The Janus of Mathematics - Burt Smith

The Invention:

It has long been known that no two snow flakes are identical,[1] that identical twins are not identical[2] and more recently, that sub-atomic particles have different mass.[3] Admittedly these are assumptions as not all snowflakes, identical twins or particles have been measured. Modern technology has the ability to measure to tenths of a billionth of a meter and beyond. While a boon to technological progress it has also noted that identities are rare, if they occur at all. Even if identities exist, their different locations in the time-space continuum would insure that whatever forces they may have, or be affected by, would be slightly different than immediate neighbors. Thus, every item is unique, a variable; or to paraphrase Heraclitus's (ca. 540-480 BC) statement of not being able to step into the same river twice;[4] the only constant is change.

Life on earth has two primary objectives: survive and procreate. In order to meet those objectives on a planet of seemingly chaos, tradeoffs have been made - there will be life and procreation but not all will survive or procreate. Aiding survival is the fact that even tho every item is unique there are a vast number of similarities; chemical elements and compounds, clays, species, etc. Life has also developed a tactic to enhance stability in that buffering of unexpected events typically increases with system's complexity.[5] Part of that complexity is the formation of social groups which in turn has led to cultures for a number of species.[6, 7] The hominid, *Ardipithecus ramidus*, the earliest known divergence from chimpanzees, had increased bipedalism, a reduction of male canines and reduced sexual dimorphism; which implies a more cooperative, equalitarian role, not only among males but also females. A crucial factor in our development.[8, 9]

The present digital preoccupation is a natural extension of several basic animal traits which humans have refined throughout their pre-history and was accelerated with the advent of agriculture and the concomitant increase in urbanization and divisions of labor. Social bonding resulted in molding these traits into uniquely humanoid characteristics. Three traits or embodiments that appear in humans, and in varying degrees of other vertebrates, are; categorization, generalization and subitization or numerousness.[10, 11]

Categorization: The grouping of similar items or events into one all-embracing category. The drawing of a line, implying that items on one side are different than those on the other. A necessary survival item as one does not have to personally encounter every lion to understand that all lions are best avoided. However, this often leads to nearly indistinguishable items being separated by an arbitrary line - a routine problem for taxonomists.

Generalization: Drawing conclusions from the shared characterizes of the items within a category. This nominally takes the form of averages, the accuracy of which is solely dependant upon the degree of variance within the category. For example, in the broad category of humans, the average human does not exist - for starters it would have to have one ovary and one testicle. On the other hand, not all bitter tasting plants are poisonous, but many are.

Subitization: The ability to rapidly ascertain if one group of items is larger/smaller than another. To correctly identify the presences of one, two, three and often four items and to know that four is more than three is more than two is more than one.[12] Numerals are not required; children, even babies have this ability.[13, 14]

Tallying, is the making a mark or notation in a one-to-one correspondence to whatever is being tallied, and is a practice dating to the Upper Paleolithic, as evidenced by the discovery of the Lembago, Wolf and Ishango bones; dated to 33,000, 28,000 and 16,000 BC, respectively.[15, 16] That the tallies were made on bone implies that the information recorded was worthy of making a permanent record. However, tallying does not require numerals, a mark or notation suffices. Tallying on wooden sticks, placing pebbles in a pile, or using the fingers is considerably easier than scratching marks on bone and quite likely pre-dated the above finds. What tallying does require is the abstract concept of a metaphor; the use of a mark to represent members of a category (set) that are considered "identical," at least for the purposes of the tally.[17] Tallying is not true counting as it has no particular order or sequence of the items tallied. However, there is an implied sequence, in that one item follows another - an offshoot of subitization.

While subitization can help explain counting up to 4 and possibly 5, something else is required to count beyond that. Various body parts have long been used to count, however fingers are more common as they are seldom covered up and are readily available. Such use was commonplace in the past and continues today, only the order varies depending upon culture, e.g. fingers up or down, sequence from left to right or reverse.[12] The arithmetical operand of addition and subtraction, are an intrinsic part of subitization, hence in tallying and counting.[18] The relationship between fingers, tallying and counting have a further commonality in that the fingers are controlled by the brain's parietal lobe, areas that are associated with subitization.[11, 12, 19]

Subitization combines categorization and generalization and allows one to quickly differentiate between one, two, three, and sometimes four separate objects as well as determining if one grouping was larger/smaller than another. In addition to reinforcing this trait, tallying advanced it by the tacit assumption that the items being tallied were identical, in some manner, and that the total was the sum total of the previous marks. The first tentative step towards the concept of an object monad (base unit) and that higher amounts could be made by simply adding another object monad.

