Forbidden Spacetimes

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Abstract

Without any additional physical inputs, Einstein's equations become almost tautological, permitting virtually all spacetimes as solutions. Typically, various energy conditions, reflecting the positivity of local matter-energy, are used to constrain these solutions. I will argue that this is a mistake: the energy conditions do not come from matter at all. Instead, I will proposal two possible alternatives that could determine which spacetimes are physically allowed and which are forbidden.

A Wild West of Space and Time

Einstein's general theory of relativity is an open invitation to the imagination. Time machines! Wormholes! The multiverse! All these spacetimes and many more are solutions to Einstein's equations. Those magnificent equations famously equate the curvature of the fabric of spacetime to the energy-momentum of matter. At first glance, it may seem remarkable that Einstein's equations permit such fantastical possibilities for spacetime.

And yet, a moment's thought indicates that Einstein's equations, at least in and of themselves, do not impose any restriction whatsoever on the kinds of spacetimes that are permissible. To see this, just take an absolutely arbitrary (suitably differentiable, symmetric) metric, calculate its Einstein curvature, and then simply define the energy-momentum tensor to equal that curvature. Voila! With no restrictions on the energy-momentum tensor, every metric is automatically an exact solution of Einstein's equations. Given that, in addition, Einstein's theory provides no guidance about the global topology of spacetime, the fact that all spacetimes satisfy Einstein's equations implies the troubling conclusion that anything goes.

A few physicists have ridden into town to impose some law and order on this Wild West of space and time. Strict laws have been posted: a spacetime is not considered physical unless it satisfies certain additional local or global constraints. For example, the cosmic censorship conjecture mandates that any singularities be cloaked by globally-defined event horizons. Naked singularities are too indecently exposed to be physically acceptable, says this prudish conjecture. Another global constraint is Hawking's chronology protection conjecture, which prohibits closed timelike curves. This eliminates the notorious grandfather paradox (in which one goes back in time to kill one's ancestors) and, as Hawking puts it, makes the universe safe for historians.

It is important to recognize that these conjectures are not conjectures in the sense that, say, mathematicians use that word. Their truth or falsity cannot be established. They are simply axiomatic prejudices regarding which spacetimes are allowed and which forbidden. For example, a super-extremal charged black hole is an exact solution of the Einstein-Maxwell equations. Such a black hole, whose electric charge exceeds its mass, has no event horizon and a singularity visible from afar – a naked singularity. Whether one should allow or disallow such a black hole depends solely on the status of one's belief in the cosmic censorship conjecture.

Energy Conditions

Besides constraints on the global properties of spacetime, local requirements have also been proposed. In particular, various ad hoc energy conditions are imposed on the types of matter considered when solving the coupled Einstein curvature/matter equations. The various energy conditions – which carry the labels null (NEC), weak (WEC), dominant (DEC), and strong energy conditions (SEC) – each express some seemingly reasonable expectation regarding matter, such as that the speed of sound in the matter be less than the speed of light. The energy conditions are inequalities that apply locally, and are asserted to hold everywhere in spacetime. They are generalizations of the notion that local energy density be non-negative and are implemented by requiring various linear combinations of components of the matter energy- momentum tensor to be non-negative. These energy conditions eliminate a vast number of the arbitrary metrics which would otherwise tautologically satisfy Einstein's equations.

Moreover, in general relativity, energy conditions play a vital role in a variety of important theorems. They are crucial, for example, to the singularity theorems that ensure that our universe began with a Big Bang singularity; almost all theoretical attempts to evade the Big Bang singularity ultimately call for violating the energy conditions at some step. Energy conditions are also invoked in the topological censorship theorem, in positive energy theorems, in prohibiting time machines, in the black hole no-hair theorem, and in whether the areas of black holes increase in physical processes. And, as already mentioned, without energy conditions, finding solutions to Einsteins equations would, in a sense, become trivial: the energy-momentum tensor could simply be defined as the Einstein tensor for any given metric.

Given their critical role in gravity, then, it is remarkable that the status of the energy conditions – why or even whether they hold – remains quite unclear. This is because the energy conditions are not derived from any fundamental principle. Well, then, where could they come from? Now, the energy conditions are implemented on the right-hand side of Einstein's equations which is the side that describes the energy-momentum tensor of matter. Lucky for us then that we have a superb theoretical framework for matter: quantum field theory. The core principles of quantum field theory are locality, microcausality, and symmetry, particularly Lorentz symmetry and gauge invariance, and of course the idea that the basic degrees of freedom are operator-valued functions of spacetime. One could add to that unitarity and perhaps renormalizability for interacting theories. Can we then derive the energy conditions from these deeper principles?

