On The Origin of Unreasonable Abstraction

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Abstract

Category theory is a type of mathematics that challenges us to rethink fundamental ideas about numbers, as experimental outcomes. Seemingly essential to modern quantum field theory, its role in gravity remains mysterious. Such nonsense is introduced under the assumptions that (i) unification is a valid goal for physics and (ii) relativistic causality holds for any local observable. Is fermionic spin analogous to Boolean truth? If so, we should remember that whatever is divided is also non separable, and this ultimate unity cares nought for all our vanity. All we can really do is count.

In the business of gathering and collating data, it is no surprise that one language, mathematics, has a linguistic advantage over others, since it was long ago selected as the guild's preferred tongue. And it is no surprise that this language, like others, has an immense literature, producing works for nothing but its own sake. Yet in fundamental physics, we feel that this language is canonical, touching some deep truths about our world.

It is time to debunk the myth that physicists consider mathematics as a tool. On the contrary, it is mathematicians who fiddle with the nuts and bolts of proof. A physicist must race through every available concept, hopelessly searching for the words that fit. A theory trusts mathematics to yield the only consistently valued measure of success: predictive power. Like the oracle at Delphi, mathematical reasoning told Dirac that antiparticles should exist. And nothing compares to the quantitative miracles of the Standard Model.

The business of efficiently arranging experimental facts automatically harbours the ongoing goal of mathematical unification. We theorise about gravity in the quantum realm not because we will be engineering the early universe any time soon, but because past experience tells us that such endeavours will lead to useful, albeit unexpected, predictions.

Over the last 20 years we have learned that the Standard Model, despite its success, is poorly understood. Enormous effort went into maintaining *locality*, while quantum physics would abandon it. At the same time, general relativity developed an urgent need for dark energy, along with dark matter. Dark matter appears to localise on galactic scales, but we cannot at present attribute particle properties to it. There is also the mystery of neutrino masses, and the potential for sterile states. In summary, we live in a time of sublime ignorance, holding a wealth of impenetrable data. Presumably, it was always thus, because we can only create, never discover.

Our present inability to envisage a relativistic quantum world, on cosmic scales, has forced us back to elementary questions about geometry and logic. There are new kinds of mathematics, that organise older concepts. Although we refer here to *category theory*, the technical idea of a category is merely one example of the developing higher order language. It is introduced to physics with the goal of transcending the abomination of Galilean ontology.[1]

With category theory [1] one can study quantum mechanical reasoning [2]. The basic idea, to combine diagrammatic logic and arithmetic, ostensibly originates with the 19th century philosopher C. S. Peirce [3], but then again, humankind always drew and counted.

We begin with our ability to count. In mathematical jargon, one talks about the ordinal numbers $\mathbb{N}$ in the context of *decategorification* [4][5]. One compares two bowls of oranges without actually bringing the bowls together, by assigning an abstract number to each set. Then such a number $N \in \mathbb{N}$ represents *every* set of things of cardinality N, by losing most of the information about the set. But we can still talk about operations on ordinals in terms of sets, using *disjoint union* for addition and *Cartesian product* for multiplication. Then there is a category of all sets, including maps between sets, called simply $\underline{\text{Set}}$.

Observe what has happened here. The idea of a *set of things* is supposedly a closer representation of an objective mundane reality than the decategorified number assigned to it. But the real advantage of defining $\underline{\text{Set}}$ is that we can now talk about things that are *not* sets, like actual atoms.

Quantum mechanics permits two particles to be *completely* indistinguishable [6]. This cannot happen in a set. Even with a perfect pair of identical billiard balls, one can in principle track the identity of one ball, wherever it goes. Identities do not morph into each other. This classical identity is as concrete, and as false, as the Newtonian spacetime that we imagine it inhabits. As d'Espagnat says [7] in proving that quantum non-separability *must* follow from experimental results (the violation of Bell's inequalities),

> What is new is that now *multitudinism* ... is contradicted by experiment independently of the formalism. Its falsehood as a philosophy thus seems unquestionable. Now no arguments, not even those based on fruitfulness, can justify the conscious dissemination of an error once it is recognised as such.

