constructions possible. And I needed a blind, stubborn faith in the vision which triggered this journey.

3.1. Inverting the uninvertible

Many operations with tensors are based on contracting with the metric. For example, the metric gives the inner product $\langle X, Y\rangle = g_{ab}X^aY^b$, and in general we can contract two upper indices: $g_{ab}T^{ab}$. But in order to have an inner product on the cotangent space, or to contract two lower indices, as in $g^{ab}S_{ab}$, we need the inverse g^{ab} of the metric g_{ab}. But how can we define the inverse of a degenerate matrix? I first tried the Moore-Penrose pseudoinverse [13, 14]. Unfortunately, each coordinate system gives another pseudoinverse, so this is not an invariant.

Fortunately, it is always possible to find in a natural way a unique vector subspace $V^{\bullet}$ of the covector space V, and a canonically defined inner product $g_{\bullet}{}^{ab}$ on this subspace [5]. I detailed how this is done and how to use it for tensor operations in [1]. After that, I crossed my fingers and hoped that all important operations and geometric objects lie within the tensor products of vector and the special covectors.

3.2. Smoothing the divergences

It was the time to extend these operations from vector spaces to curved spaces with degenerate metric. Apparently this should be easy, because at each point p of the space M there is a tangent vector space T_pM and its dual T_p^*M, so we can just define the space $T_p{}^{\bullet}M \subseteq T_p^*M$ and the metric $g_{\bullet}{}_p^{ab}$ on it. The main spaces involved in the game are shown in the end note [6].

The main difficulty occurs when the signature of the metric changes, because the inner product of covectors $g_{\bullet}{}^{ab}$ becomes infinite.

In curved spaces with non-degenerate metric, we cannot in general choose a coordinate system in which the metric has only ± 1 on its diagonal, but at least we obtain the same by using local orthonormal frames of vector fields, which can be viewed as a generalizations of the coordinates.

Well, if the metric is degenerate, an orthonormal frame would be divergent. So, I had to replace the orthonormality condition with orthogonality: the metric can be diagonalized, but not necessarily to ± 1 or 0. The problem is that for degenerate metric we cannot even guarantee the existence of local orthogonal frames – so I had to postulate this condition. I named this special type of space *almost semi-regular semi-Riemannian space* (to show that there are two conditions which separate it for being regular, *i.e.* semi-Riemannian).

In such a space we can contract lower indices, but the result is not continuous for any tensor field. For example, a covector field ω_a admits smooth norm $\sqrt{g_{\bullet}{}^{ab}\omega_a\omega_b}$, if it is *sub-metric*, that is, for each index $a \in \{1, \dots, n\}$, $\omega_a = f\sqrt{|\det g_{aa}|}$ for some smooth function f on M. Let's denote this relation by $\omega_a \prec \sqrt{|\det g_{aa}|}$. This condition is another way to say that, if g_{aa} becomes 0, ω_a needs to become 0 fast enough to allow the ratio $\omega_a/\sqrt{|\det g_{aa}|}$ to remain smooth [7]. This condition extends as well to tensors of more general type. To resume, a field admits contraction in two covariant indices if for those indices

THE FIELD ALONG A DIRECTION $\prec$ THE DISTANCE ALONG THAT DIRECTION

I wrote more about these in [2]. This condition ensures the smoothness of the covariant contraction, but would the tensors I needed be of this kind? Or was I asking for a too big coincidence?

3.3. Derivating the underivable

To derivate tensor fields in a covariant way we need a *connection*, which is a recipe to translate vectors and tensors from a point to a nearby point. For a non-degenerate metric, there is a unique connection which makes these translations without torsion and by preserving the metric – the *Levi-Civita connection*. But if the metric is degenerate, the Levi-Civita connection cannot be defined, and some difficulties occur, especially when the signature of the metric changes. This is because the Levi-Civita connection is obtained by raising an index of an expression involving the metric, the *Koszul form* [8]. To raise the index one needs the inverse of the metric, which is not defined. I realized that virtually all important operations can be done by using directly the Koszul form, without raising that index [9].