The answer is no. A counter-example will suffice. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

This is the Lagrangian for a scalar field that is tachyonic at $\phi = 0$ and which spontaneously breaks the $\phi \to -\phi$ symmetry. It is easy to check that at the minima of the potential, $\phi_\pm = \pm\sqrt{6m/\lambda}$, the weak energy condition is violated. And yet here we have almost the simplest possible field theory, dutifully respecting each of the aforementioned principles of quantum field theory.

Here is an even more basic objection. In classical field theory, one is free to add a total derivative to the Lagrangian. In particular, one can add a constant term:

$$\mathcal{L} \to \mathcal{L} + c \tag{2}$$

After a Legendre transform, this subtracts c from the Hamiltonian:

$$\mathcal{H} \to \mathcal{H} - c \tag{3}$$

allowing the weak energy condition to be violated. Put another way, quantum field theory cares only that there exists a ground state, not that it has positive energy. (Supersymmetric theories are an exception.) As these two examples show, the weak energy condition does not follow from quantum field theory, and it is no surprise that none of the classic books on quantum field theory so much as even mentions energy conditions.

A Crisis and Two Possible Resolutions

We have seen that one of our basic physical assumptions – that the energy conditions arise from a theory of matter – seems untenable and is very likely to be false. But if the energy conditions do not come from matter, where do they come from? Of course, one logical possibility is that the energy conditions simply do not hold; indeed, models violating all of them have been proposed. However, if we give up the energy conditions and don't replace them with something else, we will be back in the Wild West.

If the principles of quantum field theory do not help, and we get no help from general relativity itself, what are we left with? I will propose two possibilities. One is thermodynamics. The second law of thermodynamics, rooted as it is in statistical mechanics rather than in any particular kind of dynamics, has sweeping, awesome power. Rather than separate laws explaining why glass shatters but never unshatters, or why eggs break but never unbreak, or why black holes decay but radiation doesn't collapse, we have one all-encompassing law. Indeed, Einstein regarded the second law of thermodynamics as the physical law that was least likely to be replaced: "It is the only physical theory of universal content which, I am convinced, that within the framework of applicability of its basic concepts will never be overthrown."

Now, in relativistic theories of gravity, the second law of thermodynamics admits a remarkable generalization. Black holes are attributed entropy according to the surface area of their horizons:

$$S_{\mathrm{BH}} = \frac{A}{4G\hbar} \tag{4}$$

But how can this help us? What if there are no black holes around? A decade or so ago, Ted Jacobson of the University of Maryland wrote a fascinating paper in which he derived Einstein's equations from the first law of black hole thermodynamics. Jacobson applied the laws locally to so-called local Rindler horizons. Instead of deriving the first law from the classical equations of motion, Jacobson reversed the argument, deriving gravitational equations from thermodynamics. In the same vein, an interesting question to ask is whether the second law of thermodynamics, applied locally, might constrain solutions to Einstein's equations in a manner similar to the energy conditions. This is plausible because, in fact, the energy conditions are traditionally used to derive the second law for black holes. Perhaps the argument needs to be reversed again.

Another possibility is this. The energy conditions are expressed as conditions on matter. But in fact the whole division between matter and geometry is artificial. The Lagrangian will in general contain terms in which fields couple non-minimally to the metric, such as $e^{\phi}R$. Should we regard the piece of the equation of motion for this term as contributing to the left-hand side or the right-hand sides of Einstein's equations? And consider Kaluza-Klein compactification. As Kaluza and Klein showed, what appears to be matter in lower dimensions may actually be a component of an extra-dimensional metric. We see then that implementing conditions on an energy-momentum tensor derived from matter, as the energy conditions do, is quite unnatural; there ought to be a statement that applies equally to both matter and geometry.

Fortunately, there is a theory that offers a unified framework for both matter and geometry: string theory. In string theory, matter fields are naturally unified with gravity in that the graviton is just one of many oscillation modes of the string. Furthermore, in worldsheet string theory, the spacetime metric appears as a coupling for the worldsheet fields. One of the truly seductive results of string theory is that conformal invariance on the worldsheet leads directly to Einstein's equations in spacetime. Beautiful. But we need more. Einstein's equations are not enough; we want to constrain their solutions. Can we get constraints from string theory too?

Yes, we can! The point is, worldsheet string theory is more than just a two-dimensional conformal field theory: it's a conformal field theory coupled to two-dimensional gravity. And the equations of motion of two-dimensional gravity on the string worldsheet result in the so-called Virasoro constraints. Constraints! That's exactly what we need. The worldsheet constraints must translate into spacetime constraints. Wouldn't it be amazing if the choice of which spacetimes are physical or forbidden were determined by which were consistent with the Virasoro constraints, constraints that reflect the gravitational equations in two dimensions. Nothing could be more beautiful than if gravity were to solve its own problem.