Logic, Formalism and Reality

The ability of the mind to examine itself is the most extraordinary phenomenon that occurs in the Universe. Not only does the mind have the capacity to reflect all things, all that exists, but it also has the possibility to also reflect itself. Ever since Greek antiquity, logic stands at the foundations of all science. It is directly in contact with all the other sciences by providing them with the instruments and methods of investigation, at the same time it is the only science that thinks itself!! Through language we fix our knowledge at any given moment in history, and as such every generation that follows benefits from the experience of the past and builds on it. Logic strengthens the rational structure of language thus making possible intelligible knowledge and understanding. Consequently logic requires from us a clear, unequivocal use of the natural language.

The early Greek philosophers like Thales and Anaximander understood that the mind will be able to define itself in its own nature only when it will formulate abstract principles that can be thought independent of reality. The inner workings of that concrete reality could then be understood only if reduced to its abstract form and when this form can be thought in itself. While trying to explain the Nature and its phenomena, they discovered certain permanent relationships, viz. being, oneness, multiplicity, motion, stillness, order, disorder and the principle of all things.

Thales asserted that principle of all things in nature is *the water*. Anaximander considered the "*infinity - apeiron*" as the ultimate principle, since the essential character of this infinity is its inner motion that determines that if the same things bond together and the opposites break up, this leads to a perpetual creation and destruction process of the world.

With Pythagoras, the principle was expressed through numbers and their proportions, "*All things are numbers*". This shows that the school had <u>the idea of a pure science, based on logical principles, which came to be called **mathematics**</u>. With Parmenides and Zeno, the logic takes its first formal character, by stating and using its first two principles: *the principles of identity and contradiction*.

Heraclitus, had as principle "*pantha rei* - everything flows", and sees in *contradiction*, the dynamic becoming of all things in their struggle with their opposites, so as to only achieve unity through *logos.* Dialectics, the law of becoming of reality, was thus born and became a complete logical method.

Mathematics and physics are both methods of investigation that the logical mind creates in order to understand reality. From his humble beginnings, struggling to survive in a constantly changing and hostile surrounding, mankind used its ability to think logically in order to create means and methods that ensured his future evolution. From the invention of basic tools to the latest methods of science and technology, logic and the philosophy as phenomenology of mind has constantly evolved towards <u>increased abstraction and generality.</u>

Among these human endeavors, mathematics has gained a special place as a method of investigation and its presence is permeating almost every aspect of reality. To quantify and explain the facts of physics, established empirically through observation and experiment, one needs a precise, consistent language. Since Galileo this has consistently been proven to be the language of mathematics.

The essence of mathematics lies in its freedom as Cantor asserted, precisely because only in thinking, man is truly free. On the other hand, in physics the mind is always constrained by the facts of observation, measuring and experiment. The only ones that can validate its laws and generate understanding about the world we live in and our relation to it. It is this mutually interactive dynamic between freedom of mathematical formalism and the empirical necessity of the physical world, mediated by logical principles and reason that ultimately leads to penetrating the depths of our reality.

Given the disparate nature of both methods of investigation however, one cannot speak of a clear connection, mysterious or otherwise, between undecidability, uncomputability and unpredictability. One can only hope to find connections between these parts which do not preclude one from believing in an ideal kind of unity between logic, mathematics and physics. Perhaps one day this will be achieved. Today physics more than logic and mathematics, as the method of investigating Nature, is coming tantalizing close to achieving such a unity. It strives to conciliate the two conflicting main physical theories, viz. General Relativity with the Standard Model of quantum mechanics.

Deliberately or not, the vast majority of the mathematical community is quite frantically working side by side with the physical community, towards this goal of physics. This has been "*the cloud*" in the sky for several decades now. Ironically, by solving "*the two clouds*" in the sky of physics in 1900, viz. the black-body radiation and the electromagnetic ether problems it has hit upon a bigger and darker cloud, 100 years later, originating in the very solutions given to those previous two problems.

