

Revising Space Time Geometry: A Proposal for a New Romance in Many Dimensions

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August 25, 2012

Abstract

The ontological positioning of our existence, deeply connected with the hierarchy problem concerning the dimensions of space and time, is one of the major problems for our understanding of the “ultimate” nature of reality. The resulting problem is that mathematical concepts that need so-called “extra-dimensions” have been widely disregarded owing to a lack of physical interpretation.

This article reviews conventions, imaginations and assumptions about the non-imaginative by starting at the origins of the of 4D and 5D space-time concepts and proposing a new geometrical approach via a hyper-Euclidian path for a mentally accessible vision of continuous AND discrete complex space configuration in dimensions up to higher order.

“Time and Space... It is not nature which imposes them upon us, it is we who impose them upon nature because we find them convenient.” [Henri Poincaré, 1905]

Introduction

The general physicists’ assertion is that we are living in a three-dimensional world with one time-dimension and cosmic space is curved in the fourth dimension is based on a relativistic world view that has gained general acceptance. In fact this is a post-Copernicanian world view which does not pay attention to the consequences of Max Planck’s revolution and the development in quantum physics of the past 100 years.

The ontological positioning of our existence, deeply connected with the hierarchy problem concerning the dimensions of space and time, is essential for our understanding of the “ultimate” nature of reality. One of the biggest problems of modern natural science is based on the split in the world view that originates from the incompatibility of the four-dimensional cosmology and quantum physics, which requires a physical system with a minimum of six dimensions in the configuration space to describe the position of a particle.

How can we get a clear vision of higher space dimensions?

The problem of ontological disorientation

Above and beyond the convenience of relativistic 4D space-time, even brilliant mathematical concepts that are regarded as mathematically consistent remain widely disregarded if they need more than three space and one time dimension for the unification of all fundamental forces, owing to a lack of physical interpretation and phenomenological interpretation.

If M-theories, such as eleven-dimensional supergravity nowadays claim we are living in eleven dimensions – or even worse, E8XE8 heterotic string-model with thirty-two dimensions – it is unable to create a cognitive meaning. Since anyone who tries to follow up the topology is tempted to fall into Alice’s rabbit hole and gets lost in a labyrinth of loopholes and brane bulks.

This means that a geometry for the visualisation of the complex algebraic abstract concepts is needed in order to gain orientation and cognition for phenomenological interpretations. After that a new convention adapted to the current state of scientific knowledge can be established.

Recall the German mathematician Felix Klein (1849-1925) who first warned about imagining “higher dimensional space” in 1872 but changed his mind and declared to his students in Goettingen in 1889/90: “The developments of recent decades have meant that in Germany in many cases the focus has been on abstract, logical inquiries of geometry, whereas the training of the appropriate outlook has been neglected.” [2]

But until now not much effort has been made for this outlook, which means undertaking the development of an appropriate geometry. Nietzsche’s call, “away from formulas back to forms” has remained unheard.

The resulting problem is that all mathematical concepts that need more dimensions, so-called “extra-dimensions” have been widely disregarded owing to a lack of physical interpretation. They might be ever so brilliant mathematical thought constructions, but it remains without consequences for our cognition, and the incompatibility of general relativity and quantum mechanics remains.

We have to consider that it is almost 100 years since Gunnar Nordström, 1914 [3] and later Theodor Kaluza [4] who were both unifying electro-magnetism with gravity with the aid of the fifth dimension – and still there is and no full picture of the meaning of this finding.

Thus the time can be considered as ripe for a rethink of the foundations of the topological ideas and for research into new geometrical principles to escape from the topological labyrinth and overcome its cosmological implications with its “paradoxes” and apparent contradictions in contemporary theories.

