If time is made of numbers

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Abstract

The essay suggests a hypothesis about the nature of space and time. We assume that space is the set of all real numbers, while time is the set of positive real numbers “totally” (“linearly”) ordered in the increasing sense. We support these assumptions by considering some properties of continuous sets and by reinterpreting Dedekind’s axiom of continuity. We also suppose that under specific conditions, which can be found in systems travelling or operating at the speed of light, space and time lose continuity and become one and the same set, that of natural numbers. Lastly, we believe that our perspective may help to clarify some controversial problems of last century's physics, mainly concerning quantum theory.

1 Introduction

It is widely known that space and especially time are two aspects of the world we are part of that are very difficult to explain. They are so familiar to us, that it seems impossible to imagine a world without them and it seems, therefore, incredible that, after many centuries of philosophical and scientific investigation, we don’t know exactly what they are, though they are inseparable from our life.

From our point of view, the problem lies in the fact that space and time are not *empirical* objects. In this sense, we consider Kant’s remarks in the *Critique of pure reason* as a milestone in the investigation on that subject. According to Kant, space and time are neither empirical concepts, that is, concepts of something we have experience of; nor are they properties, qualities or features of objects.[1] Actually, space and time are necessary conditions of the possibility of dealing with objects, properties and qualities, because everything we experience is placed in space and time, whatever these may be. Kant thought that they are forms, *a priori* structures of our intuition or perception of objects, and *pure intuitions* in themselves; that is, that they do not belong to the objects but are instead inside us, not outside, following the principles of the “Copernican revolution” in philosophy. But we are not compelled to accept Kant’s theory in all his aspects. We rather know that Kant’s idealism or subjectivism of space and time and the definition of both as pure intuitions do not agree with Einstein’s theory of space and time as dimensions that are not immutable and don’t belong only to the subject.

[1] Kant, Immanuel (1787), Transcendental Aesthetic, §§ 2-4.

But what are space and time, if they are neither empirical nor subjective? In other words, is it possible that they are objective parts of the universe, provided with physical properties as claimed by Einstein, without being empirical?

We think that it is inevitable to rely on some hypotheses in any explanation of space and time we may suggest. If we accept Kant's thesis, even if only partially, we shall never find out their nature either by any sort of empirical investigation, or by means of purely logical tools. Nevertheless, many of the greatest thinkers in history, such as Pythagoras, Plato, Aristotle, Descartes, Spinoza, Kant and Einstein – only to mention a few of them – have suggested a relation between important aspects of mathematics and space, time, or both. By studying the properties of space, geometry, or better, geometries, obviously have more relevant implications on this concern. But arithmetic too has been taken into account (for example, by Aristotle and Kant), when explaining the meaning of time. The trouble is that there are probably more differences than similarities between the theories of philosophers and scientists about space and time, so that the main questions are still unresolved.

The hypothesis we want to put forward strengthens the link between space and time on one hand and numbers on the other, in order to get an identity: we suppose indeed that *space and time are numbers*, nothing else than sets of numbers.

There is no need to say that our hypothesis seems to be merely arbitrary, even more arbitrary, if possible, than any other we may find in the history of the research on these issues. What does it really mean to consider space and time as numbers? And on what ground do we identify things, such as extensionless numbers on one hand, and extended space and flowing time on the other, which, in all evidence, are very different?

Nevertheless, and against all evidence, numbers, taken not singularly, but as sets, can be thought of as having extension; while, under boundary conditions that go far beyond our usual experience, space may have not any extension and time may have not any capability to flow. Hence, we assume that, for all observers standing still or travelling at a speed inferior to that of light in vacuum, time is the set of positive real numbers, "totally" ("linearly") ordered in the increasing sense, while space is the "whole" set of real numbers. Both these powerful and compelling assumptions are bound to remain arbitrary unless we explain how they actually work.

