How Mathematics dictates All of Physics, without the hot air!

Franklin Potter

E-mail: frank11hb@yahoo.com

1. Introduction

The goal of this essay is to establish that the Universe is fundamentally mathematical. I do not mean simply a physical world described by mathematics; I mean the stronger statement that we humans too are mathematics as well as everything else in the Universe. Expressed in a different way, if the fundamental building blocks called leptons and quarks are mathematics, then so are atoms and molecules as well as we humans who are composed of molecules and atoms. Can we forge the convincing argument to support or deny this proposition that the fundamental particles are mathematics? That is, can we show that specific fundamental mathematics dictates all the specific fundamental particles and rules of physics exactly?

Yes, we can do so today, in direct contrast to those naturalists who believe there are no eternal laws or rules in Nature that are absolute. The view I propose answers Wigner's query about the "unreasonable effectiveness of mathematics in physics" by showing that the fundamental particles and rules of physics are fundamental truths of mathematics exactly. Fundamental mathematics therefore predicts the fundamental physics with the immediate consequence that there is one set of physics rules only. There cannot be multiverses with different physics rules.

I am a physicist and not a mathematician. But I know that mathematics is a discipline that encompasses a wide diversity of topics. In opposition to those who write philosophical essays about how mathematics is related to the fundamental beauty in Nature but avoid the specific details, I forego the handwaving in favor of providing the key concepts up front so that we can better understand our mathematical world.

2. The leptons, quarks, and their mixing matrices

In physics one considers a (3+1)-D spacetime for the Universe and a 4-D internal symmetry space at each spacetime location for the leptons, quarks, and their interaction bosons. Traditionally, the internal symmetry space has more dimensions, particularly for the color interactions of gluons and quarks, but I will show that 4 real dimensions suffice.

I begin by identifying the direct connection between 3-D geometry and the leptons. For the quarks I will need to go up one real space dimension to 4-D. Now that you are forewarned that geometry is required, here are some of the details.

Three reasonably familiar 3-D objects are so very important in mathematics that they permeate nearly all areas of mathematics. These are the geometrical properties of the regular tetrahedron, the regular octahedron, and the regular icosahedron, i.e., Platonic solids. They have discrete binary rotational symmetries, the word binary meaning that a 4π rotation in the 3-D space instead of a 2π rotation is required to return a mathematical function back to its initial value if that function exhibits the symmetry properties of the object. Their symmetries are expressed in the operations of the discrete (i.e., finite) binary rotational groups [1–3] labeled 2T, 2O, and 2I, or equivalently [3,3,2], [4,3,2], and [5,3,2].

I propose [4,5] that the three lepton families of the Standard Model of particle physics, the electron family, the muon family, and the tau family, each family having two particle states, represent the discrete symmetries of these three mathematical entities. It is important to note that their discrete groups are subgroups of the continuous local gauge group (i.e., the Lie group SU(2) x U(1)) of the Standard Model that describes the electroweak interaction, so I am remaining within the realm of this very successful theory and not introducing any expansion. I am simply considering discrete rotations instead of the traditional continuous rotations.

With regard to the importance of these three groups in mathematics, a quote from a paper by the famous mathematician Bertrand Kostant (1984) reveals their true value [6]:

"The ancient Greeks, especially the school of Plato, had great reverence for the regular polygons in the plane and regular solids in 3-space. The latter – the tetrahedron, cube, octahedron, dodecahedron, and the icosahedron – are often referred to as the Platonic solids. The Greeks believed that these regular figures were fundamental in the structure of the universe. If symmetry or its mathematical companion – group theory – is fundamental in the structure of the world, then one of the points of our lecture is the statement that the Greeks were absolutely right. That is, what we will be saying in a very profound way, the finite groups of symmetries in 3-space "see" the simple Lie groups (and hence literally Lie theory) in all dimensions."

Each lepton family has a charged-lepton state and a zero-charge neutrino state. We know also that each of the neutrino flavor states ν_e, ν_μ, ν_τ, is actually a linear superposition of three neutrino mass states ν_1, ν_2, and ν_3. The 3x3 PMNS neutrino mixing matrix summarizes all the relationships. One assumes that the charged-lepton flavor states, the electron, the muon, and the tau, do not mix, for there is no evidence that their mass states are different from their flavor states.