This article reviews the origins of the currently established 4D space-time concept and proposes a new geometrical approach via a hyper-Euclidian path for a mentally accessible vision of continuously countable space dimensions up to higher orders:

The development of higher-dimensional spaces in mathematics

Let’s review the development of higher-dimensional spaces in mathematics:

The discussion about higher dimensionality of space started in the middle of the 19th century. Previously Immanuel Kant (1724- 1804) had noted that “a science about all these possible sorts of spaces would be the highest geometry which a limited mind could undertake”, but he himself remained safely with his concept of one absolute space. [5]

In Germany there was the mathematician Hermann von Helmholtz (1821 – 1894) and in England the mathematicians William Rowan Hamilton (1805– 865), William K. Clifford (1845 – 1879) and James Clark Maxwell (1831 – 1879), as well as the Norwegian mathematician Sophus Lie (1842 –1899), who examined higher space dimensions. Maxwell enjoyed expressing some of its properties in a cryptic way a beautiful poem:

“My soul’s an amphicheiral knot
Upon a liquid vortex wrought
By Intellect in the Unseen residing,
While thou dost like a convict sit
With marlinspike I untwisting it
Only to find my knottiness abiding;
Since all the tools for my untying
In four-dimensioned space are lying, ” [6]

Saying this, the early knot-specialist must have had a proper understanding of its geometry. Probably his understanding was based on Lagrange’s and Hamiltonian mechanics. Joseph L. Lagrange noted, “Ainsi on peut regarder la mécanique comme une géométrie à quatre dimensions.” [7] A similar idea was expressed earlier by d’Alembert in his article “Dimension” in the *Encyclopaedia*.

It is clear that Lagrange, who said that mechanics per se is four-dimensional, had a proper vision about a 4D construction of a moving entity working on mechanical principles. Although he considered himself as devoted to analytical algebra, he warned against the exclusion of geometry which in ancient Greece was the proof for algebraic calculations. He says that all too often castles are built in the air if one relies solely on analytical calculations without being guided by geometry.

It seems he was foreseeing the upcoming development in which geometry itself became abstract, not at least with David Hilbert’s (1862 - 1943) endeavours to axiomatise mathematics by propagating a pseudo-geometry and proposing that “instead of points, straight lines and planes” one should also could say “table, chair and beer glass” – it is only all a question of fulfilling the axioms.

- Is it possible that the greatest opponent of the “ignoabimus” unwillingly paid tribute to the further development of disorientation, which led to the ontological disconnectedness within brane bulks within indefinite Hilbert spaces?

- The “meta” assumption of the non-imaginability

Leading mathematicians of that time such as August F. Möbius (1790-1868) claimed that a space of four dimensions cannot be conceived of. [8] Also Hermann von Helmholtz was convinced that we cannot have a four-dimensional experience. For this in his lecture “On the origin and meaning of geometrical axioms” [9] given in Heidelberg in 1870 and later in England, he invented the hypothetical beings of the plane, two-dimensional beings who live on the surface of a sphere and cannot imagine the third dimension. This idea was adopted by Edwin A. Abbott in his most popular novel *Flatland – a Romance in Many Dimensions*, (1884) [10]. Since then, flatland has served to demonstrate the non-availability of a mental picture of higher dimensions: Consider yourself as a person who does not know the third dimension – and your brain is on “tilt”.

Since then the “meta” assumption of the non-imaginability of the fourth dimension has been propagated by most contemporaries. For example, Rudi Rucker expands on Abbott’s idea in his popular book *The Fourth Dimension* [11] taking it into curved space, black holes and beyond by showing 4D pictures squashed down to two dimensions.

Only a few mathematicians who have served geometry in an exemplary way contradict this. The most advanced geometry of the time, the projective geometry of the Renaissance artists, was invented in 15th century Italy.

It was Charles Hinton, (1904, p.159) who was convinced that the an intuitive visualisation of the four-dimensional space is possible. He posed himself the task of constructing four-dimensional polytopes to visualise them as 3D projections on a plane. Thus he constructed the “the simplest four-dimensional solid” the 4D cube, which he called the Tesseract. With this he demonstrated its rotational invariance by drawing its eight cubes and subdivision into $81 = 3^4$ cubes. [12]

Ludwig Schläfli and Viktor Schlegel were the first to examine polytopes in \mathbf{IR}^4 (in 1850 and 1883). Schlegel discovered: “For the hypercube, the most convenient is the following: one constructs a cube inside another, such that the faces of one are parallel (situées vis à vis) and one joins the vertices of one to the corresponding vertices of the other.” [13]

According to the same principle one can imagine a sphere in \mathbf{IR}^4 as two spheres within one another. Rather imagine them connected by smaller spheres going through each point of the surface of the inner touching the outer sphere then “imagine them as glued together on their common surface” according to a common topological understanding. It is as simple as Giordano Bruno described it “to wrap a great sphere around the globe so that a continuous surface is created which extends beyond the earth.” [14] But how is it that *no one* better understands the world as a construction à la hypercube, a spherical tesseract confined by moveable spaces progressing in time?

