

A moonshine dialogue in mathematical physics

Michel Planat

Institut FEMTO-ST, CNRS, 15 B Avenue des Montboucons, F-25033 Besançon,
France. (michel.planat@femto-st.fr)

Abstract.

Phys and Math are two colleagues at the University of Saçonbon (Crefan Kingdom), dialoguing about the remarkable efficiency of mathematics for physics. They talk about the notches on the Ishango bone, the various uses of psi in maths and physics, they arrive at dessins d'enfants, moonshine concepts, Rademacher sums and their significance in the quantum world. You should not miss their eccentric proposal of relating Bell's theorem to the Baby Monster group. Their hyperbolic polygons show a considerable singularity/cusp structure that our modern age of computers is able to capture. Henri Poincaré would have been happy to see it.

Yes I'm crazy Also they say I'm lazy But I'll have my days When I do as I please You won't see the woods While you're a tree No you'll never see the woods While you're a tree (Under Moonshine, The Moody Blues [1]).

I would suggest, as a more hopeful-looking idea for getting an improved quantum theory, that one take as basis the theory of functions of a complex variable. This branch of mathematics is of exceptional beauty, and further, the group of transformations in the complex plane, is the same as the Lorentz group governing the space-time of restricted relativity... the working out of which will be a difficult task for the future (Dirac in 1939, delivered on presentation of the James Scott prize [2]).

1. Day 1: the Ishango bone, psi and moonshine

Phys: Did you hear about the mystery of the Ishango bone found in Congo by Prof. J. de Heinzelin? You can see it pictured as Fig. 1a.

Math: Yes, this is the oldest known mathematical puzzle, it dates back at least 20,000 years. The bone carries groups of notches totalizing 60 in left and right columns and 48 in the center column. A friend of mine found a good explanation of the puzzle, the ancient African of Ishango happened to use the base 12 for counting [3].

Phys: Why 12?

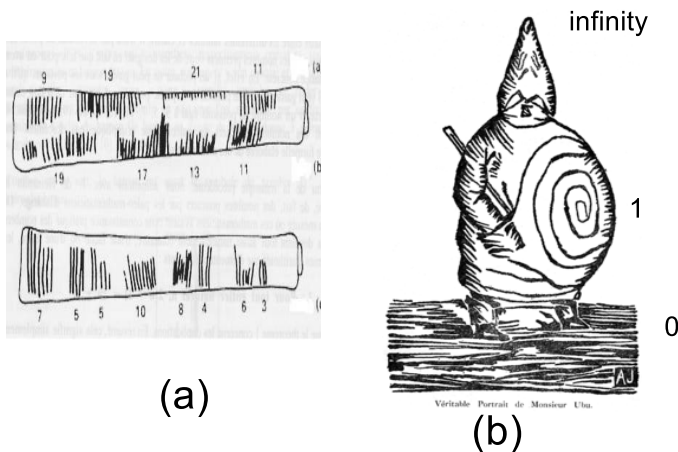


Figure 1. (a) A schematic of the notches on the Ishango bone, (b) Véritable Portrait de Monsieur Ubu, by Alfred Jarry (1896) (with a free labeling by the author).

Math: At that time, African used the thumb of a hand to count the bones in the fingers of their hands. Four fingers times three little bones on a hand yield 12 as a counting unit. As there are 24 little bones in the two hands one gets: $60 = \sigma(24)$ and $48 = \psi(24)$.

Phys: What are these strange symbols σ and ψ ? In quantum physics, $\psi(x, t)$ denotes the wave function in space time but some scientists doubt its reality, it leads to many paradoxes such as Schrödinger's cat that may be simultaneously dead or alive. One finally introduced the qubit concept $\psi = a |0\rangle + b |1\rangle$ to model the superposition, but problems arise with two qubits or more with such strange phenomena as non-locality and the non-reality of objects prior to measurements, a phenomenon called contextuality.