For covectors and other *covariant tensors* – *i.e.* of type $(0, s)$ – I could define in a canonical manner the covariant derivative, *if the Koszul form is sub-metric in the last argument* [10]. The result is smooth for sub-metric tensors.

I christened *semi-regular semi-Riemannian spaces* the almost semi-regular semi-Riemannian spaces satisfying

Koszul form along a direction $\prec$ the distance along that direction

These spaces are generalizations of the semi-Riemannian spaces, but they are not limited to non-degenerate metrics. More about this can be find in [3].

3.4. The finite face of the curvature

It was time to construct the Riemann curvature. It turned out that, for semi-regular semi-Riemannian spaces, the Riemann curvature tensor R_{abcd} can be uniquely defined by using the Koszul form and the covariant derivative of covectors, and it is smooth [11]. On the other hand, $R^{a}{}_{bcd}$ is not well defined, and it is divergent when the signature is changing.

The Riemann curvature tensor lies precisely in the tensor space which allows the contraction between covariant indices. Therefore, the Ricci tensor and the curvature scalar, and consequently the Einstein tensor, can be constructed in a unique fashion. They are smooth as long as the signature doesn't change, but unfortunately diverge when the signature changes.

I developed these ideas in [4].

3.5. Warping singularities

Given two spaces, you can obtain a third one by direct (Cartesian) product. If the two spaces have metrics, their direct product have a metric which is the sum of the two metrics. If the two metrics are non-degenerate, so is the resulting metric. But we can scale the metric of the second space by multiplying it with a function f^2 from the first space. The result is named the *warped product* of the two spaces, with *warping function* f. In [5] I showed how we can construct semi-regular semi-Riemannian spaces from other such spaces, as warped product for which f is allowed to become 0.

From this follows that there are Friedmann-Lemaître-Robertson-Walker spaces with singularities, which are semi-regular semi-Riemannian spaces. In particular, we can use the mathematical apparatus of semi-regular semi-Riemannian geometry to explore Penrose's Conformal Cyclic Cosmology and other cosmological models.

Spherically symmetric solutions of GR with singularities can also be obtained as degenerate warped products. This way, for example, evaporating black holes can have singularities which are semi-regular.

4. Singular General Relativity

4.1. The realm within the singularity

It was then the time to apply the obtained results to explicitly write a version of Einstein's equation which works for singularities. But how can we write the Einstein's equation, if the Einstein tensor is divergent? It turned out that, although the Einstein's tensor is divergent where the metric changes its signature, its densitized version is smooth. A *densitized tensor of weight* W is a tensor which is multiplied with $\sqrt{|\det g_{ab}|}^{W}$. The quantity $\sqrt{|\det g_{ab}|}$ is also used to define the volume element associated to the metric g_{ab}. The *densitized Einstein's equation* [12] takes the form

EINSTEIN TENSOR × VOL. ELEM. = STRESS-ENERGY TENSOR × VOL. ELEM.

The *Ricci tensor density* has a nice geometric interpretation – it appears in the expression of the acceleration of the volume element. The *stress-energy tensor density* is the quantity which appears naturally from the Lagrangian. So, probably the densitized Einstein equation is more natural than the purely tensorial version.

If the metric is non-degenerate, we can eliminate the density factor, and obtain the standard form of Einstein's equation. But in the presence of singularities, the metric becomes degenerate. In this case, the Einstein tensor tends to infinity, but the density factor tends to zero just enough to compensate it.

The densitized scalar curvature is also smooth – being just the Einstein-Hilbert Lagrangian density:

EINSTEIN-HILBERT ACTION = $\int$ SCALAR CURVATURE × VOL. ELEM.

which can be used to derive Einstein's densitized equation.

By applying the properties of semi-regular semi-Riemannian spaces, it turns out that we can construct the equations for matter fields such as Klein-Gordon, Maxwell, Yang-Mills, and their Lagrangians and stress-energy tensor densities, even if normally they are divergent for degenerate metric. Therefore, Einstein's equation can be defined even when singularities occur in the semi-regular semi-Riemannian spacetimes. I gave more details in [6].