First concepts about the fourth dimension

The fourth dimension – generally known as “time” – offers a series of geometrically unsolved problems of understanding. “Time” introduced as the vector ct describes a change of time multiplied by the speed of light and already escapes geometrical intelligibility, “because strictly speaking $x_4=ct$ is not time at all but the coordinate of a light signal of time.” [15]

In 1905 Henri Poincaré noted that the Lorentz transformation can be seen as a rotation in a four-dimensional space with time as the fourth dimension. [16]

“A fourth dimension one may say, is the fifth wheel on a carriage.” [17]

In his story “The Fourth Dimension” Gustav T. Fechner (1846) developed the idea that the fourth dimension could be time, and that our three-dimensional space moves parallel to itself along the time axis, so that at any point in time it was in a different three-dimensional space and thereby formed an infinite series of three-dimensional spaces – a concept that goes quite well with the hyper-sphere analogous to the tesseract.

More than half a century later, Hermann Minkowski’s words resounded: “The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.” [18]

It may be pure coincidence that in his address Hermann Minkowski used images that Fechner had already used in his “Four Paradoxes” story, where he describes three-dimensional space and time as “shadows”. Minkowski also appears to have read Melchior Palágyi (1859-1924) as there is a similar remark to the one quoted above: “As natural as it may seem, at the beginning of our investigation into time and space it must be most strongly emphasised that there is no sensory phenomenon that would manifest itself solely in space or solely in time.” The coordinates of a point in a “flowing space” Palágyi denotes as $x + it, y + it, z + it$ with a clear distinction between space and time. [19] He assigned each spatial point to a time line running through it. In fact Minkowski equation $c^2t^2 - x^2 - y^2 - z^2 = 1$ contains an imaginary factor for time t , the square root of $(-1) \sqrt{-1} \cdot t = s$, which equals the velocity of light with $i [\sqrt{-1} \cdot t = s \text{ and } 3 \cdot 10^5 \text{ km} = \sqrt{-1} \text{ sec.}]$ [18] So far he agrees with the avant-garde **thinkers** in terms of the imaginative component, which produces the higher dimensional character of four-dimensional space on which the time effect is applied. And in fact Minkowski also proclaimed the separate notion of space and time:

“Subjects of our perception are always places and times connected. No one has observed a place except at a particular time, or has observed a time except at a particular place.” In fact Minkowski also proclaimed the separate notion of space and time: “Yet I still respect the dogma that time and space each have independent significances. I will call a space-point at a time-point, i.e., a system of values x, y, z, t , as a world-point.”

But there seems to be slight indetermination in the system x, y, z, t take the variables k, l, m, n, o instead and let anyone tell which is spatial and which one should be timelike.

Therefore, in relativity theory a merging of space and time took place, which led to the ironic remark by Leo Gilbert in 1914:

“Apart from which it appears as if Wells [*The Time Machine*] caused the relativity principle, because the relativists . . . have not read Palágyi, but apparently only Wells, and were only led to confuse time with space through his interesting and brilliant joke.” [20]

In the same year, Palágyi’s book *Die Relativitätstheorie in der modernen Physik* made it clear that while both theories had actually added the fourth axis, time, they were conceptually fundamentally different, as his fourth dimension remained purely temporal, whereas in Minkowski the fourth dimension as ct became spatially equivalent to the other three. “Through his effusive urge for unity Minkowski allows himself be enraptured by the mystical striving to have space and time ‘sink like shadows’ in their special character and possibly to dissolve the difference between fundamental category terms.” [21]

The caput nili of the misguiding principle?