Alas, this persistent illusion about external reality persists today, whether in objective multiverses or in discussions of *those* atoms. One speaks of the

[1]One usually refers to *Newtonian* absolutism, but Galileo was much more that way inclined.

emergence of the necessary continuities of energy and momenta, as if the classical view is a true representation of the world.

First, we consider the indistinguishability of sets of particles, and then we move on to the internal structure of a particle. In neither case does the category Set fit the observational requirements. But to understand categories, it is good to begin with Set.

A categorical *morphism* in Set is an arbitrary function between two sets A and B. It is drawn as a directed line segment $A \to B$. Finite sets of a given cardinality are essentially equal, because one can find two morphisms, one in each direction, such that the composition of the two functions equals the identity function on the set. Sets can be either finite or infinite, but finite sets will suffice for most of the discussion.

The *Cartesian product* of sets A and B is the set of ordered pairs of elements (a, b) with $a \in A$ and $b \in B$. It is represented by the square of *projection* morphisms

$$\begin{tikzcd} A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "!"] \\ A \arrow[r, "!"'] & \mathbf{1} \end{tikzcd}$$

where $\mathbf{1}$ stands for any set with one object, and there is clearly a unique function ! from any set into $\mathbf{1}$. One can take either path from the source, with the same result. That is, the diagram *commutes*. From now on, the $\times$ symbol is omitted.

This product is universal: for any morphisms $X \to A$ and $X \to B$ (forming another square) there exists a unique morphism from X to $A \times B$ such that all squares and triangles commute. Similarly, there is a universal operation $\coprod$ for addition (see the appendix). The biggest difference between finite sets and ordinals $N \in \mathbb{N}$ is that the numbers come with an ordering. Can the order come from a category Set?

By the universal property, there exists an x in a commuting square

$$\begin{tikzcd} ABN \arrow[r, "x"] \arrow[d, "\pi"'] & BN \arrow[d, "\rho"] \\ AN \arrow[r, "\sigma"'] & N \end{tikzcd}$$

with projections π, ρ and σ. It is obvious that x is also a projection. This singles out the prime cardinalities, which sit over $\mathbf{1}$ with respect to the projections. If A and B are *coprime* (ie. sharing no common factor other than 1), then AB must be the source of the commuting square. This puts primes at the bottom of a lattice of projections in Set, giving $\mathbb{N}$ a partial order. But so far we have found no natural morphisms between primes.

The proper order on $\mathbb{N}$ really lives in another category, of *simplices*, which underlies the classical notion of space. Its objects are the numbers $N \in \mathbb{N}$, such that an N point set has its elements *labeled* by $0, 1, 2, \cdots$ and so on. One then inserts all the morphisms that carry lower ordinals to higher ordinals.[2] For example, $2 = \{0, 1\}$ is pictured as an interval $0 \to 1$. Check that the higher dimensional numbers are triangular in shape. We could almost create this category by collapsing each N point set in $\underline{\text{Set}}$ to a point, but there are too many functions hanging around.

Observe now that in a quantum set category, unlike in $\underline{\text{Set}}$, objects are truly indistinguishable. There can only be *one* number $N \in \mathbb{N}$, one identity morphism $N \to N$, and one representative $P \to Q$ for any subset P of a finite set Q. With such a limited range of morphisms, we have essentially recovered the category of simplices! The order on $\mathbb{N}$ follows from the uniqueness of the subset morphism $P \to Q$ when $P \leq Q$.[3] Indistinguishability gives all compositions of morphisms. Thus quantum sets are secretly spatial building blocks.