Math: In number theory, $\psi(x)$ may designate the second Chebyshev function, a very important concept for looking accurately at the distribution of prime numbers. But I refer to the Dedekind psi function $\psi(n) = n \prod_{p|n} (1 + \frac{1}{p})$, where the product is over the primes p that divide n . At square free numbers, $\psi(n)$ equals the sum of divisor function $\sigma(n)$. There are papers showing that both functions simultaneously arise for counting the number of maximal commuting sets of a d -level quantum system [4].

Phys: I guess you mean a multidimensional Schrödinger's cat. This may have to do with the Cheshire cat in weak quantum measurements [5]. With pre- and post-selection one has the ability to separate the location of the system (the cat) from one of its properties (the grin). Cat story in Lewis Carroll tale: Alice in Wonderland is: "Well! I've often seen a cat without a grin," thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in my life!" [6].

Math: Both arithmetic functions $\psi(x)$ and $\psi(n)$ can be used to formulate the Riemann hypothesis about the critical zeros of the Riemann zeta $\zeta(s)$. I know that the logarithmic integral $\text{li}[\psi(x)]$ does much better than $\text{li}(x)$ for counting the number

of primes [7]. Concerning $\psi(n)$, it was first introduced as the index of a particular congruence subgroup $\Gamma_0(n)$ of the modular group $\Gamma = PSL_2(\mathbb{Z})$. Congruence subgroups $\Gamma_0(p)$, for p prime, are used for defining genus zero surfaces if $p - 1$ divides 24.

Phys: Impressive, so these ‘prime groups’ and the number 24 are finally related to the explanation of the tracks on the Ishango bone. I heard about the Mathieu group M_{24} and the 24-dimensional Leech lattice used to pack spheres efficiently. I am interested in advanced maths for clarifying problems in physics. I like Wolfgang Pauli quote *There is no God and Dirac is his prophet* [8]. Do you think that God is a mathematician or, like Alfred Jarry, that *God is the tangential point between zero and infinity?* [9] It is a bit provocative, isn’t it?

Math: But the story is not finished, the groups you mention are sporadic parts of the largest finite group, the Monster group \mathbb{M} of cardinality $|\mathbb{M}| = 2^{46} 3^{20} 5^9 7^6 11^2 13^3 17 19 23 29 31 41 47 59 71 \sim 10^{54}$ -this would correspond to the mass in Kg of the known universe. Now if you add to $\Gamma_0(n)$ (n square free) the Fricke involution matrix $\frac{1}{\sqrt{n}}(0, -1; n, 0)$ you get another group with a single cusp at ∞ called $\Gamma_0^+(n)$. For prime p , the group has genus zero if and only if p is in the sequence $\{2, 3, \dots, 71\}$ occurring in the factors of $|\mathbb{M}|$. This coincidence is again a puzzle when it is rewritten as

$$\begin{aligned} 196884 &= 1 + 196883, \\ 21493760 &= 1 + 196883 + 21296876, \end{aligned}$$

and so on, in which the numbers at the left column occur in the q -expansion of the modular j -invariant and the numbers at the right column are sums of dimensions of the smallest irreducible representations of the Monster \mathbb{M} . The coincidence is known as the ‘monstrous moonshine’ (in the sense of being a crazy idea). I quote the field medalist Richard Borcherds after his proof of the puzzle with the help of string theoretical concepts: *I sometimes wonder if this is the feeling you get when you take certain drugs. I don’t actually know, as I have not tested this theory of mine* [10]. This is an illustration of the pre-established harmony between maths and physics. Concerning Jarry’s quote, I have put labels 0, 1 and ∞ on Ubu’s portrait in Fig. 1b in order to illustrate a salient feature of Grothendieck’s ‘dessins d’enfants’ [11], another way to approach the moonshine subject.

Phys: Thanks, may be this approach helps to solve the ψ -quantum puzzle. See you tomorrow.