“There has been some question in the literature about whether special relativity handles rotations correctly. The equation which generates the Lorentz transformation depends linearly on q , but the generator depends on q and q^* ”, writes Douglas B. Sweetser [22] is pointing out the caput nili of the misguiding principle. The problem of doing a “transformation” with two arbitrary independent coordinate systems is that the two spaces each of the remain THREE-DIMENSIONAL; it is the same as if a tesseract were to be torn into pieces: The four-dimensional hyper-cubic object turns into a pile of ordinary cubes – not unlike Cinderella’s carriage changing back into a pumpkin.

4D space-time approach with quaternions

It has been remarked by Arthur Cayley (1821-1895) as early as in 1854, that the rotations in a four-dimensional space may be effected by applying a pair of quaternions. [23] Ludwik Silberstein (1872 – 1948) showed how one as a prefactor and the other as a postfactor, to the quaternion q whose components are the four coordinates of a space-point, say

$$(1) \quad q' = aqb$$

where in the case of pure rotation a and b must of course be either unit-quaternions or at least such that $T^2 a \cdot T^2 b = 1$; T denoting the tensor. “ [24] While later on in relativity the linear motion happens along the unhappily chosen time axis happens the original treatise of Silberstein shows “precisely the (imaginary) angle of rotation in the plane t, x_4 of Minkowski’s four-dimensional world, corresponding to the transformation [...] we may say that one half of this rotation is effected by Q as a prefactor and the other half by the same quaternion as a postfactor.” And “the axis of Q being u , this quaternion turns r round u , i.e. in the plane y, z normal to u , while in Minkowski’s representation the rotation is in the plane x, t .”

Silberstein also formulated how the quaternionic differential equation may be written for the force-quaternion Pe , expressed by the electromagnetic bivector F and its complementary G and can be adapted to show the properties of the Maxwellian stress and of the electromagnetic momentum along with the flux and the density of energy, in correspondence to the equivalent formula of Minkowski’s four-dimensional system. The rotation is to happen along the geodesic, which corresponds to Fechner’s and Palágy’s principle. Also Arthur W. Conway pointed out that “the quaternion has the advantage of being asymmetrical, the time-scalar occupying a different position from the space-vector. It is thus more in touch with real phenomena.” So let me strike out the importance of the effect that ONLY a clearly denoted imaginary part and the rotation around R_3 creates the continuous 4D space. No, rotation about the axis with all point at the origin does NOT MAKE ANY SENSE. The rotation remains strictly spatial and 2D or 3D but would never acquire the character of a notion as TIME.

Other writers, such as Sommerfeld, subsequent to Minkowsky, have used a four-dimensional vector that involves a second kind of vector with six components.

5D

This six vector was also used by Nordström : who showed that a unified treatment of the electromagnetic field and the gravitational field is possible if one understands four-dimensional space-time as surface laid through a five-dimensional world. “In the gravitation theory that Herr Mie and I have developed, the gravitation field in the ether is given by a four-vector; if such a theory corresponds to reality, the ether state will thus be characterised by a six-vector and a four-vector.” [25] He is pointing out that it is „one of the greatest services of relativity theory that it was able to characterise the electromagnetic state of the ether by a vector, the

Minkowskist six-vector f , whereas in the old concept two field vectors were needed for this. “But this possibility to characterise the ether state by a vector was untenable as soon as one assumed another gravitational field in the ether outside the electro-magnetic field.

Independently to Nordström Theodor Kaluza found in 1919 the unification for the elctro-magnetic forces with gravity by „calling a fifth dimension for aid“. [26] Alas, due to a lack of physical interpreation the theory was widely neglected.

Problems with 5D

A problem quite often expressed in the literature, that extra dimensions are supposed to be so tiny that they cannot be seen. A student of Plato’s would have been immediately expelled from the academy if he had asked about the size of the ideal triangle. Geometrical entities do not possess any measure, weight or any other material specification.