One of the greatest challenges in particle physics is to determine the first principles origin of the quark and lepton mixing matrices CKM and PMNS that relate the familiar flavor states to the mass states. In 2013 I derived [7] the neutrino mixing as nothing more than a purely mathematical mismatch between the group generators of my proposed three discrete binary rotation groups [3,3,2], [4,3,2], and [5,3,2] for the three lepton families and the three generators for the electroweak gauge group, i.e., the three Pauli matrices of SU(2). The derived mixing angles all agree with the accepted values determined by neutrino experiments [see the Appendix for the derivation details]. In fact, my derivation predicted the most recent 2015 value [8,9] of the reactor neutrino mixing angle $\theta_{13} = 8.56$ degrees. I also determined θ_{23} is 42.85 degrees, less than the maximum absolute value at 45 degrees, thereby predicting the normal neutrino mass ordering $\nu_\tau > \nu_\mu > \nu_e$.

If there are any doubts that the three lepton families represent these three discrete symmetry groups, my derivation of the neutrino mixing angles and PMNS matrix should eliminate them. The neutrino mixing and its first principles origin reveal that the Standard Model electroweak gauge group, $SU(2)_L$ x $U(1)_Y$, which is a continuous group and assumes that its weak interaction operations occur in a continuous internal symmetry space, is not quite correct but is an excellent approximation.

The above derivation would be a hollow victory for mathematics dictating the physics if the quark mixing by the same method failed. In 2014 I derived [10] the quark mixing and its mixing matrix CKM from exactly the same first principles approach using the mismatch between generators, but the details are a bit more complicated because one must have four discrete binary rotation groups for four quark families in a 4-D internal symmetry space. The derivation produces a 4x4 quark mixing matrix CKM4 from which one extracts the 3x3 CKM submatrix.

The predicted numbers in the extracted CKM submatrix agree except for an interesting conflict arising at the up quark to strange quark mixing element V_{us} where I predicted the value

0.2203 instead of the traditional value 0.2246, which had been the accepted value for more than 40 years. However, recent experiments [11] with tau decays now predict this V_{us} value to be 0.2204, confirmation that the quark families represent the four discrete symmetry groups [3,3,3], [4,3,3], [3,4,3], and [5,3,3]. These groups in 4-D real space for the quark families are related mathematically to the lepton discrete groups in 3-D real space, as one might suspect from the notation, and have the discrete rotational symmetries of the regular 4-D polychora (also called 4-D polytopes).

There is now a potential problem. Three lepton families and four quark families are a mismatch. Particular Feynman diagrams called triangle diagrams produce unacceptable infinite values, but matching the number of lepton families to the number of quark families is an easy way to cancel these infinities. However, I showed in a recent 2015 paper [12] that the above successful derivation of the PMNS and CKM matrices by generator accommodation provides the means to avoid this problem with the triangle diagrams: the three lepton families act as one equivalent lepton family to cancel exactly the contribution from the one equivalent quark family from the four quark families acting as one. So the number mismatch is not a problem at all!

So I have quark states defined in 4-D real space R^4 and lepton states defined in its 3-D real subspace R^3, both spaces within the 2-D unitary space C^2 ($= R^4$) with its two complex axes. The physical consequences of these spatial relationships are many but I will mention just a few.

Nature has antiparticles. Why do they exist? Fundamental mathematics called Clifford algebra dictates the answer: in real spaces of dimension 4n, where n is an integer, there exist two equivalent real spaces, the normal space and the conjugate space. Therefore the leptons and quarks exist in the 4-D normal space and the antileptons and antiquarks exist in the 4-D conjugate space by convention, having the antileptons in its 3-D conjugate subspace.

Quarks have mass and so do the charged leptons. Neutrinos were once thought to be massless, but the PMNS matrix derived above reveals otherwise. The original argument can be stated: any particle state with mass must have 3 degrees of freedom (d.o.f.). The massless photon, e.g., has but 2 d.o.f., RHC and LHC polarized, or its plus and minus helicity states (i.e., spin along momentum vector or opposite). My two 4-D quark states per family have 6 d.o.f. defined on the surface of the 4-D sphere, allowing 3 d.o.f. for the up quark state and the remaining 3 d.o.f. for the down quark state. Therefore, both quark states per family have non-zero mass values.