2 What stays behind Zeno's paradoxes

Despite the fact that Zeno's paradoxes are often regarded as resolved by modern science, they continue to raise debate, as proved by the great amount of papers dedicated to them every year. This cannot depend only on the mere fact that they are extremely famous, but it must have even some theoretical meaning. Most interpreters, though agreeing on the certainty that the paradoxes have been or can be solved, do not agree on the modality of the solution. Many authors think that the infinitesimal calculus is sufficient in order to avoid or to solve at least two of the paradoxes – that of Achilles and the tortoise and that of the dichotomy – as it proves the possibility of considering a finite spatial interval as a sum of an infinite number of parts. Conversely, others, such as Bertrand Russell, Adolf Grünbaum and Gregory Vlastos, think that the simple infinitesimal calculus does not work for Zeno's paradoxes of movement because it involves, in this case, the need to perform an infinite number of steps or "tasks" in a finite period of time. They believe that only the properties of the *continuum* – mostly discovered by Cantor in the last decades of the 19th century – might reveal the nature of space and time and resolve Zeno's paradoxes. According to Cantor's set theory, there exists an *uncountable* infinity of points between any two points whatsoever of a linear continuum. This makes continuity a necessary condition of extension, since the sum of a *countable* infinity of points, within a line or a segment, is equivalent to a distance inferior to the arbitrary smallest segment; that is, a distance practically equal to zero. Therefore, it follows that anyone who is covering only a countable infinity of points cannot actually move.

We know that ancient Greeks, maybe around Zeno's age, discovered the existence of irrational numbers, but they did not understand most of their properties or the fact that, if we consider merely their amount, nearly *all* numbers are irrational. Consequently, ancient Greeks could admit that space is infinitely divisible, but not that it is continuous, according to the meaning and the properties given to the continuum in Cantor's set theory. We think, to the contrary, that a space which is infinitely divisible but not continuous is an extensionless space; and that a set of numbers, though infinite, cannot afford any extension as long as the set remains within the size of the countable. However, when the size of the continuum is reached, it becomes possible to build a geometrical representation of such numbers able to explain many properties of space. One of the most astonishing result of Cantor's theory is that all segments, all lines, all surfaces or spaces with an arbitrary number of dimension have the same amount of points, which cannot exceed the size of the continuum. This result – claimed by Cantor in a letter to Dedekind and underlined by the famous words "I see it, but I don't believe it" – makes the geometrical representation of real numbers extremely powerful (though representation does not mean identity) and may strong support the idea we want to put forward.

Yet the support given by mathematics in favour of the possible identity between real numbers and space seems to vanish if we try to establish a similar identity with time. Time is much less suitable to mathematical representation than space. This depends on the fact that spatial distances and relations can be easily geometrically represented, and geometry, as done by Descartes, can be formulated by using algebraic and analytical tools, that is in terms of relations between real numbers. There have also been several attempts to develop a numerical representation of time, for example, by viewing it as a linear continuum or as a discrete succession of moments or instants. But, on one hand, continuity seems to be incompatible with any idea of succession, because in a linear continuum there is no immediate successor of a given element. On the other hand, discreteness would leave open the question of the nature and the size of the "elementary" particles of time. Thus, the difference between past, present and future, and the running of time in only one direction are left, in any case, unexplained. For instance, Einstein was able to give a representation of time based on Lorenz's transformations and Minkowski's model but, in doing so, he had to consider time as a fourth dimension together with those of space, and had to dismiss any definite physical distinction between past, present and future.

The difficulty of explaining the properties of time in mathematical terms seems to cut off all hypothesis of considering time as a set of real numbers. Real numbers, as it is shown by our interpretation of Zeno's paradoxes, may explain extension and other properties of space. But if time is something that runs or flows, how is it possible to suppose that also real numbers "run" or "flow"? Even if it might seem impossible, we think that this is really the case.

3 Dedekind's axiom of continuity

If time is a set of real numbers, the enigma of its running must probably lie in the baffling nature of the continuum. It has not completely been explained, and maybe it will never be, even though several of its aspects have become clearer during the last centuries. One of the greatest authors of this clarifications – apart from Cantor – is the German mathematician Richard Dedekind. In his classical 1872 paper *Continuity and irrational numbers* stands the following statement, one of the most quoted among those written on the issue:

"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division into two classes, this severing of the straight line into two portions."[2]

Following Dedekind, who says that "every one will at once grant the truth of this statement", most handbooks don't raise questions about the statement. But, on a closer examination, it seems not so evident, not so much because of the truth – which cannot obviously be proved if it is an axiom – but because of the meaning, which is doomed to the ambiguousness, as it has emerged from some contemporary debate.[3]