2. Day 2: the ψ -quantum puzzle revisited

Phys: Look at what I see as the simplest ψ -quantum diagram of all, a square graph (shown in Fig. 2a) that is the paragon of two-qubit Bell’s theorem about non-locality. Imagine that Alice and Bob are spatially separated and do electron spin measurements (along the orthogonal directions x and z) with their Stern-Gerlach. The four (two-qubit) operators involved are denoted s_i , $i = 1..4$ as shown in my picture (a) where

$X = (0, 1; 1, 0)$ and $Z = (1, 0; 0, -1)$ are the Pauli spin matrices. There is an edge between two vertices if they commute. As the result of a measurement can only be the eigenvalue ± 1 of a s_i , one expects to satisfy the inequality

$$C = |\langle s_1 s_2 \rangle + \langle s_2 s_3 \rangle + \langle s_3 s_4 \rangle - \langle s_4 s_1 \rangle| \leq 2.$$

But the calculations with the operators s_i lead to the norm $\|C\| = 2\sqrt{2}$ instead of 2, and the experiments confirm this fact. There are many choices for a square graph with multiple qudits s_i and the result is always a maximal violation $2\sqrt{2}$.

Math: Do you have an explanation of the algebraic equation $C^2 = 8$?

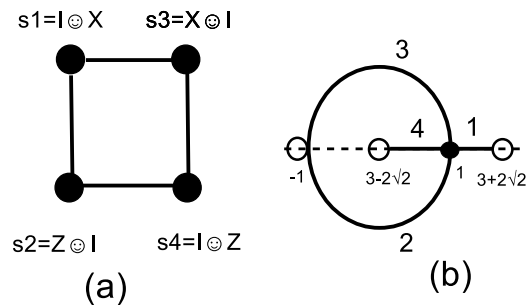


Figure 2. (a) The square graph of Bell's theorem, (b) the dessin stabilizing the square graph where the labeling 'i' of the edges corresponds to the (operator) vertices ' s_i ' of the square.

Phys: One cannot escape the quantum formalism on this matter. Observe that there is none commuting entangled pair (s_i, s_j) in my example, the proof of Bell's theorem does not refer to entanglement. But I found another argument based on the permutation group $P = \langle \alpha, \beta \rangle$ with the two generators $\alpha = (1, 2, 4, 3)$ and $\beta = (2, 3)$, this is pictured in my Fig. 2b where i means s_i . Each edge of my drawing corresponds to the same stabilizer subgroup of P .

Math: Your graph is a Grothendieck's 'dessin d'enfant' and the generators mean how you go around the black and white vertices, isn't it? And I see that the vertices live in the extension field $\mathbb{Q}(\sqrt{2})$. This means that your dessin can also be seen as a complex algebraic curve over the field of algebraic numbers. Congratulations, this is a nice use of a quite sophisticated mathematical trick. Is your graph (b) the only choice to stabilize the square?

Phys: You are right, there are essentially four choices as shown in Fig. 2 of my paper [12]. I kept the most asymmetric graph because I have the vague feeling that, as Einstein wrote: *Everything should be made as simple as possible, but not simpler* [13]. The three other choices are too symmetric, I feel one needs some breaking of the symmetry to allow a deeper explanation of Bell's theorem. I suspect that mathematics can help.

Math: You know, I met your dessin before. It corresponds to one (the case I) of the non-normal inclusions of triangle groups classified by D. Singerman [14]. The monodromy permutation P that you introduced is the subcover

$$X_0(4) \rightarrow X_0^+(2)$$

of the famous Klein quartic $X(7) = X^3Y + Y^3Z + Z^3X$. The triangle groups corresponding to the modular curves $X_0(4)$ and $X_0^+(2)$ are the congruence subgroup $\Gamma_0(4)$ and the smallest moonshine group $\Gamma_0^+(2)$ that we discussed yesterday. I suspect that your Bell's theorem is an elementary stone of a modular physical theory based on the Monster \mathbb{M} , a kind of atom. This reminds me the vortex atoms of Lord Kelvin. The modern language is knot theory and Witten developed his topological quantum field theory based on this set of ideas.