Not all indigenous tribes needed nor used numerals.[20] For those that did, numerals were typically limited to one, two, sometimes three and four, followed by a word meaning much or many.[21] In 1607 it was noted by John Smith that the Powhatan empire "had words for one, two, three and onward to one thousand."[22] The Ishango bone has three rows of grouped notches. In two rows the notches add up to 60, and one row contains four groups of 11, 13, 17, and 19 marks, which are prime numbers. The third row is said to be consistent with a numeration system based on 10, and also includes a duplication method of multiplication. Microscopic studies illuminated more markings suggesting a lunar phase counter.[23] Giving a word for individual numbers would appear to be the result of frequency of use, and lower numbers have higher rate of use.[17]

It gets complicated:

Agriculture has been called the greatest mistake humans ever made.[24] However, since it arose independently in three, if not more, separate cultures it would appear to have been inevitable. The move towards agriculture, like many cultural shifts, was long and involved. Recent evidence indicates that grinding grains for flour was taking place on the floodplains of Italy, Russia and the Czech Republic some 30,000 years ago.[25] The earliest discovered grain silo, is located in Jordan and dated to 9,000 BC, and around a 1000 years later farming settlements appeared in Mesopotamia.

Clay is common in the Near East, available everywhere, and easy to work into various shapes with bare hands. It can be harden by sunlight or baked in a camp fire; and once harden, endures. Pottery and construction bricks were likely the first uses of clay, and sometime during the 9th millennium BC, clay was shaped into various small tokens that represented different commodities. Their actual uses at that time is unknown but is suspected to have played an accounting role, likely for tax purposes.[26, 27]

Trading in animals was, and is, typically conducted on a one-to-one basis (discrete units). However, commodities such as grains and oil require a unit of capacity that is acceptable to all parties concerned. The earliest Mesopotamian texts imply that standards for area and capacities were in use prior to writing.[28] How long before is unknown, but the amount of intra-city trade that developed would imply that some kind of standards were in use early on. In any event, variously shaped tokens were in use for various commodities on a one-to-one basis; e.g. one ovoid token equaled one unit of oil. Tokens quickly became numerical counters and were used for record keeping, trade and occasionally in funerary offerings.[26, 27] Compared to tallying, this was an increase in abstraction, as tokens indicated the product as well as the quantity one; which furthered the conception that the one (average) reflected an equality on all members.

Settlements increased, and with it came kings, priests, merchants, scribes, craftsmen and the inevitable bureaucracy for collecting taxes and disbursing monies for various projects. The 6th millennium BC saw pottery that was imbued with a rich assortment of animals and geometric designs; squares, circles, arcs, etc.[28] Tokens also grew in complexity and later, some city-states made clay envelopes to enclose the tokens with wedge marks inscribed on the envelope to indicate the type and number of tokens inside. Eventually the envelope morphed to a rectangular clay tablet upon which the pertinent information was inscribed with a stylus; but still on a one-to-one basis. Clay seals and tags were also placed on jars and containers of various commodities and began to be marked with various symbols denoting the owner, person of interest and/or contents,[26] a further increase in abstraction. The increasing volume of commodities grown, traded and recorded during the later part of the 4th millennium BC, and construction of large, 40 by 80 ft, temples,[29] plus irrigation and field measurements would imply that there were verbal words for some numbers, perhaps tied to a specific commodity or measuring stick.

Around 3100 BC, but before writing, there arose another kind of symbol, likely invented over a relatively short period of time by various unknown scribes, that of written numerals. This all but completed the abstract concepts of quantity (numerals), independent of the physical items they described. Shortly thereafter, the above described accounting system evolved into a cuneiform text[27, 28] and much later alphabets. The concepts of numerals and the written word, but not the Mesopotamian symbols, quickly spread throughout the region, entering Egypt around 2900 BC and, somewhat later, the Indus Valley and Mediterranean ports and eventually Greece.[26, 30]

Compared to the nearly 5000 years between the first use of tokens and the advent of written numerals what followed was indeed swift. The Royal Cubit was defined in Egypt around 3000 BC,[31] and within 600 years there was a comprehensive written language in both Mesopotamia and Egypt, mathematical tables, scribal schools, ziggurats were being built in Mesopotamia and the pyramids in Egypt; the right angle triangle (the so-called Pythagorean theorem) was solved and applied to surveying, and much more.[32] Either literacy had greatly increased human cognitive abilities[33] and new leaders arose to lead and educated the people, or there was a massive backlog of oral tradition (memes) just waiting for a new media, such as the oral Homeric epic poems - or perhaps all of the above and a bit more. In any event the Genie was out of the bottle.

