Undecidability, Fractal Geometry and the Unity of Physics

T.N.Palmer
Department of Physics, University of Oxford, UK
tim.palmer@physics.ox.ac.uk

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Abstract

An uncomputable class of geometric model is described and used as part of a possible framework for drawing together the three great but largely disparate theories of 20th Century physics: general relativity, quantum theory and chaos theory. This class of model derives from the fractal invariant sets of certain nonlinear deterministic dynamical systems. It is shown why such subsets of state-space can be considered formally uncomputable, in the same sense that the Halting Problem is undecidable. In this framework, undecidability is only manifest in propositions about the physical consistency of putative hypothetical states. By contrast, physical processes occurring in space-time continue to be represented computably. This dichotomy provides a non-conspiratorial approach to the violation of Statistical Independence in the Bell Theorem, thereby pointing to a possible causal deterministic description of quantum physics.

The Disunity of 20th Century Physics

Three of our greatest theories of physics were formulated in the 20th Century: general relativity theory, quantum theory and chaos theory. There is hardly any aspect of human endeavour in the 21st Century that has been untouched by the consequences of at least one of these theories. However, each is remarkably disparate from the others, the very antithesis of the unity to which most physicists aspire in their search for laws which govern the universe. To be specific:

- Our inability to synthesise general relativity theory and quantum theory into a satisfactory quantum theory of gravity is legendary and is widely regarded as the single biggest challenge in contemporary theoretical physics.
- There are profound differences between quantum theory and chaos theory despite the fact that unpredictability lies at the heart of both theories. In conventional interpretations of quantum theory, unpredictability arises from the randomness of the measurement process in what is otherwise a linear theory. By contrast, unpredictability arises in chaos theory from the instability and nonlinearity of its deterministic equations of motion. However, there is more than this. By virtue of its determinism, chaos has not been seen as a route to understand the phenomenon of quantum entanglement: in order to violate the Bell inequality a conventional chaotic model of quantum physics would have to be explicitly nonlocal, a property inimical to the goal of synthesising with a causal theory of gravity.

- The way chaos is typically defined is incompatible with the principles of relativistic invariance. In particular, a defining characteristic of a chaotic system is instability, characterised by the fact that two states which are initially close can diverge exponentially in time, implying the existence of positive so-called Lyapunov exponents [15]. However, such divergence can be eliminated by a logarithmic reparametrisation of time suggesting that, in terms of the standard definitions at least, the phenomenon of chaos is not coordinate independent [3].

The purpose of this essay is to provide some basis for believing that these theories can be brought closer together through the unifying concept of non-computability.

Chaos and the Undecidable Geometry of Fractal Attractors

Although unpredictability is a familiar if not defining characteristic of chaotic systems such as the famous three-component Lorenz equations [11]

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= x(\rho - z) - y\\ \frac{dz}{dt} &= xy - \beta z,\end{aligned} \tag{1}$$

chaotic unpredictability is not a direct manifestation of non-computability. The reason is as follows: although chaotic systems exhibit sensitive dependence to initial conditions, they do nevertheless exhibit continuous dependence on initial conditions [19]. Such continuous dependence means that in a chaotic system it is possible to predict reliably as far ahead as you like, providing the initial conditions are known sufficiently accurately. This implies that such predictions are computational (i.e. can be performed to arbitrary accuracy by finite computing machines in finite time).

However, in the infinite future, two interesting things happen. Firstly, no matter what the initial condition, the state of a time-irreversible chaotic system such as described by the Lorenz equations settles down on its fractal attractor, sometimes referred to as a dynamically invariant subset of state space, or invariant set for short. Secondly, in the infinite future, the property of continuous dependence on initial conditions finally breaks down. This suggests the interesting question: are the fractal attractors of chaotic dynamical systems uncomputable?