4.2. Through the eye of a needle

The Einstein's equation, in its densitized form, shows that spacetime can have singularities, and its geometry and physics still make sense. But Einstein's equation provides an atemporal view, of a block universe. How can we describe the time evolution?

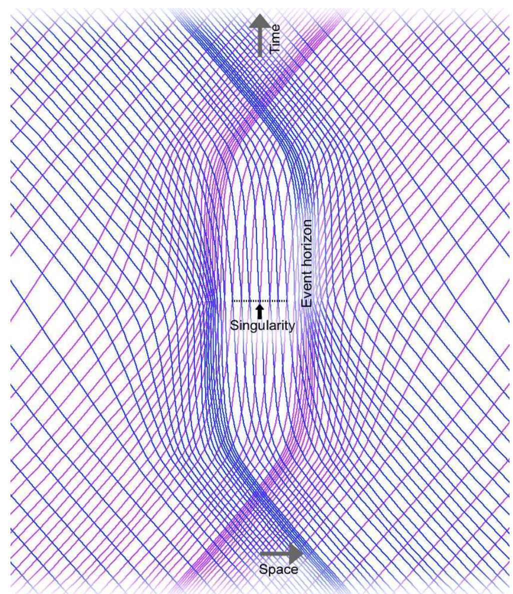
In General Relativity, there are several ways to replace Einstein's equation with equations describing the time evolution. Probably the most known is due to Arnowitt, Deser and Misner [15], named the *ADM formalism*, or *geometrodynamics*. It consists in decomposing the spacetime in a product:

SPACETIME = SPACE × TIME

All the fields appearing in Einstein's equation are decomposed in their space and time components accordingly. The three obtained a system of constraints – which tell what kind of fields are allowed to exist on the three-dimensional space, and a system of evolution equations – which describe how the fundamental variables change in time.

Apparently, the ADM equations cannot describe well singularities, because they involve the inverse of the metric. It was suggested that Ashtekar's "new variables" [16, 17, 18] can solve the problem of singularities. Ashtekar's variables are a frame field and a connection. The metric can be obtained from the frame field, which is allowed to become degenerate. Naturally, I tried to apply Ashtekar's method to the problem of singularities. Unfortunately, when the frame variable becomes degenerate, the connection variable becomes in general divergent. The same problem occur in the Einstein-Palatini method, due to Einstein [19, 20].

But the properties of semi-regular semi-Riemannian metrics allow a reformulation of the ADM equations so that the divergences disappear.



It is known for some time a version of the ADM equations, which uses as fundamental variables the metric and the *extrinsic curvature* of the three-dimensional space. The intrinsic curvature of a subspace is the curvature defined by the metric of that subspace, and it is independent of the way the subspace is embedded in the larger space. The extrinsic curvature describes the way the subspace is curved inside the larger space, without regard to its intrinsic metric. So, these two tensors complement each other. They can be used to express the time evolution in GR, by describing how space "moves" in the spacetime.

I realized that the version of the ADM equations, based on the 3-metric and the extrinsic curvature, can be arranged so that the tensors involved are not divergent. Therefore, they can be used to describe how time evolution can pass through the singularities, without being stuck.

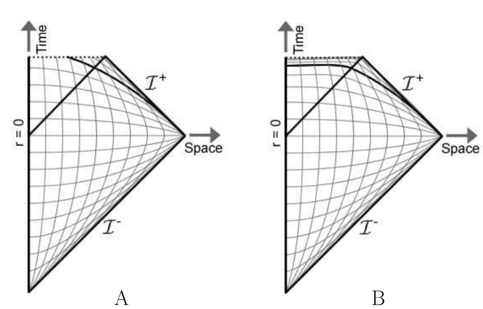
For more details, please see [7].

4.3. Resurrection of information destroyed in singularity

According to Stephen Hawking [21, 22] the black holes can evaporate, leaving no trace of the information swallowed by the singularity. But the laws of physics, especially Quantum Theory, require the information to be preserved, so it seems that there is a paradox.

But the time evolution can take place even in the presence of degenerate metrics, as we have seen before. If the black hole evaporates completely, the singularity disappears and the information is retrieved.