The problem lies in the fact that Dedekind affirms the possibility of dividing the straight line in two parts, or sets of points, with only one point "producing" that division. But where does the dividing point of the straight line lie? According to Dedekind, it must lie either in the first or in the second class of points. There is no other possibility, because the line is the geometrical representation of the set of real numbers, which is, technically, a "complete metric space", in the sense that it has no "hole" (no missing point in it), and contains only real numbers. But how can we say that the dividing point is unique? If it is the last point of the first class, then there is another point, the first of the second class, which can be taken as the dividing point as well. Conversely, if

[2] Dedekind, Richard (1872), p.11.
[3] A very interesting discussion on Dedekind axiom of continuity can be found at the following web address: http://www.physicsforums.com/showthread.php?t=83018.

we assume that the dividing point is the first one of the second class, then the last point of the first class can also fulfill the same task.

Shall we conclude that Dedekind's axiom is logically incoherent and has to be dismissed? Not at all. We think that it may well represent not only the continuity of real numbers, but also some essential properties of them, as long as we consider the point producing the division in two classes not really as a "severing" point, but as a *binding* or *uniting* one. In other words, we consider it as a single *intersection-point*, that is the only point which is common to both classes. Therefore Dedekind's axiom of continuity can be rephrased in the following terms:

"If all points of the straight line fall into two classes so that every point of the first class lies on the left of every point of the second class, then one and only one point exists, which is *common* to both classes, thus producing the *union* of them in a linear continuum."

The present formulation of the axiom expresses well the meaning of continuity ("continuum" etymologically derives from the Latin "contineo", which means "to hold together"), maybe better than Dedekind's original form, and it is so simple that it seems impossible that he failed to conceive it. In fact we believe that Dedekind surely conceived it – or an equivalent of it – but he did not put it forward. Why didn't he do that? Of course we shall never be sure of getting the right answer, but we can reasonably suppose that he was forced to avoid the easiest expression of his "principle", because otherwise he would have to refuse another principle, which is known as the *axiom of choice*.

The axiom of choice is surely the most discussed axiom of set theory. It says that, given any class of mutually disjoint non-empty sets, there is at least one set, which has one and exactly one element in common with each of the non-empty sets; in other words, there is a choice function that associates every set with a single element of it. Until the late 19th century, the axiom of choice was mostly used implicitly, though Ernst Zermelo formally stated it only in 1904. Actually, Cantor assumed as evident the existence of a choice function from an arbitrary number of sets, and so, substantially, did Dedekind, as reported by Gregory E. Moore in the most comprehensive and detailed study written on the issue so far[4].

In any case the axiom of choice is incompatible with our reading of Dedekind's axiom of continuity. Given the two sets of points of which the straight line is made, there is no choice function that assigns every point to the left or to the right set, because the sets are not disjoint, but united in just one point. But it is very unlikely that Dedekind – along with Cantor – could dismiss the principle of choice, first of all because he couldn't know how questionable that principle really is. The great debate about the axiom of choice rose after the first 1904 Zermelo's formulation, and lasted until 1963, when Paul Cohen was able to prove that the axiom of choice cannot be derived from the other axioms of classical Zermelo-Fränkel set theory. Several years before, in 1938, Kurt Gödel had proved the opposite (and complementary) result, that the *negation* of the axiom of choice cannot be deduced from the same theory. The combined outcome of Gödel's and Cohen's proofs is

[4] Moore, Gregory H. (1982), pp. 13-16.

the complete independence of the axiom of choice from the other axioms of set theory. To accept or not this axiom is fundamentally a question of *choice*.

Thanks to the twentieth century achievements, we can do what for Dedekind and Cantor was very difficult to uphold: the removal of the function of choice from the numerical explanation of time. In our view of Dedekind's statement about continuity, all numbers standing on the left side of the intersection number are the past, while those standing on the right side are the future. The intersection number, which belongs neither completely to the past, nor to the future, is the present. It is the passage-point from the past to the future, allowing us to affirm that time passes, or flows, or runs.