Subset arrows are just as special as the projections. These morphisms in $\underline{\text{Set}}$, now drawn with a feathered tail, fit into a commuting square

$$\begin{tikzcd} P \arrow[r, tail, "r"'] \arrow[d, "!"'] & Q \arrow[d, "\chi"] \\ \mathbf{1} \arrow[r, tail, "\text{true}"'] & \mathbf{2} \end{tikzcd}$$

with the universal property: for any other morphism $f : X \to Q$ for which X is sent to 1, there exists a morphism $X \to P$ so that everything commutes. Here χ is the characteristic function that defines P as a subset of Q, sending elements of P to 1 and every other element in Q to 0. The true arrow sends the single point to the value 1 in a two point set $\{0, 1\}$ of possible truth values.[4] In this way, $\underline{\text{Set}}$ does classical logic [5].

Is there an empty set of particles? Certainly not. The empty set is a unit 0 for the addition of sets, but particles are not sets. The vacuum is not empty. Although addition is clearly important for traditional quantum Fock spaces, we can only get there slowly, because Fock categories [9] start with the underlying number field, as if it already exists.

For our sets of particles, the commuting square above is unique for every pair P, Q of ordinals such that $P \leq Q$. A special object $\mathbf{2}$ is only utilised when P is strictly less than Q, suggesting that $N \in \mathbb{N}$ should be distinguished from P/Q in the rational interval $(0, 1)$. Indeed, for the simplex

[2] This does not cover the technical definition, which is in [1].

[3] The object $\mathbb{N} \cup \infty$ in $\underline{\text{Set}}$ is also used in arithmetic logic [8], whereby ∞ should exist because $\mathbb{N}$ is itself a set.

[4] The zero secretly stands for the empty set. While $\mathbf{1}$ is the terminus of all the ! maps, the empty set is the *initial* object, sitting inside every set. Quantum mechanically, this is probably nonsense.

interval, 0 and 1 are objects, while the interval is a morphism. A rational P/Q such that $P < Q$ has many representations PN/QN, requiring a canonical coprime pair, sitting near the bottom of a lattice of product projections [10].

In numerical categories the primes should be special, so that any projection not belonging to a product is derivative. This includes all those characteristic functions χ into $\mathbf{2}$ for which Q is odd. Quantum sets naturally partition $\mathbb{N}$ into odds and evens, with the even Q providing the value $P/Q = 1/2$. There are exactly Q morphisms $Q \to \mathbf{2}$ for $Q \geq 1$, namely the morphisms that pick out 1, 2, 3, $\cdots$ objects in Q. In the degenerate case $Q = \mathbf{1}$, the object $\mathbf{2}$ behaves like $\mathbf{1}$.

One can now guess that the objects 0 and 1 represent the two states of fermionic spin (usually labeled *up* and *down*). A point in the interval $[0, 1]$ determines a pair of probabilities $(p, 1 - p)$. So simplices are built out of probabilities, extending the two state case to any probability set $(p_1, \cdots, p_N)$ with $\sum p_i = 1$.

What about complex amplitudes? Quantum states collect the set of $N \geq 2$ outcomes into some kind of vector. Look at $\mathbf{2}$, the case of fermionic spin. The collection of all possible amplitude pairs is usually given by the complex sphere $\mathbb{C} \cup \infty = \mathbb{CP}^1$. The basic message, given the logic of $\underline{\mathrm{Set}}$, is that classical *truth itself* is blurred by quantum amplitudes. If the truth arrow still exists, it is probably a *complexification* map into $\mathbb{CP}^1$. Unfortunately, we no longer know what a complex number is!

Restricting attention to finite sets of measurement outcomes, what is a *minimal* representation for the quantum states? For fermionic spin, there is an easy answer: the three Pauli operators σ_X, σ_Y and σ_Z. That there are exactly *three* operators is incidental, at this point. They are defined canonically using an alternative version of $\mathbf{2} = \{0, 1\}$, namely the field $\mathbb{F}_2$ with two elements. In $\mathbb{F}_2$, $1 + 1 = 0$. This rule makes $\mathbb{F}_2$ closed under addition and multiplication.