Even more intriguing, the moonshine group $\Gamma_0^+(2)$ is related to the Baby Monster group \mathbb{B} of order $2^{41} 3^{13} 5^6 7^2 11 13 17 19 23 31 47 \sim 4.10^{33}$ through another puzzling series [15, p. 20]

$$\begin{aligned} a_1 &= 4372 = 1 + 4371, & a_2 &= 96256 = 1 + 96255, \\ a_3 &= 1240002 = 2.1 + 4371 + 96255 + 1139374, \end{aligned}$$

and so on, where the numbers a_i correspond to the q -expansion of the main modular invariant T_{2A} (also called a Hauptmodul) for $\Gamma_0^+(2)$ and the numbers $1, 4371, 96255, 1139374, \dots$ are all dimensions of smallest irreducible representations of \mathbb{B} . One has the property: $\sum_{i=1}^{24} a_i^2 \bmod(70) = 24$ as for the q -expansion of $j(q)$ and that of the modular discriminant $\Delta(q)$ [16].

That moonshine for \mathbb{B} relates to Bell's theorem is a remarkable coincidence!

Phys: My approach is the application of a simple axiomatic with only two letters a and b [17]. Mathematically, my language is also called a free group $G = \langle a, b | b^2 = e \rangle$. Elements in the group are words u , any combination $uu^{-1} = e$ (the neutral element) is annihilated and I also write that b is an involution that is, $b^2 = e$.

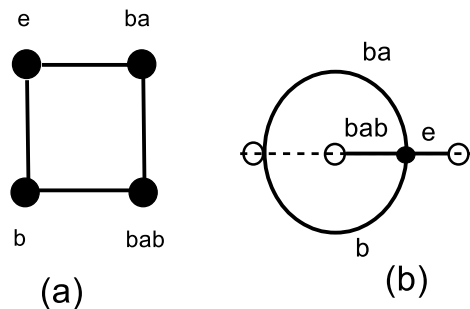


Figure 3. (a) The representatives of cosets for the dessin (b) stabilizing the square graph (a). For this special case, mutually commuting operators correspond to mutually commuting cosets.

The relative size of a subgroup H of G is called the index which means that there are n inequivalent copies (called cosets) of H that fill up G . The action of generators on these cosets creates the permutation group P by the Todd-Coxeter algorithm. You can name the cosets by a word representative, the other elements in the coset are conjugate to the representative. I did it in Fig. 3 for the case just discussed where the index is 4.

With a little effort, you can check that pairs of cosets on a edge are commuting in the group sense that is, the commutator $(u, v) = u^{-1}v^{-1}uv = e$. In this special case, coset commutation respects quantum commutation.

But I can show that it is not always the case for a geometry stabilized by a higher index group. Following Gottlob Frege quote: *Never ask for the meaning of a word in isolation, but only in the context of a sentence* [18].