My two 3-D lepton states per family are defined on the surface of the familiar 3-D sphere by 4 numbers, hence only 4 total d.o.f. are available. Therefore, Nature had a choice! There could be two lepton states, each with 2 d.o.f. and massless, or there could be one massive lepton state (the electron) per family with 3 d.o.f. plus one massless state (the left-handed neutrino) with 1 d.o.f. Recall that the conjugate space exists for the anti-electron and the right-handed anti-neutrino states. The 2 d.o.f. states are known as sterile neutrino states. Today, the sterile neutrinos are still viable.

The parity violation of the weak interaction is maximum, i.e., its preference for only left-handed helicity states for the leptons and quarks, is a direct consequence of quarks and leptons existing in C^2. Mathematically, one should consider using unit quaternions to represent both the weak interaction bosons, W^+, W^-, and Z^0, and the lepton and quark weak isospin states, i.e., the flavor states. The Irish mathematician W. R. Hamilton defined the quaternion $q = a + bi + cj + dk$, where a,b,c,d are real numbers and $i^2 = j^2 = k^2 = ijk = -1$. But quaternions acting on quaternions automatically results [1] in maximum parity violation by producing left-handed doublets and right-handed singlets (and vice-versa in the conjugate space). So maximum parity violation is a mathematical property expressed by the weak interaction.

3. Monster Group, its j-invariant, and particle masses

Fundamental mathematics also dictates the lepton and quark mass values via an interesting connection to the famous j-invariant function of elliptic modular functions and the finite simple groups. As you might recall, a group that is not simple can be broken into two smaller groups, a normal subgroup and the quotient group. The sporadic finite simple groups have had a special place in the minds of mathematicians ever since the discovery of the largest one, the Fischer-Griess Monster Group in 1982, with its first non-trivial irreducible representation needing a 196,883-dimensional space.

The whole collection of finite simple groups are considered to be the basic building blocks of all finite groups, in a way similar to the way prime numbers are the basic building blocks of the natural numbers. Prime numbers never end but the Monster is the last and largest finite simple group, so many mathematicians have speculated that it should occupy a fundamental place in the behavior of Nature. I can show [13] how the Monster Group dictates all of physics, but in this short essay I can introduce only part of the whole picture.

The j-invariant of elliptic modular functions is the direct connection from the Monster Group to the lepton and quark families via the above groups [3,3,2], [4,3,2], [5,3,2], and of the [3,3,3], [4,3,3], [3,4,3], and [5,3,3]. If the Monster is the basis for the ultimate quantum field theory, its partition function is the j-invariant [14].

This j-invariant function is related to the linear fractional transformation, a name most often shortened to being the linear transformation, and also known as the Möbius transformation, which one finds everywhere in physics. Space translations, time translations, rotations, Lorentz transformations, and the linear superposition of fields are all examples of Möbius transformations. All the conservation laws follow from the symmetries associated with linear transformations by applying Noether's theorem: space translation and linear momentum; time translation and energy; rotation and angular momentum; Lorentz transformation and relativistic 4-momentum; etc.

But I discovered [4] that there exists an unexpected connection between the j-invariant and the mass values of the leptons and of the quarks. And thereby hangs the important clue that further supports the hegemony of mathematics over fundamental physics. For each of the lepton binary rotational groups, one can write down three functions in two complex variables, with each function invariant under the operations of the group. The three functions for these 3-D objects give the position of their vertices, face centers, and edge centers. Only two of the three functions are independent, and the ratio of the two functions is proportional to the j-invariant in a mathematical syzygy [15] with proportionality constant N_i. This mathematical result is in the famous 1884 book by Felix Klein called "Lectures on the Icosahedron and Solutions of Equations of the Fifth Degree" which has been republished by Dover Publishing.

For the lepton groups [3,3,2], [4,3,2], [5,3,2], the ratios of the three proportionality constants N_i are 1:108:1728. I argue that this j-invariant connection dictates the lepton mass ratios to be 1:108:1728, values very close to the actual charged-lepton mass values 0.511:105.7:1776.8. Note that this numerical similarity is fortuitous because the mass units of MeV/c^2 just happen to make the electron mass close to 1.

One can refuse to accept this origin of the mass values by noting the numerical discrepancies. However, the position is false. The two basis states of each of the assigned binary rotational groups are degenerate, so one must form their linear superposition to make the two physical flavor states per family. These non-degenerate flavor states can have mass values in agreement with the empirical ones. Non-zero neutrino mass values are therefore possible because one has a quantum mechanical amplitude A_i to flip from one base state to the other, but this A_i can be sensitive also to the local environment.