The idea of an additional dimension became more attractive in 1926, when Oskar Klein relaxed the condition that the fields are independent of the fifth coordinate, and assumed instead that the fifth dimension is rolled up in a tiny circle so that the fields are periodic in that coordinate. [27] Nevertheless, the theory remained almost forgotten until it was reactivated and extended by String theorists.

Nowadays the theoretical physicist Lisa Randall categorises it as supremely important, as she states that if it were possible to prove the higher dimensionality of space, then Kaluza’s idea would trigger a revolution in physics comparable to the Copernican”. [28] This text assumes that the determination of the dimensions represents a philosophical problem concerning conventionalism that will be logically, geometrically defined, rather than concerning an experimentally provable physical phenomenon.

6D

So far, the only well-known, often quite alluring CAD depictions of the 6-D space have derived from the Calabi Yau spaces (named by Philip Candelas et al. in 1985 after the Calabi conjecture by Eugenio Calabi (1954, 1957) and proven by Shing-Tung Yau (1977, 1978).)

In recent decades Calabi Yau manifolds has also been utilized to “compactify” the ten space dimensions of string theory to unify the elementary forces.

The 2-D slice of this 6-D manifold, which is embedded in a 4-D space, is best described with the following equation containing five complex variables:

$$(2) \quad z1^5 + z2^5 + z3^5 + z4^5 + z5^5 = 0$$

The reason Calabi Yau manifolds do not look like Penrose patterns, which are also a 2-D slice of 6-D space, can be found in the mathematical concept that states that $z5$ is divided and $z3/z5$ and $z4/z5$, is constant.

This leads to a simplification of a normalised inhomogeneous equation a second time, and plotting the solutions to

$$(3) \quad z1^5 + z2^5 = 1$$

the resulting surface with a ten-fold Rosetta is embedded in 4-D, and please note that it is only projected into 3-D for visualization. [29]

A proposal for an appropriate geoemtry for continuous spaces up to higher dimensions

A propiate geometry has to put aside all the trains and spaceships going past one another on de-coupled coordinated crosses that have contributed to the current confusion.

And it has to take another early critique into account: In 1917 Erich Kretschmann clearly formulated what it takes for a space-time theory to satisfy a genuine relativity principle. [30] He first pointed out that a theory does not satisfy a relativity principle simply by virtue of being cast in a form that is covariant under the group of transformations associated with that principle. In the spirit of Klein's Erlangen programme, Kretschmann's proposal was to characterise relativity principles in terms of symmetry groups of the set of geodesics of all space-times allowed by the theory. In special relativity, this would be the group of Lorentz transformations that map the set of all geodesics of Minkowski space-time, the only space-time allowed by the theory, back onto itself. [31]

There is no need to compactify dimensions. All higher dimensional spaces are three-dimensional and the hierarchy is clearly distinguishable by their movement, connexions and interferences of straight lines and circles on a system of interconnected complex coordinates in movement interfering and creating elements together in a way that the "virtual space-construction" of higher-dimensional spaces evolves itself by the simultaneous movement should obey the mechanisms of the 6-dimensional Lorentz group $O(1,3)$ as well as 10-dimensional Poincaré symmetry group.

The four dimensional manifolds can thus be established as a hypersphere with spheres on each point of the surface of a 2-sphere. The whole space around R^3 the three-dimensional spheres becomes the four-dimensional, surrounding bigger sphere with a hypothetical countermovement, the first symmetry-breaking creates the illusion of rest. Each point on the hypersphere can be considered as a north pole – there are no antipodes with their heads hanging downwards, as the speculation against the spherical worldview predicted and as it would be the case, if the world itself were three-dimensional. It is a strange fact that in the time of space travel and flights from continent to continent the experience of the fourth dimension became a daily and common habit, but its geometrical understanding of it was lost.

One of my proposals is that – analogous to the concept of lines of forces of Mawell [32] and Faraday – creative effects of spaces and lines may be hypothetically established in order to achieve a geometrical construction of logically interconnected spaces acting upon certain rules. Only by taking the imaginary part into account can the counter rotations between the spheres express mechanics that are four-dimensional, and which opens the knot of a string in a way that it is untied, as Maxwell knew. These imaginary parts are the extended vectors of the coordinate system x_i, y_i, z_i above the surface of the sphere and create the moving hyperbolic interfering spaces.