What if the black hole survives forever? In this case, we can think at the information as surviving somewhere inside the black hole. Two problems seem to appear if the black hole is eternal. First, this time even the metric can tend to infinity. Second, an object falling in a black hole reaches the singularity after a finite proper time (according to its watch).



But this is not a problem, because the time evolution is not labeled by the proper time. For example, we can describe the Schwarzschild black hole in a coordinate system in which the singularity is moved at the future infinity. The time evolution described above in §4.2 is consistent with this view, because it uses a time parameter compatible with the densitized lapse function, and not some proper time. Also, if the metric diverges at the future infinity, this poses no problem to the time evolution, which remains smooth and undisturbed.

More about this can be find in [8].

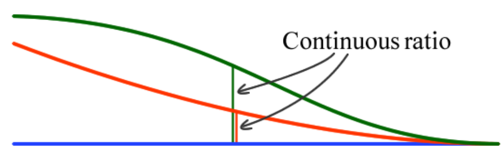
Epilogue: Infinite resolution

Singular General Relativity is an extension of the standard GR – it reduces to this when the metric is non-degenerate. When singularities appear, the fields which are important for writing Einstein's equation, Einstein-Hilbert equation, and the ADM equations in 3-metric and extrinsic curvature remain smooth. Smooth matter fields and stress-energy tensor densities can be constructed. The new fields don't have "true singularities" and even remain smooth, although the metric can be degenerate.

The key reason is the condition that the Koszul form is sub-metric in its last argument:

Koszul form along a direction $\prec$ the distance along that direction

This condition requires that, when the length of a vector field from the orthogonal frame becomes degenerate, the corresponding components of the Koszul form tends to zero fast enough to maintain the continuity of the curvature and of the covariant derivative of covariant tensors.



Another manifestation of this principle is in the densitized form of Einstein's equation, which remains continuous. To maintain the continuity, as the Einstein tensor diverges, it is compensated by the metric's determinant, which converges to 0.

Although several difficulties needed to be solved along the way, the core idea is simple:

(1) allow the metric to be degenerate,
(2) but make sure that the Koszul form remain sub-metric.

Therefore,

***Singular General Relativity* is essentially based on the fact that some infinitely small quantities compensate the divergences of other quantities.**

In my opinion, this approach is deeply rooted in the continuity of spacetime and of the fields. Definitely, any continuous theory can be "digitized" in principle, but I think that in this case the discrete alternative would be more complicated, requiring different rules for different regimes.

***Singular General Relativity* is based in an essential, irreducible way, on the necessity that spacetime and the values of the fields are divisible *ad infinitum*.**

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Notes

1 One can make a parallel between information loss in a black hole, and information loss in a computer. A way to lose data in a computer is by overwriting that data, or a *pointer* to that data. In General Relativity, the information contained by the fields is labeled by points in space and time – the points play the role of the pointers. Overwriting the points causes a loss of information – a *memory leak*.

2 We start with the more familiar notion of distance in the Euclidean Geometry. In plane, the distance between the origin and another point is given by the theorem of Pythagoras, and it is $AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$, where the points have the coordinates $A = (x_A, y_A)$ and $B = (x_B, y_B)$. For three Euclidean dimensions it becomes $AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2$. In the Minkowski four-dimensional spacetime of Special Relativity, the distance becomes

$$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2 - c^2(t_B - t_A)^2. \tag{1}$$