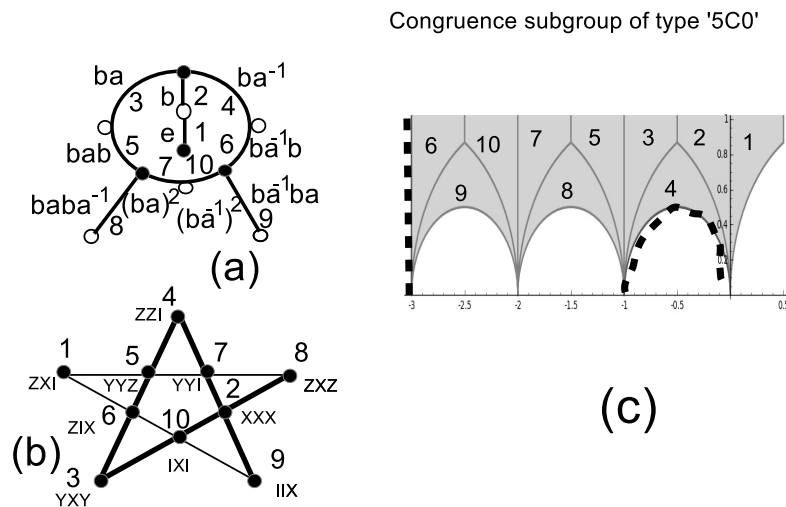


Figure 4. A dessin (a) stabilizing Mermin's pentagram (b) and (c) the fundamental domain of congruence subgroup '5C0'. The pictures have coordinates in the G -set $\{1..10\}$ with the corresponding cosets shown on (a) and a set of 3-qubit operators shown on (b).

Look at Mermin's pentagram shown on Fig. 4b [19], I labeled the vertices from 1 to 10 and with 3-qubit coordinates. The product of operators on a thin line is III and $-III$ on a thick line so that, as shown by David Mermin, this pentagram is a contextuality proof [12].

Remember that I was able to stabilize a square with the permutation group (i.e. the dessin d'enfant) shown in Fig. 2b. With the same reasoning, I stabilize the pentagram with a permutation group P generated by the permutations $\alpha = (2, 3, 4)(5, 7, 8)(6, 9, 10)$ and $\beta = (1, 2)(3, 5)(4, 6)(7, 10)$ as shown in Fig. 4a. Each point on a selected line of the pentagram corresponds to the same stabilizer subgroup S of P (all the S are isomorphic but they act on different sets of points). Then in Fig. 3b, I have put the

coset representatives for labeling the edges. In this way you can check that on a thick line of the pentagram not all cosets are commuting [17]. Of course the cosets on a (thin) line containing the identity element e are commuting.

Math: You didn't comment on your Fig. 4c but I recognize the tiling of a fundamental domain in the upper-half plane \mathbb{H} . I see that the tiles of \mathbb{H} reflect the edge (coset) structure of your dessin and, of course, the tiles also correspond to your three-qubit operators.

Ree-Tits hyperbolic polygon
with 1755 tiles

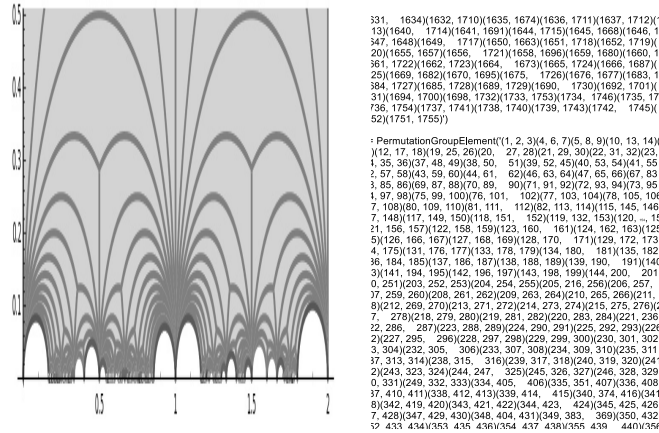


Figure 5. The fundamental domain for the index 1755 permutation group stabilizing the Ree-Tits octagon, details can be found at <http://brauer.maths.qmul.ac.uk/Atlas/exc/TF42/mag/TF42G1-p1755B0.M>.

Phys: Yes, the ‘modular’ representation of the pentagram is in the spirit of what you explained yesterday. As the generators α and β have order three and two respectively, they build a subgroup Γ' of the modular group Γ that happens to be a congruence subgroup of level 5. The set of cusps for Γ' consists of the Γ' -orbits of $\{\mathbb{Q}\} \cup \{\infty\}$, cusps are at -3 and ∞ and they have width 3. My Γ' is of type ‘5C0’ in Cummins-Pauli classification (<http://www.uncg.edu/mat/faculty/pauli/congruence/congruence.html>). I used the software Sage to draw the fundamental domain of Γ' thanks to the Farey symbol methodology. I am not the first to play with Sage on modular aspects of dessins, you can read the essay by Lieven le Bruyn [20].