To answer this question, we must first define what is meant by the term 'uncomputable'. One can take the definition from the seminal work of Turing [21] who famously showed, by an extension of the Gödel incompleteness theorem, that no algorithm exists that can decide whether, from the set of all possible pairs of computer programs and program inputs, a given program-input pair will halt.

In their seminal book 'Complexity and Real Computation' [2] (co-authored by Steve Smale one of the pioneers of chaos theory), Blum et al set about answering the question of whether membership of a fractal, such as the famous Mandelbrot Set, is decidable. Their argumentation applies equally to the fractal attractor $\mathcal{A}$ of a chaotic system. We consider a putative algorithm/machine on the real numbers, which takes as input a point $\mathbf{x}$ in the state space of the chaotic system, and halts if $\mathbf{x} \in \mathcal{A}$ (Fig 1). Blum et al's Path Decomposition Theorem implies that such an algorithm does not exist if $\mathcal{A}$ does not have integer dimension. The very definition of a fractal is one whose (e.g. Hausdorff) dimension is not an integer. Hence we can conclude that indeed the fractal invariant

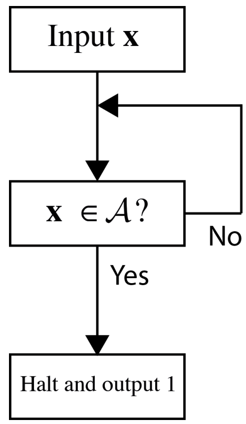

Figure 1: *Adapted from Fig F of [2]. Blum et al consider a putative algorithm/machine that takes as input a point* $\mathbf{x}$ *in the state space of a dynamical system with attractor* $\mathcal{A}$*, and halts if* $\mathbf{x} \in \mathcal{A}$*. Their Path Decomposition Theorem implies that no such machine exists if A has fractional dimension. The fractal attractors of chaotic systems are therefore non-computational, and* $\mathbf{x} \in \mathcal{A}$ *is undecidable.*

sets of chaotic systems are uncomputable. This notion was further developed by Simant Dube [4], who showed that many of the classic undecidable problems of computing theory (e.g. the Post Correspondence Problem named after one of the pioneers of computing theory, Emile Post) can be recast in terms of geometric properties of fractal attractors (e.g. does a given line intersect the attractor).

It is possible that this property of non-computability may also arise in finite time, in the initial-value problem for the Navier-Stokes partial differential equations of classical fluid mechanics. The physics behind this assertion lies in the possibility that the e-folding time, associated with the linear instability of a particular turbulent eddy, increases without bound as the spatial scale of the eddy goes to zero, implying a finite-time breakdown of continuous dependence on initial conditions [19]. However, such a property has not be proven rigorously and indeed is closely related to one of the Clay Mathematics Millenium Prize Problems. For this reason, we do not pursue it here. As the author has discussed in [19], this finite-time breakdown in the computability of the Navier Stokes equations (rather than unpredictability in low-order chaos) is what Ed Lorenz actually meant by 'The Butterfly Effect' [12].

Before returning to the principal theme of this essay, a number of important points need to be made about properties of fractal attractors. The essence of a fractal attractor is the Cantor Set. In Fig 2a we show a simple ternary Cantor set $\mathcal{C}_2$ (remove the middle third from the interval $[0, 1]$ and iterate). In Fig 2b is shown a generalisation $\mathcal{C}_p$ to p iterated pieces (to which we return in the Appendix). In both cases, the fractal is itself the intersection of all fractal iterates. A point on $\mathcal{C}_2$ can be represented by a base-3 real between 0 and 1 whose base-3 expansion does not contain the digit 1, e.g. $p = .02022020\ldots$. If we perturb this number by adding to it a number drawn from the unit interval (e.g. $\delta p = .0000010\ldots$), then almost certainly the perturbed number $p + \delta p$ will not lie on $\mathcal{C}_2$.