The same j-invariant approach to getting mass values for the quark families requires the four 4-D discrete symmetry groups listed above, each of which has two degenerate basis states defined in 4-D real space. The ratios of their proportionality constants are 1/4:1:108:1728, obtained by projecting their symmetries onto the unitary plane. Again, linear superpositions form the actual quark flavor states in each family with the correct mass values. Figure 1 shows a geometrical

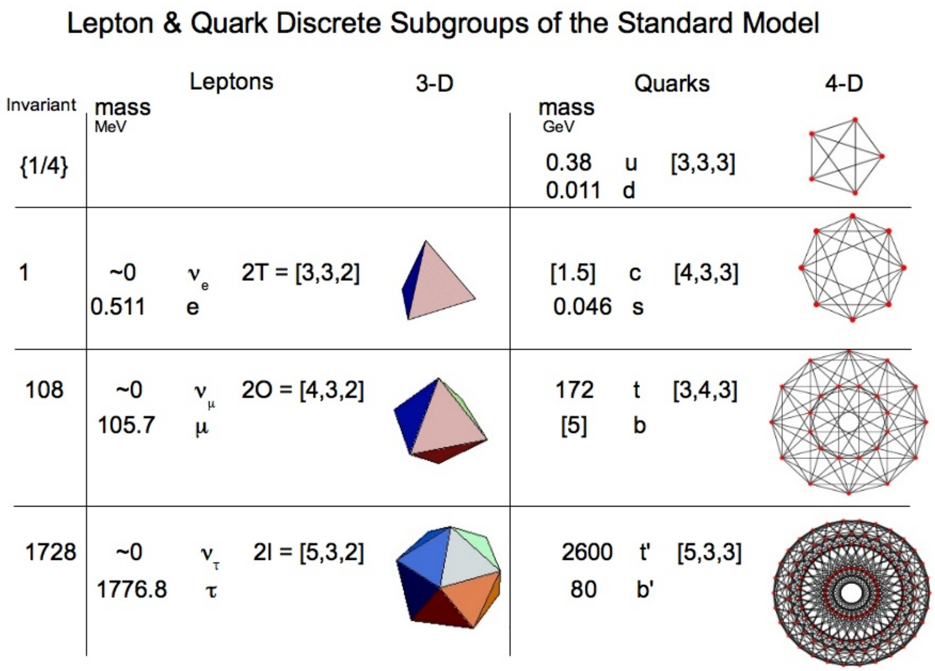

Figure 1. Discrete symmetries for the lepton and quark families [4]

summary.

The still-to-be-discovered 4th quark family would have mass values of about 65-80 GeV for b' and about 2600 GeV for t', values derived using the mass ratios with the charm quark and bottom quark masses as references. With the predicted mass value for the b' quark at 65-80 GeV, its production at Fermilab and the LHC is expected, but the b' has not shown itself yet. I do not know why not. The prominent decays are flavor changing neutral current (FCNC) decays to a b quark plus either a gamma or gluon, which have relatively low production rates. Also, its estimated lifetime is long enough to form a b' quarkonium with its antiparticle. Both positive and negative parity bound states are possible, so if the positive parity state is preferred, then the 125 GeV resonance could be evidence for the b' quarkonium. And, most of the quarkonium bound states could have long lifetimes, so angle, momentum, and energy cuts are critical in selecting the data at the colliders.

In order to establish the complete connection from mathematics to fundamental physics, I must discuss the color interaction of its gluons operating on the quark states. Recall once again that I have the lepton and quark states defined in 3-D and 4-D real internal symmetry spaces, both real spaces being subspaces of the unitary plane C^2. Traditionally, the color interaction operates in a space C^3 of three complex dimensions for the local gauge group $SU(3)_C$.

Surprisingly, rotations in a 4-D real space can accomplish the same physics. Rotations in 4-D occur simultaneously in two orthogonal planes, of which there are exactly three pairs which I claim correspond to the three color charge values traditionally labelled red, green, and blue. Since these color charge assignments are arbitrary, color is an exact 4-D rotational symmetry. Again, fundamental mathematics dictates an important physical property, that color is an exact symmetry. Furthermore, one can show that 8 special 4x4 real matrices are equivalent to the traditional 8 gluon SU(3) matrices for the color interaction, which is now a 4-D rotation from one color charge to another, perhaps in a discrete space.