There was likely no one inventor of numerals, writing, or mathematics. The development took long enough that involved individuals were forgotten, so that later generations considered such momentous inventions gifts by various gods and goddesses. In Mesopotamia the first was Nisaba (patron saint of scribes) and by Old Babylonian times the goddess Istar was the giver of the rod, ring, and rope (measurements) to kings. In Egypt writing was introduced prior to the first dynasty (ca. 2925 BC) apparently by one of the kings, possibly Narmer or Memes;[34] however, eventually Thoth was designated the inventor of writing, languages, etc. The influence of Thoth continued in the Hermetic writings as Hermes Trismegistos (first to fourth century AD), and was cultivated by the Arabs and re-introduced to the West in late medieval and Renaissance literature, furthering the connection with divinity.

In the 7th and 6th centuries BC, Greeks were serving in the Egyptian army as mercenaries and grain was being shipped to Greece in exchange for silver.[35] Pythagoras (ca. 580-500 BC) was said to have journeyed to Egypt and beyond and upon his return introduced mathematics as well as its accompanying mysticism to Greece. Pythagoras and the group he founded, the Pythagoreans, considered integers divine and much was made of the various relationships integers made with one another, as well as their influence on humanity. Odd numbers were male and even numbers female. The monad or unit one was not considered a number since it was singular and all other numbers plural, as they could be constructed by the addition of monads. The Pythagoreans attempted to construct a foundation of mathematics based on their arithmetic, however, the discovery of irrational numbers (e.g. $\sqrt{2}$) that resulted from the calculations of the hypotenuse in geometry, threw their concept into disarray.[36] Euclid (ca. 300 BC) compiled earlier work, in addition to his own, and produced the thirteen books of “The Elements.” A monumental task as his work covered all of the known mathematics and built a rigorous and logical geometry that effectively became the foundation of mathematics for the next 2,000 years.[37]

However, it remained for Plato (428/427-348/347 BC) to put a major spin on, if not the divinity, then the celestially of mathematics. Influenced by both Heraclitus and Pythagoras, he noted that no two things in the visible world are perfectly equal, just as there is no all-good, or perfect beauty. Yet, equality is a fundamental requisite for mathematics and also everyday life; hence the notion of good, beauty, equality and mathematics must come from another world, a world beyond the senses.[38] St. Augustine of Hippo (354-430 AD) adapted his version of Platonic thought to Christian ideas and formulated a strong Christian theology with mathematics as an abettor incorporating it in his book "The City of God."[39] With the fall of the Roman Empire the emphasis of mathematics shifted to India and Arabia where it was nourished and improved upon. After its return to Europe in the 14th century it soon became part of God's plan and more or less remained that way into the 19th century, where the German mathematician Leopold Kronecker summed it up by claiming that "God made the integers, all else is the work of man." Presently, the relationship between mathematics and some form of supernatural intelligence appears to be either the language of God, its own reality (Platoism), or the work of humans; which many appear to believe, but few publicly espouse.

It should be noted that original documents concerning early mathematics in Egypt are limited to two papyri and a handful of fragments. Most of what is known about early Greek philosophers is what much later writers said they said, along with their own notions and biases, as only a few fragments exist prior Alexander The Great (356-323 BC). Euclid's book, "The Elements," is known only from translated copies from the 8th to 10th century AD.[40] However, in Mesopotamia roughly half a million clay tablets have thus far been uncovered with hundreds containing mathematical notations that, if nothing else, dispelled the notion that the Greeks invented mathematics. Mathematics in ancient China also pre-dated the early Greeks.[41]

Euclidian geometry was the foundation of mathematics until the 1700's when the spell was broken by the development of several non-Euclidian geometries. This prompted a search for a more rigorous foundation and an arithmetical one was considered but led to a seemingly irresolvable paradox. In 1873, Georg Cantor introduced set theory, which few took seriously until the early 1900's when it was axiomatized by Ernst Zermelo, however the "Axiom of Choice" could not be proved, which resulted in controversy. In the 1930's Kurt Gödel proved his "Incompleteness Theorem" which essentially said that mathematics is incomplete in that in any axiomatic system there is a true statement that cannot be proved. This essentially liberated set theory and it is now the de rigueur approach in a totally axiomatized mathematics.[42] Theoretical mathematics is presently an endeavor defined by postulates (assumed and/or defined truths) which may or may not have a dependency on reality. Its results have no meaning outside of the boundaries of the postulates that define it; similar to chess and other organized games.