This simple equation contains, like a seed, the principle stating that light always travels with a velocity equal to c, the formulae giving the variation of length, time and mass with the speed, the symmetries between the electric and magnetic fields, the spin, and other important physical phenomena.
In addition, it shows that the distance AB can be zero even for distinct A and B – if A and B can be joined by a photon. In this case, the distance is considered in spacetime. By contrast, degenerate metrics allow other kinds of distances which are 0, *e.g.* even when $t_A = t_B$.
Now, let's go back to the Euclidean plane, and consider that the coordinate system is orthogonal, but not orthonormal. The two coordinates need to be rescaled, and the theorem of Pythagoras contains now the scaling factors: $AB^2 = g_{xx}(x_B - x_A)^2 + g_{yy}(y_B - y_A)^2$. If, in addition, the two axes are not perpendicular, together with the square terms, in the theorem of Pythagoras should appear terms containing the projections of one axis on the other. Pythagoras's theorem takes now the form $AB^2 = g_{xx}(x_B - x_A)^2 + g_{yy}(y_B - y_A)^2 + 2g_{xy}(x_B - x_A)(y_B - y_A)$. The coefficients form a matrix $\begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}$ which represents the *metric* or the *inner product* in that particular coordinate system. If we go a step further and express the distance in curvilinear coordinates, the metric is represented as a matrix whose entries are different from point to point. Of course, in Euclidean and Minkowski spaces it is always possible to avoid such complications by using orthogonal coordinates.
General Relativity explains gravity by allowing the spacetime to be curved. The objects follow the straightest line in a curved spacetime, yet their trajectories turn out to be bent, exactly as if they would be curved by gravitation. The spacetime being curved, it is not possible in general to find orthogonal coordinates in which to express the distances by the simple formula (1). The best we can do is to write the theorem of Pythagoras in its infinitesimal form:

$$\mathrm{d}s^2 = \sum_{a=0}^{3}\sum_{b=0}^{3} g_{ab}\mathrm{d}x^a\mathrm{d}x^b. \tag{2}$$

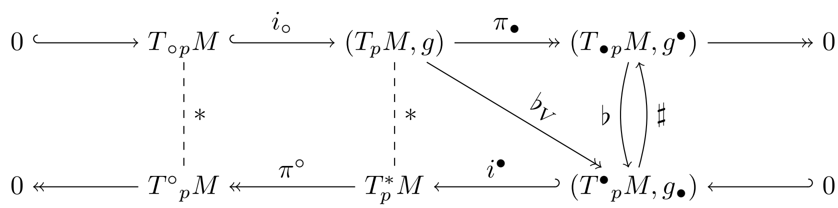
3 In fact, tensors are objects associated to a vector space V. We write the vectors from V in the form v^a, where the upper index a runs through the labels of the frame in which we express the vector. The linear functionals $: V \to \mathbb{R}$ form the *dual vector space* V^*. They are named *covectors*, and are denoted with a lower index, as in ω_a. They act on V by $\omega(v) = \sum_{a=1}^n \omega_a v^a$. Einstein observed that we can in general drop the $\sum_{a=1}^n$ sign and sum whenever the same index label is used both as an upper and lower index. So we can write $\omega(v) = \omega_a v^a$.
A tensor of type (r, s) is a function defined on $\underbrace{V^* \times \ldots \times V^*}_{r \text{ times}} \times \underbrace{V \times \ldots \times V}_{s \text{ times}}$, linear in each argument.
Vectors, covectors and scalars are tensors of type $(1, 0)$, $(0, 1)$, respectively $(0, 0)$.
Like vectors and covectors, a tensor of type (r, s) is denoted using indices, as an array $T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}$ containing its components of T in some reference frame.
Tensors of the same type can be added. They can be multiplied without regard of their type. A tensor $T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}$ of type (r, s) can be *contracted*, obtaining a tensor $\sum_{c=1}^n T^{a_1 \ldots a_{r-1} c}{}_{b_1 \ldots b_{s-1} c}$ of type $(r-1, s-1)$. For example, $\omega_a v^a$ $(= \omega(v))$ is the contraction of $\omega_a v^b$.
In the case of a curved space M of dimension n, there is a vector space of dimension n associated to each point – named the *tangent space*, whose dual is the *cotangent space*. Starting with these spaces we

can construct tensors at each point. A function on M which associates to each point $p \in M$ a tensor at p is named *tensor field*.

4 Little work has been done in the field of Singular Semi-Riemannian Geometry, *e.g.* [23] and [24]. Demir Kupeli obtained important results in [25]. Unfortunately for what I needed, these results concern exclusively the particular case when the metric doesn't change its signature.