Thus the interference of 4D spaces can be thought as creating the 5D space – small circles as proposed by Klein. These circles may interfere in a way that again 4-dimensional spaces may evolve in order to interact in a 7- or 8- dimensional and 9-dimensional way. The six-dimensional case closes the system in which all higher-dimensional processes up to infinity may occur.

The same for ten-dimensional systems, which were regarded as a full description of the ether by Nordström.

Therefore the dimensions of our inhabited globe, including 4D "time-spaces" and 5D "light-spaces", together result in six – since Faraday and Maxwell the phenomenon of light can be considered as space warp of the electro-magnetic field, and since Kaluza and Klein this belongs in the fifth dimension.

Added to this are another six dimensions of the quantum world, which arises in the fifth dimension through the second symmetry break, making 11 dimensions in which it is possible to live in a "unified cosmic surrounding" and new age of enlightenment concerning the ontological position.

Rethinking the 4D conception within the theory of relativity and its cosmological implications

What will the consequences be if one puts back the geodesics where they belong?

Was the cosmological constant not fixing the rest of the universe on our planet – as Ptolemy did?

Could its removal actually let all its parts burst in all directions – away from the obscure OBSERVER ?

- How long will the convention hold that the universe was created ex nihilo from a point in space (!?) from where it could revolve on the time axis by opening itself like an umbrella in the rain?

- Or could there be one day another interpretation for the observed redshift of some cosmic objects?

- Dark energy

- Dark matter – the notion is pre-Platonian and well being 2,500 years old: fire as form in the world of becoming and passing away is the principle operating within the Chora, which is “a difficult and dark principle” the wet-nurse of becoming.

According to Plato, elementary matter comes into existence when undetermined, loose agitation tendencies strive against each other. They are balanced by the wet-nurse of becoming whereat the four elements will be separated by each other in a rocking motion, the precondition for matter, which is composed of the four elements by the most beautiful triangles in all possible symmetries.

“Romantic” outlook

- What about after all reinstalling the ancient Greek sparkling jewellery cosmos with crystalline spheres held in eternal motion by the fifth element, the quintessence, known as the ether of Maxwell and Faraday, Lorentz and Tesla – adding Lie group behaviour up to E8 instead of Pythagorean number tricks ?

- Or what about inhabiting the Poincaré doedecahron universe?

- Anything would be more sparkling than the dark ages of the grainy pancake universe, neither flat nor topologically deformed. Let's escape the wormholes of Marcuse's one-dimensional man.

Acknowledgements

Thanks to all collaborators supporting the Quantum Cinema project and the Austrian Science Fund (FWF) for its funding within the framework of the PEEK programme.

References

- [1] Henri Poincaré, Intuition and Logic in mathematics as part of "La valeur de la science ", Paris (1905) translated by George Bruce Halsted (1907)
- [2] Daniela Wuensch, Der Erfinder der 5.Dimension, Theodor Kaluza. Leben und Werk, Termessos, Göttingen (2007)
- [3] Nordström, Gunnar (1914). "Über die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen". *Physikalische Zeitschrift* 15: 504–506. OCLC 1762351.
- [4] Theodor Kaluza, Zum Unitätsproblem der Physik, In Sitzungsberichte der Preußischen Akademie der Wissenschaften zu Berlin, (1921), S.966-972
- [5] Immanuel Kant, Gedanken von der wahren Schätzung der lebendigen Kräfte" Dorn, Königsberg (1746)
- [6] Daniel S. Silver, The Last Poem of James Clerk Maxwell, Notices of the AMS Volume 55, Number 10, p.1266-1270 original quotation of Maxwell's letter To Hermann Stoffkraft, Ph.D. A Paradoxical Ode After Shelley, 1878.
- [7] Théorie des fonctions analytiques, quatrième edition, Partie III, p. 337, §.1, FIRST EDITION (1797).
- [8] August F. Möbius, Der Barycentrische Calcul: Ein neues Hilfsmittel zur analytischen Behandlung der Geometrie, Georg Olms Verlag, (1827)
- [9] Hermann von Helmholtz. In 'On the Origin and Significance of Geometrical Axioms,' Popular Scientific Lectures< Second Series (1881), 57-59.
- [10] Edwin Abbott Abbott, Flatland: A Romance of Many Dimensions (1884)
- [11] Rudi Rucker The Fourth Dimension, : A Guided Tour of the Higher Universes, Houghton Mifflin Harcourt, 1985
- [12] Charles Hinton, The Fourth Dimension (1904), p.159)
- [13] Victor Schlegel, Einige geometrische Anwendungen der Grassmann'schen Ausdehnungslehre, Quandt, (1882), p 194
- [14] Giordano Bruno, De Immenso et Innumerabilibus Liber I–VI, (1591). germ.: Das dreifache Minimum und das Mass, Skorpion Verlag, Preißenburg 2002, p.76
- [15] Gerold von Gleich, Invariantentheorie und Gravitation, Astronomische Nachrichten, volume 236 (1929) p.165-178,
- [16] Henri Poincaré, (1906), On the Dynamics of the Electron Rendiconti del Circolo matematico di Palermo 21 (1): 129–176
- [17] Gustav Theodor Fechner, Der Raum hat vier Dimensionen, Aufsatz (1846)