The present view of time and of its flowing may appear to be simple, even too simple, especially if one is aware of the difficulties that a mathematical representation of time has met along history. However, we don't think that our explanation is in any way exhaustive. It is perhaps a good starting point, but many crucial questions are left to be answered. One fundamental question is the following: if the present-point is unique – while the past and the future ones are uncountably infinite – how can we identify it? The answer is obligatory: in no way. In order to find that point we need a choice function; that is, the axiom of choice. But this axiom is not valid for the totally ordered set of positive real numbers, which form time. Actually, every point of the set *is* the present, as it is the past and it is the future. However, it is the present – as it is the past, or as it is the future – not in an absolute way, but only for some observers or frames of reference. More precisely: for every point (or number x), there is a frame of reference in the universe for which the point is the present, while all the points on the left side of it – if there are any – are the past and all those on the right side of it are the future. The infinite multiplicity of all different presents lies in the different speed of the observers – or frames of reference – according to the special Relativity theory.

4 Time without future

According to his "Autobiographical Notes", Einstein's first approach to Relativity came about when, at 16, he tried to imagine how the world would appear to him while "pursuing" a beam of light at the same speed c. Einstein thought he would see the beam as "an electromagnetic field at rest though spatially oscillating". Because "there seems to be no such thing", he realised that a new pattern of thought was required, beginning a route that would lead to the theory of Relativity in about ten years.[5]

Even though the meaning of this famous mental experiment is mostly held as controversial, one point seems to be assured: Einstein couldn't believe, even since he was adolescent, that a beam of light might be seen at rest from any frame of reference whatsoever. Some years later, the theory of Relativity, stating the constancy of the speed of light in vacuum, would rule out the possibility of

[5] Einstein, Albert (1949), pp.52-53.

seeing the beam at rest, with the consequence that space and time would have to lose their steadiness and absoluteness.

But how would the world appear to an "observer" travelling at the speed of light? We know, following Lorentz's equations used by Einstein, that, at the speed of light, the spatial length is reduced to null, while time stops running. That does not resolve our question of what they are at that speed. The question rather becomes, if possible, even more difficult. Scientists have often seen space and extension as synonyms, as for example did Descartes. Otherwise, speaking of a no running time seems to be – to common people and to many theorists too – simply meaningless. The question may then assume the following form: is it possible to speak of space and time at the limit-condition of the speed of light in vacuum? And if yes, in which terms?

We have suggested that, for every observer travelling at a speed inferior to that of light in vacuum, space and time are sets of real numbers, and it is the continuity of these sets that explains both the extension of space and the running of time. But, what happens if both spatial extension and flow of time are removed as it is for a frame of reference travelling at the speed of light? Our answer is that also the continuity has to be removed, thus implying that space and time are no longer sets of real numbers, but mere sets of whole numbers. Moreover, we suppose that, under these circumstances, it is impossible to distinguish space from time and they are just the same thing, that is, the set of natural numbers (positive whole numbers).

About a century after the special Relativity theory, we may not only affirm that a world seen from an "observer" travelling at the speed of light would be a discrete world, but we also have the possibility of displaying a *model* of that world, a model which, unlike Einstein, we are acquainted with: that is the world of *information* as represented and transmitted by digital systems such as our ordinary personal computers. For these systems, as for any other reference-frame, the speed of light is c. But that speed is the only thing – along with logical and numerical structures – that would appear the same, as within and without the digital system. The rest of the world, compared with ours, would appear "as frozen" as the beam of light of Einstein's mental experiment. However, that happens not because the system reaches the beam of light, as in the "paradoxical" Einstein's example, but because space and time deeply change their structure when "observed" at the speed of light. Deprived of continuity, of extension and of the running of time, the world we are dealing with becomes essentially a world of numbers, more properly of *whole* numbers.

One may easily think that this world of whole numbers, which is the only possible world that can be "observed" travelling at the speed of light, is a mean or a merely virtual world. Of course it is a world where several properties, such as mass, size, and change are missing. But it is a world that is able to represent or describe all that happens in our world, it is a "model" of our world.

A model is a structure that, in a different and usually smaller scale, maintains many properties of what it represents. From a logical point of view, a model maintains the "truth-values", in the sense that all that is true of the object is also true of its model. But we know that universal Turing machines, such as our personal computers, can theoretically develop a model of every object or fact we come across in daylife. We may then affirm that, in a space made of pure whole numbers, we can represent the world that exists in "our" space of real numbers, provided that *information* about this world is available.

This is about space. But how about time? We have supposed that, at the speed of light, space and time are the same thing. But this involves the risk of wiping out the peculiarity of time and reducing it to space. It is therefore important to find a way of representing time in a logical or numerical dimension. Our suggestion is that time may be the set of all "states" or configurations of a digital computer, where by "state" we mean what Turing called "inner state" of his universal computing machine. The totality of the inner states may be held to be equivalent to the past, while the current state of the system may be taken as the present.