In an uncanny similar way mathematics had its foundation vigorously shaken in 1900 by an earthquake brought about with the arrival of Cantor-Dedekind's set theory. Their subsequent axiomatic treatment led in a remarkable twisted way towards more rigor and unity in mathematics. It was Hilbert's criticism of the Euclid's 2300 year old axiomatic method in geometry in 1899 that brought new guiding light into the foundations of mathematics and into the power of this method as the future paradigm for truth and objectivity in mathematics and physics alike.

Euclid's axiomatic method in geometry was one among many remarkable lasting gifts that Greek genius bestowed on mankind. Its lasting influence throughout the millennia, along with the Aristotelian logic principles, was not in the slightest an accident. Even though the modern physics and mathematics began with Descartes, Galileo, Newton and Leibnitz, by continuously rejecting and adding to the Greek heritage, the power of the deductive axiomatic system and the principles of logic could not be fully discarded. Only reformed, developed and applied further to other parts of mathematics and physics.

Even if not fully accepted, today the axiomatic set theory is at the foundation of most modern mathematics which beginning with 17th century permeates most of physics. Whereas physics has always been a source of motivation and inspiration for mathematics, ideas streaming out from the advances in physics in last century have in turn highly motivated and inspired mathematicians for the last 30 years.

In classical times, physics' relation with mathematics was mainly via analysis, in particular through differential equations. This provided the main tools to formulate the physical laws. Hilbert extended that relation with the applications of integral equations to the kinetic theory of gases, which led him to propose the axiomatization of physics in 1900 Paris Congress. Hilbert's 6th problem was to apply the axiomatic method to those branches of physics in which mathematics is prevailing. He did this by means of axioms in the same manner as his investigations on the foundations of geometry in 1899.

Cantor's Paradise: *One of the most vigorous and fruitful branches of mathematics [...] a paradise created by Cantor from which nobody shall ever expel us [...] the most admirable blossom of the mathematical mind and altogether one of the outstanding achievements of man's purely intellectual activity. (**Hilbert** on set theory)*

Cantor was the creator of set theory, which has become a fundamental in mathematics. Before Cantor there were only finite sets and "the infinite" (which was considered a topic for philosophical, rather than mathematical, discussion). By proving that there are infinitely many possible sizes for infinite sets, Cantor established that set theory was not trivial, and it needed to be studied.

Indeed, just a few decades after his seminal papers, set theory has come to play the role of a foundational theory in modern mathematics. It interprets propositions about mathematical objects from all the traditional areas of mathematics (such as algebra, analysis, topology, etc.) in a single theory, and provides a standard set of axioms to prove or disprove them. The basic concepts of set theory are now used throughout mathematics. The Continuum hypothesis, introduced by Cantor, was presented by David Hilbert as the first of his twenty-three open problems in his famous address at the 1900 International Congress of Mathematicians in Paris.

Hilbert's formalism: *"Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise"* **Hilbert [1919]**

A very important feature of Hilbert's axiomatic method, namely, its logical consistency, received a major blow in 1931 when Gödel proved that every axiomatic system is either inconsistent (leads to false theorems) or is incomplete (doesn't prove all true theorems). However, this did not discourage many of the believers in the virtues of the method. They pursued it further and applied its rigor pattern to various branches of mathematics and physics. While hopes to establish consistency were lost, subsequent work done by new generations, especially the Bourbaki group made it nonetheless the prevailing paradigm for both sciences.

"*Physics is too hard for physicists*", Hilbert used to say, implying that the necessary mathematics, without which the physical insight and experimental facts cannot take shape, was generally beyond them. Under the influence of

Minkowski, in 1909 Hilbert dedicated himself to the study of differential and integral equations; his work had direct consequences for important parts of modern functional analysis. In order to carry out these studies Hilbert introduced the concept of an infinite dimensional Euclidean space, later called Hilbert space.

His work in this part of analysis provided the basis for important contributions to the mathematics of physics in the next two decades, though from an unanticipated direction. Hilbert spaces are an important class of objects in the area of functional analysis, particularly of the spectral theory of self-adjoint linear operators that grew up around it during the 20th century. Additionally, Hilbert's work anticipated and assisted several advances in the mathematical formulation of quantum mechanics. His work was a key aspect of Hermann Weyl's and John von Neumann's work on the mathematical equivalence of Heisenberg's matrix mechanics and Schrödinger's wave equation. The eponymous theory of Hilbert space plays an important part in quantum theory.