Math: Do you encounter the moonshine group $\Gamma_0^+(5)$ here?

Phys: Not yet. There are papers about the connection of modular subgroups to dessins d’enfants, K3 surfaces and supersymmetric gauge theory (this is part of the so-called ‘umbral moonshine’) [21]. Look at such an elegant formula as $G(q) = \phi(q)^{24}$, that generates the states of a bosonic string oscillator from the Euler phi function $\phi(q)$ [21] (still the number 24 occurring). Tomorrow, I tell you more about my work inspired by the moonshine topic.

3. Dessins d'enfants, hyperbolic polygons, Rademacher sums

Phys: It seems not to have been recognized that many sporadic groups in the ‘Atlas of finite group representations’ are defined as dessins d'enfants that is, with only two generators. They may be used to stabilize the corresponding graph/geometry (this is a different approach of that followed by J. Tits). Lieven le Bruyn writes in [20]: *It would be nice to have (a) other Farey-symbols associated to the second Janko group, hopefully showing a pattern that one can extend into an infinite family as in the iguanodon series and (b) to determine Farey-symbols of more sporadic groups.*

This is precisely what I did for several ‘sporadic’ iguanodons (dessins d'enfants [11],[12]). In table 1, I list a few of them (those related to J_1 , J_2 , J_3 , McL , M_{24} , T , Suz , Fi_{23} and Co_1) and give the main characteristics of their fundamental domain \mathcal{P} in the upper-half plane \mathbb{H} . All the groups are represented as non-congruence subgroups of the modular group Γ . The number n of edges of the dessin \mathbb{D} is the index of the (sporadic) group representation. The genus g of \mathbb{D} equals that of the hyperbolic polygon \mathcal{P} , a face of \mathbb{D} corresponds to a cusp of \mathcal{P} , the number of black points (resp. of white points) of \mathbb{D} is $B = f + \nu_3 - 1$ (resp. $W = n + 2 - 2g - B - c$), where f is the number of fractions, c is the number of cusps, ν_2 and ν_3 are the number of elliptic points of order two and three of \mathcal{P} , respectively. For instance, the fundamental domain for the index 1755 representation of the Tits group T [that stabilizes the Ree-Tits octagon $GO(2, 4)$] is shown in Fig. 5.

Table 1. Characteristics of a few small ‘sporadic’ fundamental polygons.

Graph/geometry	Group	n	g	ν_2	ν_3	cusps	f
Hall-Janko	J_2	100	0	0	4	$1^2 7^{14}$	33
Livingstone	J_1	266	0	10	5	7^{38}	88
McLaughlin	$McL : 2$	275	10	11	14	$11^1 22^{12}$	88
$T(24)$	M_{24}	276	10	12	15	23^{12}	12
Ree-Tits $GO(2, 4)$	T	1755	57	91	0	13^{135}	586
Suzuki	Suz	1782	70	42	0	$1^1 3^{137}$	595
Janko	J_3	6156	321	76	36	19^{324}	2041
Fischer	Fi_{23}	31671	1876	695	27	$2^1 4^2 7^9 14^{45} 28^{1106}$	10549
Conway	Co_1	98280	6922	0	27	$10^{12} 20^{108} 40^{2400}$	32752
Fischer	Fi'_{24}	306936	19409	3512	0	29^{10584}	155225

Math: Excellent, the next step would be to compute the modular symbols, the relative homology of the extended upper half plane and the corresponding modular forms of weight two and higher. I am curious to see if the noncongruence cusp forms for these ‘sporadic’ polygons have unbounded denominators, as conjectured [22].