In other words, we can say that, from the point of view of the system, only the current and all the foregoing states exist, stored in the computer's memory, while for an observer standing outside the system there is also an unlimited "tape" – like that of the original 1936 Turing machine – where the machine may write the states that are passing through. But this tape is blank, because no one can *a priori* know what the machine will eventually write on an arbitrary "square" of the tape. That is all we can say. No Turing machine can "see" the unlimited tape in front of it. For it, time is without future.

5 Breaking the waves

A proper way to value a hypothesis is to consider its explicative power. There are hypotheses that are formulated in order to get out of an impasse (as it is for instance the Cartesian pineal gland). Such hypotheses, also called "hypotheses *ad hoc*", have a low explicative power, because they can account only for the problem for which they have been conceived. There are also hypotheses that are powerful, in the sense that they are able to explain a wide range of phenomena. One of these is the theory of the corpuscular nature of light. Nevertheless it is not powerful enough, because even the alternative wave theory is able to explain a large amount of phenomena. That doesn't mean that both theories are wrong – they have been confirmed uncountable times – but that probably they are both incomplete, in the sense that a more comprehensive view is needed.

What is our hypothesis for? It cannot be experimentally confirmed, since space and time, unlike light, are not empirical object, but we can look for indirect witnesses, by testing its explicative power in relation to some facts or phenomena we already know. We may begin by considering the problem of the nature of light just mentioned above. Supposing that space and time are sets of numbers, the wave-particle dualism may be expressed in the form of the double nature of space-time, considered either as discrete or as continuous set. Let us imagine how this correspondence may work.

Suppose light is travelling in a "space" of natural numbers. Then it can be described as a collection of punctiform particles, because in such a space no extension is allowed. But for us, who are moving very slowly compared to light, space is extended, and a beam of light travelling in it takes a wavelike shape. But why does light travel just like a wave and not like a beam of straightforwardly moving particles? Because the straightforward movement is impossible without the actual travelling of the particles, while a wave can be described as a vibration or oscillation of space that does not require any actual displacement of the involved particles (it is the wave that travels, not the particles forming the medium in which it travels). The wave-particle dualism might be a sort of a "way out" from the puzzle of representing as travelling in an extended space and in a time lapse something that in itself experiences no extension and no duration.

The hypothesis that space and time are sets of numbers might also offer an explanation of a phenomenon often seen as the most puzzling of the modern physics, that is the phenomenon of the "entanglement". However strange it may appear, such a phenomenon is not difficult to understand, in our context of a space-time made of numbers. Within this context, space does not coincide with extension and distances or lengths are always relative to the status of the observer, according to Einstein's theory. That implies that spatial proximity is not essential for interactions between particles, and that *non-local* effects are fully possible, especially when we know that the involved particles travel at a speed equal or close to that of light. Indeed, at this speed, space has no length (or a slight length), and time does not flow (or flows at a very slow rate). Consequently, it is not relevant to speak of local or non-local conditions, because extension is no more an essential property of space, which holds only its proper numerical nature.

Although we are not able within the present essay's bounds to develop these points, we believe that the idea of space and time as sets of numbers is in agreement not only with primary trends of physics, but also with important aspects of modern and contemporary mathematics. Only to mention one example, the great development of 20th century's topology may support our idea that space can be, in some cases, unextended. A fundamental property of the "simply connected manifolds" – to which Poincaré's conjecture, recently resolved by Gregory Perelman, applies – is that every path or loop traced on their surface can be reduced to a point. Moreover, if the manifold is "contractible", the entire object can be reduced to a point, according to the rules of homotopy.

In agreement with the topological approach, it is possible to define speed as the rate of the decrease of distance and of the expansion of time. This means that, while speed increases, the geodesic between two points shrinks proportionally, following the relativistic equation:

$$s^1 = s\sqrt{1 - v^2/c^2}$$

On the other side, time lengthens (in the sense that it shortens its flowing) according to the equation:

$$t^1 = t\,/\sqrt{1 - v^2/c^2}$$

If speed reaches the highest limit – which is the speed of light, in the relativistic theory – the distance shrinks to null and time expands itself to infinity (that is, it stops running).

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