In 1926 von Neumann showed that if atomic states were understood as vectors in Hilbert space, then they would correspond with both Schrödinger's wave function theory and Heisenberg's matrices. Throughout this immersion in physics, Hilbert worked on putting rigor into the mathematics of physics. While highly dependent on higher math, physicists tended to be "sloppy" with it. To a "pure" mathematician like Hilbert, this was both "ugly" and difficult to understand. As he began to understand physics and how physicists were using mathematics, he developed a coherent mathematical theory for what he found, most importantly in the area of integral equations.

Einstein's realism: *„If it is true that the axiomatic foundations of physics cannot be derived from experience but have to be freely invented, can we at all hope to find the right way? Or worse still, does this „right way" exists only as an illusion...To this I answer with complete confidence that this right way exists and we are capable of finding it. In view of our experience so far, we are justified in feeling that Nature is the realization of what is mathematically simplest...It is my conviction that we are able, through pure mathematical construction, to find those concepts and the law-like connections between them, which yield the key to our understanding to the natural phenomena. The really creative principle is in mathematics. In a certain sense, I consider therefore to be true –*

as was the dream of the Ancients – that pure thought is capable of grasping reality.” ***Einstein [1934]***

From a fierce empiricist and logical positivist in his first half of his scientific career, Einstein ended up to be fully converted by mathematics in his latter part of his life when he was searching for a unified field theory. Feynman used to say that his focus onto mathematics at the expense of giving up his amazing physical intuition might have caused Einstein not to produce other significant discoveries in his later work.
The mathematical reformulation of Special Relativity by Minkowski was Einstein’s first encounter with the remarkable insights induced by the Minkowski’s four dimensional space-time continuum formalism. The reaction of Einstein at the time was rather critical, declaring that the theory lost its physical meaning and that Minkowski’s reformulation was nothing but “useless erudition”, only to amend that later when he realized that it was this formulation which guided him on the correct path to General Relativity. It is well known that the mathematical aspect of the GR was done by Einstein’s friend Marcel Grossmann, a mathematician at Zurich Polytechnic and former colleague of Einstein. It was during the formulation of GR that Einstein got into contact with the works of Gauss, Riemann, Ricci and Levi-Civita, from where the whole mathematical apparatus of the theory was drawn. By contrast, Newton was not so lucky to have all mathematics available at the time he wrote the Principia, so that he had to invent the Calculus. Hilbert, in his race with Einstein to creating the GR, he did find the field equation in only 6 weeks compared to a 10 hard working years of Einstein. But Hilbert had a different program in his mind that had started 15 years before with the axiomatization method and in particular for physics and probability theory. Guided by the axiomatic method, he went on to try to axiomatize Einstein’s GR and this quest, lead Emmy Noether to discover the remarkable and fertile relation between symmetries and conservation laws, that together with gauge symmetries introduced by H. Weyl, was later to be a pivotal results in quantum physics. This was probably the moment of rupture for Einstein when he realized the power of mathematical axiomatic thought and its amazing efficiency in dealing with reality. However, by not embracing the quantum revolutionary ideas, all his quest for a unified theory came to no avail.

Weyl’s structuralism: *“All beginnings are obscure.[...] from time to time, the mathematician above all must be reminded that origins lie in depths darker than he is capable of grasping with his methods. Beyond all the knowledge* *produced by the individual sciences, remains the task of comprehending. Despite philosophy endless swing from system to system, back and forth, with must not dispense with it all together, lest knowledge be transformed into a senseless chaos."(from **Space, Time, Matter, 1918**)*