For any finite field, a *character* is a map that sends each element to a complex phase in $\mathbb{C}$, in such a way that arithmetic is preserved. For a finite field $\mathbb{F}_q$ [11] the characters[5] are organised into q separate basis sets of quantum vectors. The physical requirement is that any pair of vectors x and y from two distinct bases X and Y should pair to the same absolute value $|\langle x|y\rangle|$, equal to $\sqrt{q}^{-1}$. Theoretically, it seems natural to start with these character sets, whose pair sums then *define* the required real square roots.

Such a basis pair is said to be *mutually unbiased* [12][13], meaning that with respect to X, every outcome in Y is equally likely. Each basis vector gives a column of a $q \times q$ (unitary) matrix. Since the identity matrix is mutually unbiased with respect to every basis of complex phases, there are

[5] We restrict attention to either $q = 2$ or $q = p^n$ for p an odd prime.

a total of $q+1$ mutually unbiased bases. For $\mathbb{F}_2$, these three unitary matrices are exactly the eigenvectors associated to the Pauli operators σ_X, σ_Y and σ_Z. The dimension of macroscopic space is derived from the number **2**.

With mutually unbiased states, it is less surprising that the number **1** is difficult to perceive. If there is only one possible measurement outcome, with any phase amplitude, then any two phases in $\mathbb{C}$ are mutually unbiased. This naively implies a continuum of observables. As with a string, nothing can be resolved to a point.

In summary, axioms for quantum sets are secretly the study of numbers. For a truly non separable concept of existence, we must reinterpret the continuum of $\mathbb{C}$. Unfortunately, none of the above yields the relativistic causality of the Dirac equation. It is easy to see the three dimensions of space, but very difficult to see the four dimensions of spacetime. Presumably, negative numbers are the key.

We should also consider categories associated to spacetime symmetries, such as the Poincare group and its relatives. A *group* is precisely a category with only one object and invertible morphisms. A representation is then a morphism from one category into another, known as a *functor*. Building more and more structure into categories of representations, we can approach the actual dimension of spacetime *with the categorical diagrams* (see for example [14]).

Other sciences are also in the business of quantifying data. They too become increasingly abstract. What was once an algorithm in computer science may well be considered a mathematical proof today. In sublime ignorance, we go on seeking unification.

Appendix: More About Sets

In Set, two ordinals N and M are added using a disjoint union $\coprod$,

$$\begin{tikzcd}
 & X & \\
 & N \coprod M \arrow[u, "x"] & \\
N \arrow[uur, "f"] \arrow[ur] & & M \arrow[uul, "g"'] \arrow[ul]
\end{tikzcd}$$

where x exists for any f and g because $\coprod$ is universal in the opposite sense to the product. The empty set is a unit for $\coprod$. The distribution law

$$MA + MB = M(A + B)$$

on $\mathbb{N}$ morally follows from the obvious fact that M copies $\coprod^M A$ of A equals MA.

A surjection $Q \to P$ is a morphism for which every element of P comes from something in Q. If we have these, the primes are no longer special. Imagine the category built from surjections, subsets, disjoint unions, ordinal bijections, and given universal morphisms. In fact, this covers *all* reasonable functions $f : X \to Y$ on finite sets, since we can partition X by the range $f(X)$, so that f equals the composition of an inclusion

$$i : P \to Y = f(X) \coprod (\text{not} F(X))$$

and a projection $p : X \to P$ that sends each partition to a point.

For any set P, there is also a set of all subsets of P in Set, including the empty set [5]. Pairs of subset morphisms $X \to P$ and $Y \to P$ produce a source object that creates yet another universal commuting square, and this special set is the intersection $X \cap Y$. Similarly, unions exist.

A *functor* F between two different categories $\mathcal{C}$ and $\mathcal{D}$ (with morphism composition denoted by $\circ$) is a map that sends every object in $\mathcal{C}$ to an object in $\mathcal{D}$, and every morphism to a morphism, such that $F(f \circ g) = F(f) \circ F(g)$. Number theory is truly about pairs of functors $\mathcal{C} \to \mathcal{D}$ and $\mathcal{D} \to \mathcal{C}$ that themselves compose to *something* like an identity functor on each category. It is this something that allows us to draw higher dimensional pictures.

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