5 I used instead of the dual space V^* endowed with the metric given by the inverse of g_{st}, its subspace $V^\bullet$, given by the linear functions which can be written as $\omega_s = g_{st}v^t := (v^\flat)_a$ for some v^t. The metric g_{st} induces on $V^\bullet$ the metric $g_\bullet{}^{st}$ defined by $g_\bullet{}^{st}(u^\flat)_s(v^\flat)_t := g_{st}u^s v^t$.

6 The following diagram summarizes the relations between the *radical* (or the degenerate) subspace $T_{\circ p}M \subseteq T_pM$, the factor space $T_{\bullet p}M$ of T_pM which it determines, the *radical annihilator* subspace $T^\bullet{}_pM \subseteq T^*_pM$ of linear functionals which annihilate $T_{\circ p}M$, the factor space of T^*_pM which it determines, and the corresponding induced inner products $g^\bullet$ on $T_{\bullet p}M$ and $g_\bullet$ on $T^\bullet{}_pM$.



7 The covariant contraction of two covector fields (and also their inner product) $\omega, \tau \in \mathscr{A}^{\bullet 1}(M)$ is

$$\langle\!\langle \omega, \tau \rangle\!\rangle_\bullet = \sum_{a=1}^{n} \frac{\omega(E_a)\tau(E_a)}{\langle E_a, E_a \rangle}, \tag{3}$$

in a local radical orthogonal frame $(E_a)_{a=1}^n$. If $\omega(E_a) \prec |E_a|$ and $\tau(E_a) \prec |E_a|$, it is smooth.

8 *The Koszul form* is defined as $\mathcal{K} : \mathfrak{X}(M)^3 \to \mathbb{R}$,

$$\mathcal{K}(X,Y,Z) \quad := \quad \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\}. \tag{4}$$

The *covariant derivative* for a non-degenerate metric is defined as $\langle \nabla_X Y, Z\rangle = \mathcal{K}(X,Y,Z)$.

9 The *lower covariant derivative* of a vector field Y in the direction of a vector field X is the differential 1-form $\nabla^\flat_X Y \in \mathcal{A}^1(M)$ defined for any $Z \in \mathfrak{X}(M)$ as

$$(\nabla^\flat_X Y)(Z) := \mathcal{K}(X,Y,Z) \tag{5}$$

10 I introduced the covariant derivative of a radical-annihilator 1-form $\omega \in \mathcal{A}^\bullet(M)$ in the direction of a vector field $X \in \mathfrak{X}(M)$ by $\nabla : \mathfrak{X}(M) \times \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$,

$$\left(\nabla_X \omega\right)(Y) := X\left(\omega(Y)\right) - \langle\!\langle \nabla^\flat_X Y, \omega\rangle\!\rangle_\bullet. \tag{6}$$

It extends to other covariant tensor fields by using the Leibniz's rule:

$$\nabla_X(\omega_1 \otimes \ldots \otimes \omega_s) := \nabla_X(\omega_1) \otimes \ldots \otimes \omega_s + \ldots + \omega_1 \otimes \ldots \otimes \nabla_X(\omega_s) \tag{7}$$

11 I defined the *Riemann curvature operator* as $\mathcal{R}^\flat : \mathfrak{X}(M)^3 \to \mathcal{A}^1(M)$, where for all $X, Y, Z \in \mathfrak{X}(M)$

$$\mathcal{R}^\flat_{XY}Z := \nabla_X \nabla^\flat_Y Z - \nabla_Y \nabla^\flat_X Z - \nabla^\flat_{[X,Y]} Z \tag{8}$$

12 The densitized form of the Einstein equation is

$$G_{ab}\mathrm{d}_{vol} + g_{ab}\Lambda\mathrm{d}_{vol} = \kappa T_{ab}\mathrm{d}_{vol}. \tag{9}$$

Actually, in [6] I showed that for any semi-regular semi-Riemannian space the densitized Einstein tensor $G_{ab}\det g$ is smooth, so $G_{ab}\sqrt{|\det g|}^W + \Lambda g_{ab}\sqrt{|\det g|}^W = \kappa T_{ab}\sqrt{|\det g|}^W$ with weight $W = 2$ is smooth, but for many natural examples encountered in Physics we can safely take $W = 1$.