Hermann Weyl was fully enthralled by Einstein's work from its early days. 1918 was Weyl's *annum mirabilis* when, he published both his monograph on the continuum and his *Space, Time, Matter*, the first comprehensive treaty on General Relativist and the first unification of gravitation and electromagnetism fields, that inaugurated the program of unified field theory, that subsequently occupied the last three decades of Einstein's life. In the unification paper, he introduced the notion of gauge, and gave the first example of what is now known as a gauge theory. Weyl's gauge theory was an unsuccessful attempt to model the electromagnetic field and the gravitational field as geometrical properties of space-time. In 1929, he introduced the concept of the vierbein into general relativity. Inspired by the quantum theory, Weyl developed the theory of compact groups, in terms of matrix representations, results that were foundational in understanding the symmetry structure of quantum mechanics, which he put on a group-theoretic basis. Together with the mathematical formulation of quantum mechanics, due to von Neumann, this gave the treatment familiar since about 1930. Non-compact groups and their representations, particularly the Heisenberg group, were also streamlined in that specific context, in his 1927 Weyl quantization, the best extant bridge between classical and quantum physics to date. From this time, and certainly much helped by Weyl's expositions, Lie groups and Lie algebras became a mainstream part both of pure mathematics and theoretical physics.

Summary:

Nowadays, there are two conflicting theories in physics: the standard model of particle physics and general relativity. Many parts of these theories have been put on an axiomatic basis, even if the Standard Model is not logically consistent with General Relativity, indicating the need for a still unknown theory of Quantum Gravity, which, if found by axiomatic method will indirectly provide, the solution to Hilbert's 6^{th} problem. Nevertheless, both general relativity and quantum mechanics brought a new life and the controversy it has generated was quite fertile for mathematical creativity leading to new fields of mathematics viz. distribution theory, non-commutative geometry.

Many parts of mathematics such as algebra, geometry and topology, complex analysis and algebraic geometry enter naturally into physics and get new insight from it. In spite of being an ill-defined object from the point of view of rigorous mathematics, Feynman functional integral proved to be a powerful tool in quantum physics. It was gradually realized that it is also a convenient mathematical means. The geometrical objects such as loops, connection, metrics, are natural candidates for local fields and geometry produces for them interesting action functional. The Feynman integral then leads to important geometrical or topological invariants.

The past 30 years has seen a remarkable rebirth of the interaction between logic, mathematics and physics. This has been mainly due to the increasingly sophisticated mathematical models employed by particle physicists, and the consequent need to use the appropriate mathematical machinery. In particular, because of the strongly non-linear nature of the theories involved, topological ideas and methods have come to play a prominent role. The mathematical community has benefited from this interaction in two ways. First, and more conventionally, the mathematicians have been spurred into learning some of the more relevant physics and in collaborating with theoretical physicists. Second, and more surprisingly, many of the ideas emanating from physics have led to significant new insights in purely mathematical problems, and remarkable discoveries have been made in consequence.

The main input from physics has come from quantum file theory. While the analytical foundations of QFT have been intensely studied by mathematician for many years, the new stimulus has involved the more formal (algebraic, geometric, topologic) aspect. By its own nature and by historic heritage, mathematics lives on an interplay of ideas. The progress of mathematics and its vigor has always depended on the abstract formalism helping the concrete reality and the concrete feeding the abstract. We cannot lose the awareness that mathematics is but one part of the great flow of ideas.

In dealing with the modern upsurge of physics into mathematics, Sir Michael Atyah identified 20 years ago, four strategies that mathematicians can adopt towards the flood of ideas emerging from physics community.

"First, mathematicians should take the heuristic results " discovered" by physicists and try to give a rigors proof by other methods. Here the emphasis is on ignoring the physics background and only paying attention to the mathematical results that emerge from physics. The second approach is to try

to understand the physics approach and enter in dialogue with the physicist concerned. The third approach is to try to develop the physics on a rigorous basis, so as to give a formal justification of the conclusions. This approach is sometimes too slow to keep au with the development. The forth and the most visionary idea is to ***"try to understand the deeper meanings of the physics-mathematics" connection****. Rather than view mathematics as a tool to establish physical theories or physics as a way of pointing to the mathematical truths, we can try to dig more deeply into the relation between them."*

Where should we search for these deeper meanings of the physico-logico-mathematical connection? The answer that the modern mathematics provides today is in topology and category theory both founded on axiomatic set theory. The answer that the physics provides is in quantum field theory which combines special theory of relativity and quantum mechanics successfully and it is hoped that through the power of topology to also include the 100 years old General Relativity.

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