One can also verify that the traditional quark combinations to make hadrons are the only ones that produce no net 4-D rotations. That is, the particular mathematical combinations of three quarks, three antiquarks, or quark-antiquark are the physical hadron states of QCD, the quantum theory of the color interaction. Even quark confinement is mathematically dictated because the 4-D quark states cannot exist in our 3-D spatial world. However, intersection theory will combine these 4-D entities into 3-D hadrons in the correct QCD combinations. Finally, we now know that leptons do not experience the color force because they are defined in 3-D, not in 4-D, so they cannot have a color charge.

At a conference physicist Cecelia Jarlskog asked me why the weak interaction exists at all! She said that QCD and its color interaction via gluons produces a self-contained world and no additional particles or interactions are required. My answer was that Kuratowski's theorem in graph theory states that only the K_5 and $K_{3,3}$ graphs are stable, i.e., do not reduce to planar graphs. Of the quark families, the up/down family is exactly the K_5 graph corresponding to [3,3,3], so all other 4-D quark families do not maintain their integrity and will decay. But in QCD the quarks experience the color interaction only, an interaction that can change the quark color but not the quark flavor. Therefore, one requires the weak interaction to change the flavors. The need for the weak interaction is simply the dictate of graph theory.

The above examples outlining how fundamental mathematics dictates fundamental physics are just the beginning of a whole related set of examples. Many more direct consequences could be listed to indicate how all other physical properties of the Standard Model are actually just the dictate of fundamental mathematics.

4. Weinberg angle, icosians, and Weyl E8 x Weyl E8

I would be remiss if I didn't introduce you to some of the other connections of mathematics to the Standard Model. In particular, one should see how the 4-D internal symmetry space of the leptons and quarks gets connected to (3+1)-D spacetime via discrete groups. Mathematically, the two spaces can be considered to be discrete at the Planck scale, so one would consider a 2-D hexagonal lattice of nodes in C^2. The nodes would have no measurable physical properties but their combinations into entities with the correct discrete symmetries would be the fundamental particles such as leptons and quarks.

In the traditional interpretation of the Standard Model the vector bosons and the gluons perform continuous mathematical rotations from one fermion state to another in their appropriate mathematical spaces. However, I have shown that the existence of the PMNS and CKM matrices reveals that the continuous rotations are not correct because the leptons and quarks have specific discrete symmetries, different for each family. So let's now consider the electroweak interaction and ignore the color interaction via gluons. Instead of the continuous $SU(2)_L$ x $U(1)_Y$, we need the correct discrete subgroup to perform the discrete rotations in its two dimensional complex space C^2. By examining the seven binary rotation groups for the leptons and quarks, one determines that the product group 2I x 2I' is the discrete electroweak group required. Why? Because 2I cannot handle the rotations of the [4,3,2] completely [16]. The first 2I is the binary icosahedral group, i.e., [5,3,2], with three generators, and the second is the same 2I group but with one of its three generators (i.e., one Pauli matrix) conjugated.

We are now in a very powerful position because we can derive two very important results that cannot be done with the traditional continuous group. First, we can derive [12] the weak mixing angle that relates the Z^0 to the photon γ. Also called the Weinberg angle, I determine its value to be 30 degrees exactly. One uses the 2I x 2I' group generators to bridge the mismatch between continuous rotations and discrete rotations. This first principles derivation of the Weinberg angle is in the Appendix.

Second, we can telescope mathematically [5,13] with icosians, special quaternions which are also special octonions, from the real space R^4 ($= C^2$) in which our fundamental particles exist upward to R^8. So there is more mathematics to examine based upon the 120 group elements of

2I. Beginning with the 2I x 2I' group, which is the discrete equivalent to the electroweak gauge group, one can telescope up from 4-D to 8-D with the 120 icosians of the first 2I to define the D8 lattice. Then another 120 icosians of 2I' make another D8 lattice that fills in the gaps of the first D8 lattice. Together, D8 + D8 is the famous E8 lattice in R^8.

Then we connect to spacetime. Normally, we assume that our (3+1)-D spacetime is a continuous space, but if we assume that the space is discrete at the Planck scale, then one will get another E8 lattice from the Lorentz transformations. One must use the Penrose 'heavenly sphere' for the Lorentz transformations, tessellate its 2-D surface, and then realize that the required discrete group for the transformations is again 2I x 2I'. That is how another E8 lattice arises.