You are close to the ‘philosophy of cusp forms’ of Harish-Chandra. He was studying under Dirac himself but turned to mathematics when he learned in Princeton that *not every function is analytic* [23]. The keyword for the link between moonshine and the

Langlands program is VOA (for vertex operator algebra). The Monster vertex algebra is conjectured (by E. Frenkel and collaborators) to be the unique holomorphic VOA with charge 24 and partition function $j - 744$ (where j is again the modular invariant).

I quote E. Frenkel [24]: *Mathematics is not about studying boring and useless equations: It is about accessing a new way of thinking and understanding reality at a deeper level. It endows us with an extra sense and enables humanity to keep pushing the boundaries of the unknown.*

I suggest that we look at a promising line of moonshine research by I. Frenkel ('not E. Frenkel') and J. Duncan [25] based on the use of a Rademacher sum for the modular invariant $j(\tau) = q^{-1} + 196884q + 21493760q^2 + \dots$, with $q = e^{2i\pi\tau}$ as

$$j(\tau) + 12 = e^{-2i\pi\tau} + \lim_{K \rightarrow \infty} \sum_{\substack{0 < c < K, \\ -K^2 < d < K^2 \\ (c, d) = 1}} e^{-2i\pi \frac{a\tau+b}{c\tau+d}} - e^{-2i\pi \frac{a}{c}},$$

with a, b relative integers and $ad - bc = 1$, $\tau \in \mathbb{H}$.

The authors show that the McKay-Thomson series $T_g(\tau)$ of an element $g \in \mathbb{M}$ coincides with a generalized Rademacher sum. They expect to get a version of the simplest chiral $3d$ quantum gravity with 24 charges, as the original construction of the chiral $2d$ conformal field theory by McKay based on the partition function $j(q)$.

Phys: I am familiar with Rademacher work. Some time ago, I investigated the thermodynamics of the Euler gas whose partition function is that of the number of unrestricted partitions $p(n)$. This was useful to model the low frequency fluctuations (of the $1/f$ type) occurring in a gaz of bosons, like the phonons in a quartz crystal resonator [26]. The mathematics involves the Hardy-Ramanujan circle method in analytic number theory and this was improved by Rademacher.

Math: Do you think that our mathematics is the real world?

Phys: As a provisional response, I offer you a quote of Stephen Hawking from his lecture "Godel and the end of the universe" [27]: *In the standard positivist approach to the philosophy of science, physical theories live rent free in a Platonic heaven of ideal mathematical models... But we are not angels, who view the universe from the outside. Instead, we and our models are both part of the universe we are describing. Thus a physical theory is self referencing, like in Godels theorem. One might therefore expect it to be either inconsistent or incomplete. The theories we have so far are both inconsistent and incomplete.*

Math: Yes, self reference, consciousness and contextuality are isomorphic at the bottom and a quote attributed to Darwin is: *A mathematician is a blind man in a dark room looking for a black cat that isn't there* [28]. But in Dyson's words in 1981: *I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the universe. This is of course only a wild speculation, almost certainly wrong* [29].