This raises the question: How would we actually do mathematics on fractal attractors, for example so that when we add or multiply two points on such an attractor, the sum or product

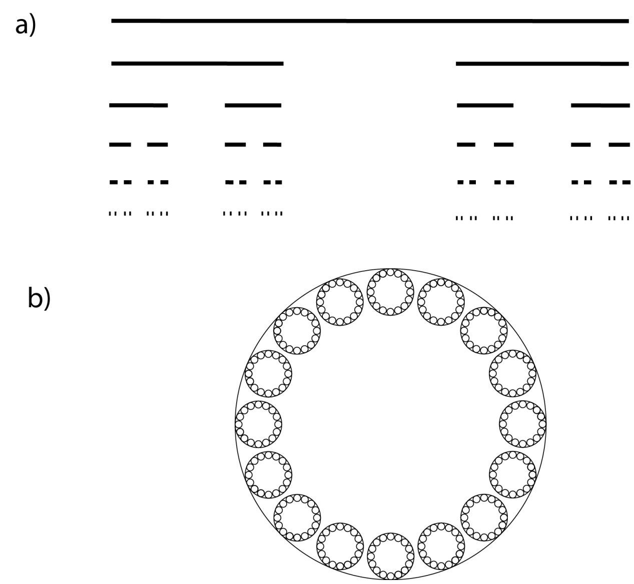

Figure 2: *Iterates of a) the familiar Ternary Cantor Set $\mathcal{C}_2$; b) a representation of the Cantor Set $\mathcal{C}_{16}$. In both cases, $\mathcal{C}_p$ is the intersection of all iterates. The set $\mathbb{Z}_p$ of p-adic integers is homeomorphic to $\mathcal{C}_p$.*

remains on the set? This is nontrivial because if, for example, we take two points $p_1 = .020220\ldots$ and $p_2 = .002020\ldots$ on $\mathcal{C}_2$ (written in base 3) and add them together, then $p_1 + p_2 = .100010\ldots$ which contains the digits 1 and therefore does not lie on $\mathcal{C}_2$. A similar issue arises if we multiply p_1 by p_2. One might imagine that simply replacing each 2 with a 1, so that p_1 and p_2 are represented in binary, would do the trick. However, it does not as the example $p_1 = .10010\ldots$, $p_2 = .11110\ldots$ shows (in this case $p_1 + p_2$ no longer lies in the unit interval $[0, 1]$).

In fact, there is a way to ensure that addition and multiplication on Cantor Sets are arithmetically closed. However, instead of using familiar real-number representations of p_1 and p_2 on $\mathcal{C}_2$, we must instead use so-called 2-adic integer representations (and p-adic integers for $\mathcal{C}_p$). p-adic numbers are bread and butter for pure mathematicians [9]. However, they are typically considered exotic by physicists. This may have to change if we are to exploit the notion of non-computability in physics. There is a rich theory of nonlinear dynamical systems based on mappings of p-adic numbers [23]. Going beyond this, it is possible to do calculus, complex analysis, Lie group theory, indeed much of the usual sorts of mathematics performed by physicists, using p-adic numbers. Like the real numbers, the set of p-adic numbers forms a completion of the set of rational numbers - however, with respect to a different metric: the p-adic metric rather than the more familiar Euclidean metric. There is an important consequence of this: points which lie in the fractal gaps of the Cantor Set (corresponding to p-adic numbers which are not p-adic integers) are, p-adically distant from points on the Cantor Set, even when from a Euclidean perspective they may seem arbitrarily close. This has conceptually implications discussed below.

Despite all this, the notion of fractals and non-computability may invoke a sense of uneasiness for physicists who believe that the world around us should be describable using finite mathematics. As

Hilbert famously noted: 'the infinite is nowhere to be found in reality, no matter what experiences, observations, and knowledge are appealed to.' In this regard, we should note that non-computability can leave an imprint in finite approximations D' of chaotic systems D. In particular, the proposition $x \in \mathcal{A}_{D'}$, although algorithmically decidable, can nevertheless be computationally irreducible: it cannot be decided reliably by an algorithm which is itself a simplification of D'. We will refer to this again in the discussion of the Bell Theorem. In can be noted that finite representations of fractals can be represented simply by finite truncations of the corresponding p-adic integers.