Pragmatic or applied mathematics works as advertized, since numerals are strictly adjectives operating on base units (object monads) that are defined as being equal to one another. Today a meter is a meter the world around - being one of the seven international system of units (SI) from which other units are derived. The meter is defined as the distance light travels in roughly one-three hundred millionths of a second. For comparison, the Egyptians likely used the Royal Cubit, the distance between two scratch marks on a black granite slab (as opposed to the common cubit which was the distance from the elbow to the extended finger tips). The Royal Cubit, along with thousands of workers, built The Great Pyramid of Giza; the sides of which varies less than 0.05% from the mean length.[43] Quite impressive when compared to the allowable error of 0.1 to 0.2% in much of today's home construction. It doesn't greatly matter how the base measurement is derived as long as the original is on, or of, durable material, everyone involved uses it, and copies are periodically checked against the standard.

Cognitive scientists have been searching for the locus of mathematics and while there doesn't seem to be one particular location, they've found several areas of the brain that light up in MRI scans when subjects are doing math. It appears that the brain functions somewhat differently when doing approximate arithmetic (subitization) than when doing exact arithmetic. The first relies on numerical magnitudes while the second ties into a language specific format; essentially the language locations, which enlarge. Together they result in mathematical intuition. The numerals are not represented by individual neurons, but rather clusters of them. Smaller numerals are represented by larger clusters than the more distant ones; an analog format.[44, 45]

Categorization and generalization gave rise to the concept of an average or norm, which in turn became internalized as a commodity or object monad. Similarities are accentuated and dissimilarities discounted or ignored. The object monad allows for quick evaluations: friend/foe, edible/non-edible/, safe/unsafe, etc. It costs little to detour around a suspicious looking coppice compared to ignoring it and being surprised by a hidden predator. Adult vervet monkeys of East Africa have three calls of alarm; one for eagles, one for leopards, and a third for snakes - each requires a different defensive reaction. The young learn these calls by noting the action of the adults when the call is given.[46] Subitization allows further refinement of the object monad for quickly determining which food source is larger/smaller than another or evaluating the potential danger from one or more adversaries. For example, in chimpanzees, fights between neighboring groups only occur if one or the other has an overwhelming numerical advantage.[47] A single female lion at a kill ignores the presence of one hyena, will share the kill with two, and leave if three appear. However, if the lion is a mature male the kill is typically uncontested;[48] categories within categories.

Problems:

The assumption of equality within members of a category has become a very successful, world-wide cultural meme; varying only as to the degree of categorization. For example, many urbanites consider the category sheep, as "seen one, seen them all." However, most categories can be further subdivided. To a knowledgeable sheep producer they can be further segregated as to breeds, ewes, rams, lambs, hoggets, two-tooths, pregnant, culls and occasionally, individuals. The problem is not that there are sub-categories but that a sub-class, a part, is often used to define the whole. A 2010 survey of papers published in psychological journals showed that 96% of the subjects came from Western Industrialized countries and of those, nearly 68% were undergraduates.[49] A sub-set of a sub-set was often being used to characterized the behavior of all humans.

There are over a dozen definitions of the word system. As used in this essay it refers to life in general and any activity that life engages in; schools, governments, clubs, agriculture, ecosystems, to name a few. A system is composed of parts, some of which must be sensors and monitors to keep tabs on both the system's inner and outer environments. Parts do not maximize their output, but rather optimize it with respect to the need of other parts. If all goes well, the system may give rise to an emergent property - something that is not obvious from an inspection of the parts alone. Since sheep have already been mentioned; two, three and sometimes four sheep will scatter in different directions when confronted by a sheep dog, whereas more than four sheep will flock and move as one. The flocking is an emergent property.

A primary characteristic of life is that it is not "all one or another." In every action there is at least a very small interval of time before the reaction; there is no instantaneous reaction. As the American physicist John Wheeler put it; "Time is nature's way to keep everything from happening at once." Nature flows not only with time but with substance. Life, in particular, is in continual flux from one form to another; there is nothing digital in nature, only in the minds of humans.

Zeno of Elea (495-430 BC) argued against infinities by posing his famous paradoxes. Essentially that if one started to go from "here to there" and first went half way, then half the remaining distance, then half that, etc, one would get very close but never actually reach "there." This has been a bur in mathematician's shorts ever since. It has been variously claimed to have been solved,[50] and since mathematics became axiomatized, it's relatively easy to define one's way out of an intractable problem; much as a^{-1} was defined to be 1/a so that it was then possible to prove that $0^0 = 1$.[51] Obviously, one can get from here to there - just get up and walk over there - but try doing it Zeno's way. No one really knows what Zeno had in mind with his paradoxes, since none of his original works have survived. All that is know is what other claim he said; which of course, had fueled endless speculations. I suspect Zeno was attempting to show the irrelevance of theoretical mathematics (infinite series) when applied to the natural world. Others have arrived at similar conclusions.[52] The Universe is finite, if not then the Big Bang theory fails. A finite universe can not have anything infinite, and without infinite series much of modern mathematics collapses.