Each E8 lattice obeys the operations of the discrete symmetry group Weyl E8, not the E8 Lie group of continuous symmetry for superstring theory. Together, combining the Weyl E8 from the Standard Model with the Weyl E8 from the Lorentz transformations produces a 'discrete' SO(9,1) = Weyl E8 x Weyl E8 group in 10-D spacetime. I must admit that I was surprised that two 8-D spaces combine to make a 10-D Lorentzian space!

Going in the opposite direction, back down to discrete 4-D, we have that the 'discrete SO(9,1)' = Weyl E8 x Weyl E8 uniquely telescopes downward to the Standard Model discrete operations and the Lorentz transformations in discrete spacetime, both existing in discrete 4-D space. One does not have the 10^{500} ways as in M-theory! This discrete connection is UNIQUE!

There is much more mathematics [13] to enjoy in our mathematical Universe, but I will just mention a few of the connections here. The particles of the Standard Model can represent E8 lattices, and when an interaction occurs in a triality Feynman diagram the three E8 lattices can momentarily form a 24-D Leech lattice. The Leech lattice is connected to the Golay-24 binary code of information theory and to the Monster Group. These connections would take us into interesting mathematical concepts that further strengthen my argument for mathematics dictating all of physics, specifically by limiting the number of fundamental particles in the Universe to 72. The details here are too encumbering for this essay.

5. Possible origin of Quantum Mechanics

We live in a Universe that obeys the rules of quantum mechanics. Earlier I stated that the lepton and quark flavor states are linear superpositions of the base states of the discrete symmetry groups. That is, there is an oscillation, i.e., a flipping, between the two base states in each family dependent upon A_i so that each particle has its own characteristic frequency $\omega_i = m_i c^2/\hbar$. The collections of quarks we call hadrons would have their characteristic frequencies, too.

With two assumptions, I can derive the three rules of quantum mechanics. The assumptions are: (1) each fundamental particle acts as an antenna sending out signals and receiving signals, i.e., the vibrations of the lattice of discrete spacetime at the frequency ω, with the particle's new location being the position of the maximum signal summation; and (2) the vacuum is a transponder scattering the signals identically for any ω with no phase shift and no signal loss. The details will be put in a future paper.

This feedback signal mechanism for particle behavior in spacetime can be shown to be equivalent to Feynman's path integral approach which describes all of quantum and classical physics. One can verify this equivalence by examining single slit diffraction and double slit interference to derive the rules [16]: (1) the probability P of an event is the square of the absolute value of the probability amplitude ϕ; (2) when the event can occur in several alternative ways, the total probability amplitude ϕ is the sum of the amplitudes over each way, such as $\phi_1 + \phi_2$; (3) if the experiment is capable of determining which alternative is taken, even in principle, the total probability P is the sum of the individual probabilities.

In this approach, there is an added bonus, for the gravitational interaction appears to be an emergent phenomenon from the action of the feedback signals. A preliminary analysis indicates that gravitation is a very weak interaction that originates from small phase shifts. One would need to examine the particulars here in order to verify or refute this possibility.

6. Final comments

I have tried to provide the details as well as a *feeling* for how the mathematics dictates the fundamental physics without using all the precise mathematical expressions. All the above ideas can be found in my published papers online where the mathematical details are included. I hope that my outline of the mathematics and the resultant physics has been good enough to bring you to the realization that our understanding of Nature is about to take a tremendous leap forward. Put in a different way, we are not doomed to multiverses, mathematical landscapes, and the anthropic principle! There exist eternal laws or rules in Nature that are absolute.

Have I convinced you that the fundamental particles as well as we humans are mathematics instead of simply being described by mathematics? If my predicted 4th quark family is discovered, I would like to hear your argument denying so! Therefore, you can understand my anxiety over the restart of the LHC this year because I have high hopes that two more quarks will make their appearance in the detectors.