Towards a Unification of 21st Century Physics

Using the concept and properties of uncomputable fractal attractors, let us return to the issue raised at the beginning of this essay: the disunity of the three great theories of 20th Century physics. We discuss possible ways to resolve the disunity in reverse order to that described above.

Chaos Theory and Relativity Theory

We can easily overcome the obstacle between chaos and relativity theory discussed above. The answer [3] is to define chaos in terms of the geometric properties of its fractal invariant set. For example, as discussed, one defining geometric characteristic of a fractal is its non-integer dimension. An approach based on an analysis of the invariant sets of a parametrisation of the cosmological Mixmaster model [13] allows one to talk meaningfully about coordinate-independent chaos in a relativistically invariant cosmological setting. This allows us to introduce a concept which is central to the discussion of quantum entanglement below: the notion of the universe evolving on some uncomputable fractal invariant set I_U.

Chaos Theory and Quantum Theory

One seeming obstacle between quantum theory and chaos theory - the linearity of the former and the nonlinearity of the latter - is easily overcome. Fig 3 shows, using Monte Carlo techniques based on the Lorenz equations, the evolution of some contour of a probability distribution. The evolution of probability density ρ in classical physics satisfies a linear Liouville equation which in Hamiltonian form can be written

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} \tag{2}$$

where $\{\ldots\}$ is the Poisson bracket. This is remarkably close in structure to the von Neumann-Dirac form

$$i\hbar \frac{\partial \boldsymbol{\rho}}{\partial t} = [\mathbf{H}, \boldsymbol{\rho}] \tag{3}$$

of the Schrödinger equation where $[\ldots]$ is the operator commutator. (A reason for the appearance of $i\hbar$ in (3) and not in (2) is discussed in the Appendix.) Now the linearity of the Liouville equation is simply a consequence of conservation of probability. In particular, the linearity of the Liouville equation says nothing whatsoever about the underlying nonlinearity of the dynamics which generates ρ. The close formal similarity between the Hamiltonian form of the Liouville equation and the von Neumann-Dirac equation is strongly suggestive (to the author at least) that there must also be some deterministic framework underpinning quantum physics. If this is so, then, by analogy with the Liouville equation, the linearity of the von Neumann-Dirac equation says nothing about the nonlinearity of this underpinning deterministic dynamic.

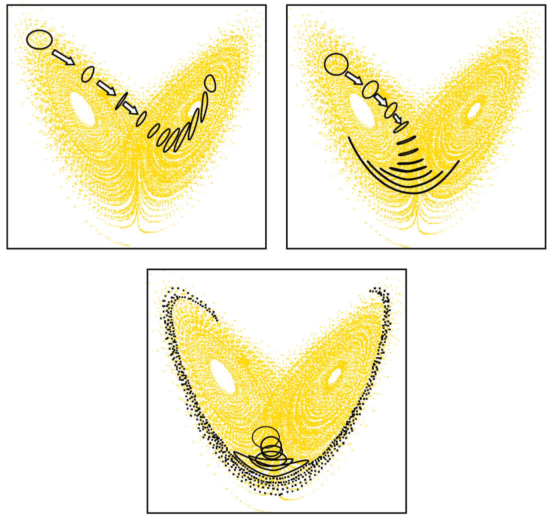

Figure 3: *Evolution of a contour of probability, based on Monte Carlo integrations of (1), is shown evolving in state space for different initial conditions, with the Lorenz attractor as background. The linearity of the Liouville equation exists peaceably with the nonlinearity of the underlying dynamical equations.*