Euclidian geometry was the foundation of mathematics until the 1700's when it was noted, among other things, that it didn't work when transferred to a sphere. One might wonder why it took the Europeans so long to figure that out, but it must be remembered that they had recently come out of the dark ages. So dark, that on July 4, 1054 AD, when the light from an explosion of a supernova, the Crab Nebula, appeared and remained visible during daylight for 23 days, and at night for almost two years, and was recorded in China, India, the Americas as well as by many indigenous tribes, nobody in Europe made note of it.[53] Europeans finally saw the light with the arrival of mathematics from the Arab world, which they attributed to divine will. Mathematics has been on that pedestal ever since.

Mathematics require identities to function; hence monads. Divide 1 by 3 and one obtains 3 lesser versions of the original; no problem in metrology which has their own defined bases. However, divide a complex system by 3, such as an automobile or living dog, one obtains 3 messy parts with little relationship to the original. Without identities, addition and subtraction have minimal value and the other operands are of little, if any, value. A human in our economy can have zero money or be in debt, but the concept of zero and negative values does not apply to the natural world. There is no place in the universe where there is nothing, or less than nothing; there is always something, if only radiation or gravitation.

Mathematics, with its linear foundation (dependant upon the equality of monads) has problems with more than three variables and can only approximate; which it does by using iteration and renormalization techniques that incorporates new in-coming data. Models are constructed and computers are programmed so that the model can simulate past events. However, when events are recorded someone makes a decision as to which variables will be recorded along with the event. Equations are developed and refined so that they meet or almost meet all the recorded events, but fail when predicting future events. The reason is each event is the result of a multitude of variables, anyone of which is capable of changing the outcome (the butterfly effect).

Uniqueness presents unique problems. The mathematical operand factorial (n!) is a method of determining the total number of linear combinations that can be made from a group of unique items. Three distinct objects can make 6 possible combinations, 5 items can make 120 combinations and 10 items over 3.6 million. However, the natural world is three dimensional plus time, so there are actually many more combinations than indicated by a factorial. In a universe full of unique items the possible combinations are quite large indeed. All of which makes statistics meaningless, since the sample size is always one. Which is good enough for some jobs, but not when truth is the objective.

Contemplations:

Our quantized vision of nature is the result of efforts to make sense out of a random world to survive and procreate. We've done that. But in the process we have created an environment that is no longer conducive to our tried and true methods of resolving problems. Our digitized version of mathematics is one tool, which has pragmatic value when dealing with items that can be digitized. But unlike a carpenter with only a hammer, not all problems are nails. Time flows, life flows and so does the physical world. We digitize it only because doing so worked in the past. Attempting to find the truth of nature requires a different approach. As such, it requires tools that are amenable to nature, rather than bending nature to fit our ancient survival paradigm. If one can't step into the same river twice, one can walk with the river - go with the flow. It may not be exactly the same river one steps into each time but it's certainly closer than if one proceeds cross-wise.

Managing large complex systems requires continual attention, competent sensor and monitors, quick response to problems, and the ability to recognize when to change and when not to. There must be objectives that are possible, but can be changed if the situations demands. A system can be guided towards feasible objectives, but likely by a path that was not one of those initially considered. It's doable, I've done it and so have many others. But how its done is unique to each particular system - there is no cookbook; at least not now.

Unfortunately, analog computers lost out to digital some time ago, not because they didn't do the job, but primarily because of linear economic thinking - they didn't make as much money. Much like the marketing battle over Sony's Betamax and Matsushita's video home system (VHS). The Betamax was the better, but more expensive system and so was eventually abandoned. Analog thought and computation is not dead, as any Google search will demonstrate. Work by such groups as "Brains in Silicon" at Stanford University, shows promise of things to come.[54] It may not lead to absolute truth, but it will be closer.

The problems arising from a mathematics that cannot duplicate the natural world would be subtle and possibly not contentious, if only we were aware of its faults and planned accordingly. Unfortunately, our mathematics reflects our core behaviors; that of sociality, categorization, generalization and subitization. We draw lines around similar item, call them the same, project the traits of the average on all members, decide if a category is larger or smaller than another, and then ardently discuss what we’ve done; creating memes that now circle the globe in seconds. Changing our core behaviors is not likely to happen, changing our mathematics to better conform to reality is possible.

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