- [18] Hermann Minkowski, Raum und Zeit, address to the 80th Congress of German Natural Scientists, Cologne, 21 September 1908, Space and Time in The Principle of Relativity (1920), Calcutta: University Press, 70-88.
- [19] Dr. Melchior Palágyi, Neue Theorie des Raumes und der Zeit, Die Grundbegriffe einer Metageometrie, Verlag von Wilhelm Engelmann, Leipzig (1901)
- [20] Leo Gilbert, Das Relativitätsprinzip, die jüngste Modernität d. Wissenschaft Volume 1 of Wissenschaftl. Satyren, Breitenbach, (1914), p.51
- [21] Dr. Melchior Palágyi, Die Relativitätstheorie in der modernen Physik. Vortrag gehalten auf dem 85. Naturforschertag in Wien. Berlin: G. Reimer, (1914)
- [22] Douglas B. Sweetser, Doing Physics with Quaternions (2005)
<http://world.std.com/~sweetser/quaternions/ps/sr.pdf>
- [23] Arthur Cayley Phil. Mag. vol. vii. (1854), and Journ. f. reine u. angew. Mathem. vol. 50 (1855); or 'Papers,' vol. ii.]
- Ludwik Silberstein (1872 – 1948)
- [24] Ludwik Silberstein, Quaternionic Form of Relativity. Communicated by Dr. G. F. C. Searle, F.R.S. Phil. Mag. S. 6, Vol. 23, No. 137 (May 1912), 790-809.
- [25] Nordström, Gunnar "Über die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen". *Physikalische Zeitschrift* XV, 1914, 504–506. OCLC 1762351
- [26] Theodor Kaluza, Zum Unitätsproblem der Physik, In Sitzungsberichte der Preußischen Akademie der Wissenschaften zu Berlin, 1917, S.966-972
- [27] Oskar Klein, "Zur fünfdimensionalen Darstellung der Relativitätstheorie" *Zeitschrift für Physik*, p. 895-906, 1937
- [28] *Warped Passages: Unravelling the Universe's Hidden Dimensions*, Ecco, Harper Collins, New York, 2005
- [29] C.-Z. Quehenberger, P. Weibel, H. Rauch, H. Katzgraber, A New Digital 3-D Dynamic Geometry for the Visualization of Complex Number Space, Proceedings of ICOCENT 2012, Jaipur
- [30] Erich Kretschmann, *Über den physikalischen Sinn der Relativitätspostulate. A. Einsteins neue und seine ursprüngliche Relativitätstheorie*. Annalen der Physik, 53: 575–614.
- [31] Michael Janssen, Einstein's first systematic exposition of general relativity,
<http://philsci-archive.pitt.edu/2123/1/annalen.pdf>
- [32] J. C. Maxwell, On Physical Lines of Forces. Philosophical Magazine and Journal of Science, London 1861