Bibliography

- [1] The Moody Blues, Octave 02 under moonshine, 1978, can be listened on <https://www.youtube.com>.
- [2] P. A.M. Dirac, The Relation between Mathematics and Physics *Proc. Roy. Soc. (Edinburgh)* **59** 1938-39, Part II pp. 122-129 (1939).
- [3] V. Pletser et al, The Ishango artefact: the missing base 12 link, *Forma* **14** 336 (1990).
- [4] M. Planat, *J. Phys. A: Math. Theor.* **44** 045301 (2011).
- [5] T. Denkmayr et al, Observation of a quantum Cheshire Cat in a matter-wave interferometer experiment, *Nature Comm.* **5** 4492 (2014).
- [6] L. Carroll (Charles Lutwidge Dodgson), Alice's Adventures in Wonderland (1865) chapt 6 (http://en.wikipedia.org/wiki/Alices_Adventures_in_Wonderland).
- [7] M. Planat and P. Solé, Improving Riemann prime counting, Preprint 1410.1083 [math.NT].
- [8] W. Heisenberg, Physics and beyond; encounters and conversations, New York, Harper & Row (1971), also http://en.wikiquote.org/wiki/Wolfgang_Pauli.
- [9] A. Jarry, Exploits and Opinions of Doctor Faustroll, Pataphysician, translated by S. W. Taylor, Exact Change (Boston, 1996), p. 114; available at http://72.52.202.216/fenderse/Dr_Faustroll.pdf.
- [10] S Singh, Interview with Richard Borchers, The Guardian (28 August 1998), available at <http://simonsingh.net/media/articles/maths-and-science/interview-with-richard-borchers/>.
- [11] A. Grothendieck, Sketch of a programme, written in 1984 and reprinted with translation in L. Schneps and P. Lochak (eds.), Geometric Galois Actions: 1. Around Grothendieck's Esquisse d'un Programme (Cambridge University Press, 1997).
- [12] M. Planat, Drawing quantum contextuality with 'dessins d'enfants', *Frontiers Collection: It from Bit or Bit from It*, ed. by A. Aguirre et al (Springer, March 2015), chap. 4.
- [13] A quote attributed to Einstein, see the Quote Investigator (Exploring the origins of quotations) for more information: <http://quoteinvestigator.com/2011/05/13/einstein-simple/#more-2363>.
- [14] K. Hoshino, The Belyi functions and dessins d'enfants corresponding to the non-normal inclusions of triangle groups, *Math. J. Okayama Univ.* **52** 45 (2010).
- [15] T. Gannon, Monstrous moonshine: the fist twenty-five years, *Bull. London Math. Soc.* **38** 1 (2005).
- [16] Y. He and J. McKay, Moonshine and the meaning of life, Preprint 1408.2083 [math.NT].
- [17] M. Planat, Geometry of contextuality from Grothendieck's coset space, Preprint [1411.7704] [quant-ph].
- [18] G. Frege. Die Grundlagen der Arithmetik. Eine logisch-mathematische Untersuchung uber den Begriff der Zahl. W. Koebner, Breslau, 1884. Reprint published by G. Olms, Hildesheim, 1961.
- [19] N. D. Mermin, *Rev. Mod. Phys.* **65** 803-815 (1993).
- [20] L. le Bruyn, available at <http://win.ua.ac.be/lebruy/b2hd-LeBruyn2012a.html>.
- [21] Y. He and J. McKay, Eta products, BPS states and K3 surfaces, Preprint 1211.1931 [hep-th].
- [22] W. Winnie Li and Ling Long, Atkin and Swinnerton-Dyer congruences and noncongruence modular forms, Preprint 1303.6228 [math.NT].
- [23] R. Howe, A biographical memoir, National Academy of sciences, Washington , DC (2011).
- [24] The Atlantic, 9 October 2013, available at E. Frenkel, <http://www.theatlantic.com/technology/archive/2013/10/the-nobel-prize-in-physics-is-really-a-nobel-prize-in-math/280430/>
- [25] J. F.R. Duncan, I. B. Frenkel, Preprint 0907.4529 [math.RT].
- [26] M. Planat, *Physica A* **318** 371 (2003) (also math-ph/0307033).
- [27] S. Hawking, Godel and the End of the Universe, a lecture available at <http://www.hawking.org.uk/godel-and-the-end-of-physics.html>.
- [28] This quote is incorrectly attributed to Charles Darwin: more infoemations at http://en.wikiquote.org/wiki/Charles_Darwin and <http://www.darwinproject.ac.uk/six-things-darwin-never-said>.
- [29] S. Roberts, Curiosities: Pursuing the Monster, Institute of Advanced Study (Fall 2013 Issue); available at <https://www.ias.edu/ias-letter/roberts-monster>.