[1] Du Val P 1964 *Homographies, Quaternions, and Rotations* (Oxford: Oxford University Press) pp 49–54
[2] Conway J H and Smith D A 2003 *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry* (Wellesly, Massachusetts: A.K. Peters) pp 33, 41–47
[3] Coxeter H S M 1974 *Regular Complex Polytopes* (Cambridge: Cambridge University Press) p 92
[4] Potter F 1994 Geometrical Basis for the Standard Model *Int. J. of Theor. Phys.* **33** 279–305; http://www.sciencegems.com/gbsm.html
[5] Potter F 2006 Unification of Interactions in Discrete Spacetime *Prog. in Phys.* **2** 3–9; http://www.ptep-online.com/index_files/2006/PP-04-01.PDF
[6] Kostant B 1984 In Asterisque (Proceedings of the Conference "Homage to Elie Cartan", Lyons) p 13
[7] Potter F 2013 Geometrical Derivation of the Lepton PMNS Matrix Values *Prog. in Phys.* **9(3)** 29–30; http://www.ptep-online.com/index_files/2013/PP-34-09.PDF
[8] Naumov D V (on behalf of the Daya Bay collaboration) 2014 Recent results from Daya Bay experiment *arXiv* 1412.7806v1
[9] Haigh M D 2015 Results from T2K *arXiv* 1501.04283v1
[10] Potter F 2014 CKM and PMNS Mixing Matrices from Discrete Subgroups of SU(2) *Prog. in Phys.* **10(1)** 1–5; http://www.ptep-online.com/index_files/2014/PP-38-03.PDF
[11] Lusiani A 2014 Determination of $|V_{us}|$ from τ branching fractions *arXiv* 1411.4526v1
[12] Potter F 2015 Weinberg Angle Derivation from Discrete Subgroups of SU(2) and All That *Prog. in Phys.* **11(1)** 76–77; http://www.ptep-online.com/index_files/2015/PP-40-14.PDF
[13] Potter F 2011 Our mathematical Universe: I. How the Monster Group Dictates All of Physics *Prog. in Phys.* **7(4)** 47–54; http://www.ptep-online.com/index_files/2011/PP-27-11.PDF
[14] Witten E 2007 Three-Dimensional Gravity Reconsidered *arXiv* 0706.3359
[15] Klein F 1956 Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree. Dover, New York [originally published 1884] pp. 1-151
[16] Baez J 1995 This Week's Finds in Mathematical Physics http://math.ucr.edu/home/baez/week59.html
[17] Feynman F, Leighton R B and Sands M 1965 The Feynman Lectures on Physics: Quantum Mechanics, vol. 3. Addison-Wesley, Reading, MA page 1-10

7. Appendix

A. *Neutrino mixing angles derivation*

This section reviews my 2013 derivation [7] of the neutrino mixing angles from first principles. The three quaternions i, j, and k, can generate all rotations in R^3 about a chosen axis or, equivalently, all rotations in the plane perpendicular to this axis. For example, the quaternion k is a binary rotation by 180° in the i-j plane.

One now must construct the three SU(2) generators, U_1 = j, U_2 = k, and U_3 = i, from the three quaternion generators from each of the discrete subgroups [3,3,2], [4,3,2], and [5,3,2] for the three lepton families. The complete mathematical description [13] for the generators R_s operating on the unit vector x in R^3 extending from the origin to the surface of the unit sphere S^2 is given by R_s = i x U_s where s = 1, 2, 3 and

$$U_1 = j,\ \ U_2 = -i\,cos\frac{\pi}{q} - j\,cos\frac{\pi}{p} + k\,sin\frac{\pi}{h},\ \ U_3 = i, \tag{1}$$

with h = 4, 6, 10 for the three lepton family groups [p,q,2], respectively. Their U_2 generators are listed in the table, with the quantity $\phi = (\sqrt{5}+1)/2$, the golden ratio.

My three lepton family binary rotational groups, [3,3,2], [4,3,2], and [5,3,2], all have the same SU(2) generators U_1 = j and U_3 = i, but one sees immediately that each U_2 is a different quaternion generator operating in R^3. One obtains the correct neutrino PMNS mixing angles from the linear superposition of their U_2's by making the total U_2 = k, agreeing with the SU(2) generator. This particular combination of three discrete angle rotations is now equivalent to a rotation in the i-j plane by the quaternion k.

Table 1. Lepton Family Quaternion Generators U_2

Fam.	Grp.	Generator	Factor	Angle°
ν_e,e	332	$-\frac{1}{2}i - \frac{1}{2}j + \frac{1}{\sqrt{2}}k$	-0.2645	105.337
ν_μ,μ	432	$-\frac{1}{2}i - \frac{1}{\sqrt{2}}j + \frac{1}{2}k$	0.8012	36.755
ν_τ,τ	532	$-\frac{1}{2}i - \frac{\phi}{2}j + \frac{\phi^{-1}}{2}k$	-0.5367	122.459

The total contribution of all three U_2 generators should be k, so there are three equations for the three factors: -5.537, 16.773, and -11.236, which are normalized to the values in the table. The resulting angles in the table are the arccosines of these factors, i.e., their projections to the k-axis, but they are twice the rotation angles required in R^3, a property of quaternion rotations.

Using one-half of these angles produces

$$\theta_1 = 52.67°,\ \theta_2 = 18.38°,\ \theta_3 = 61.23°, \tag{2}$$

then taking differences results in predicted mixing angles

$$\theta_{12} = 34.29°,\ \theta_{13} = -8.56°,\ \theta_{23} = -42.85°. \tag{3}$$

The absolute values of these mixing angles are all within the 1σ range of their values for the normal mass hierarchy as determined from several experiments:

$$\theta_{12} = \pm 34.47°,\ \theta_{13} = \pm 8.5°,\ \theta_{23} = \pm(38.39° - 45.81°). \tag{4}$$

The ± signs arise from the squares of the sines of the angles determined by the experiments.

B. *Weinberg angle derivation*

Also considered in the essay was the statement that the Weinberg angle is determined by experiment only and not derivable from first principles. However, by using my quaternion generator approach that successfully derived the CKM4 and the PMNS mixing matrices, the Weinberg angle θ_W, i.e., the weak mixing angle, is derivable from first principles [12].

The four electroweak generators of the SM local gauge group $SU(2)_L$ x $U(1)_Y$ are typically labeled W^+, W^0, W^-, and B^0, but they can be defined equivalently as the quaternion generators i, j, k and b. But we do not require the full SU(2) to act upon the flavor states $\pm\frac{1}{2}$ for discrete rotations in the unitary plane C^2 because the lepton and quark families represent specific discrete binary rotational symmetry subgroups of SU(2). That is, we require just a discrete subgroup of $SU(2)_L$ x $U(1)_Y$.

One might suspect that the large 2I subgroup would be able to perform all the discrete symmetry rotations, but 2I omits some of the rotations in 2O. Instead, one finds that 2I x 2I' works for the leptons and the quarks, where 2I' provides the "reciprocal" rotations, i.e., the third generator U_2 of 2I becomes the third generator U'$_2$ for 2I' by interchanging ϕ and ϕ^{-1}:

$$U_2 = -\frac{1}{2}i - \frac{\phi}{2}j + \frac{\phi^{-1}}{2}k, \quad U_2' = -\frac{1}{2}i - \frac{\phi^{-1}}{2}j + \frac{\phi}{2}k. \tag{5}$$

Consider the three SU(2) generators i, j, k and their three simplest products: i x i $= -1$, j x j $= -1$, and k x k $= -1$. Now compare the three corresponding 2I x 2I' discrete generator products: i x i $= -1$, j x j $= -1$, and

$$U_2\, U_2' = -0.75 + 0.559i - 0.25j + 0.25k, \tag{6}$$

definitely not equal to -1. The reverse product U'$_2$U$_2$ just interchanges signs on the i, j, k, terms.

One needs to multiply this product quaternion U$_2$U'$_2$ by

$$P = 0.75 + 0.559i - 0.25j + 0.25k \tag{7}$$

to make the result -1. Again, P' has opposite signs for the i, j, k, terms only.

Given any unit quaternion q $=$ Cos θ $+$ n̂ Sin θ, its power can be written as q^α $=$ Cos $\alpha\theta$ $+$ n̂ Sin $\alpha\theta$. Consider P to be a squared quaternion P $=$ Cos 2θ $+$ n̂ Sin 2θ because we have the product of two quaternions U$_2$ and U'$_2$ (since we need the first term in P only). Therefore, the *quaternion* square root of P has Cos $\theta = \sqrt{0.75} = 0.866$, rotating the U$_2$ (and U'$_2$) in the unitary plane C^2 by the quaternion angle of 30° so that each third generator becomes k. Thus the Weinberg angle, i.e., the weak mixing angle,

$$\theta_W = 30^\circ. \tag{8}$$

Therefore, the Weinberg angle derives from the mismatch of the third generator of 2I x 2I' to the SU(2) third generator k.

The empirical value of θ_W ranges from 28.1° to 28.8°, values less than the predicted 30°. The reason for the discrepancy is unknown, although one can surmise either (1) that in determining the Weinberg angle from the empirical data perhaps some contributions have been left out, or (2) the calculated θ_W is its value at the Planck scale at which the internal symmetry space and spacetime could be discrete instead of continuous.