The principal obstacle in drawing together chaos and quantum theory is therefore not the linearity of the Schrödinger equation, but the Bell Theorem. Unless it explicitly violates local causality (i.e. is nonlocal), a conventional computable deterministic model of quantum spin must satisfy the Bell Inequality

$$C(0,0) + C(0,1) + C(1,0) - C(1,1) \leq 2 \tag{4}$$

and thus be inconsistent with experiment. As usual, we imagine a source of entangled spin-1/2 particles prepared so that the total angular momentum of any pair of particles is zero. The spins of the particles are measured by remote experimenters Alice and Bob who can each choose to orient their measuring apparatuses in one of two ways (relative to a reference direction): conventionally these are referred to as $X=0$, $X=1$ (for Alice) and $Y=0$, $Y=1$ (for Bob). $C(X,Y)$ denotes the correlation in spin measurements for ensembles of particles, as a function of the measurement settings. In the discussion below, $X'=1$ if $X=0$ and *vice versa*, and similarly for Y.

In a local deterministic theory, each pair of entangled particles is described by a supplementary variable λ, often referred to as a hidden variable (though in the uncomputable type of theory proposed here, there is no need for λ to be hidden; as discussed below, the ontic characteristics of a putative quantum state are inaccessible to the experimenter [22]). For each λ in a conventional hidden-variable theory, a value of spin (here ± 1) is defined for each of the four values of X and Y.

One property which a locally causal deterministic theory must conform to if it is to satisfy the Bell Inequality is that of Statistical Independence

$$\rho(\lambda|XY) = \rho(\lambda) \tag{5}$$

The assumption (5) ensures that when the individual correlations in (4) are estimated from separate sub-ensembles of particle pairs (as happens in any real-world experimental test of the Bell inequality), then the hidden variables associated with these sub-ensembles are statistically equivalent to one another. A theory which violates (5) is referred to as superdetermistic [6], and the existence of statistically inequivalent real-world sub-ensembles is almost universally seen as implausibly conspiratorial and even unscientific.

However, in a non-computable theory, it is possible to violate (5) without negating the statistical equivalence of real-world sub-ensembles of particles/hidden variables. To see this, suppose indeed that the universe is evolving on some uncomputable fractal invariant set I_U in cosmological state space, as discussed above. Hence, if a pair of entangled particles, represented by some unique λ, is measured with some particular choice of measurement settings (X, Y), then, by hypothesis, the state $\mathbf{x}(\lambda, X, Y)$ of the world associated with the triple (λ, X, Y) lies on I_U.

Now if $\mathbf{x}(\lambda, X, Y)$ is a real-world state, then $\mathbf{x}(\lambda, X', Y)$ or $\mathbf{x}(\lambda, X, Y')$ are counterfactual states: they describe hypothetical worlds where measurement pairs such as $(X'Y)$ or (XY') might putatively have been performed on the same particle pair, even though the measurements (X, Y) were performed in reality. Now as discussed above, a random perturbation to a point on a Cantor set almost certainly takes the point off the Cantor Set - whence in such a model it is plausible that neither the counterfactual states $\mathbf{x}(\lambda, X', Y)$ nor $\mathbf{x}(\lambda, X, Y')$ lie on I_U. In the Appendix we describe a particular uncomputable model where such counterfactual states *definitely* do not lie on I_U. That is to say

$$\rho(\lambda|XY) = 1 \implies \rho(\lambda|X'Y) = 0 \text{ and } \rho(\lambda|XY') = 0 \tag{6}$$

which is a manifest violation of (5).

On the other hand, since it is undecidable whether $\mathbf{x}(\lambda, X, Y) \in I_U$, then there is no algorithm for determining which of $\mathbf{x}(\lambda, X, Y), \mathbf{x}(\lambda, X', Y), \mathbf{x}(\lambda, X, Y')$ lies on I_U. Hence, (6) does not itself prevent

$$\rho_c(\lambda|XY) = \rho_c(\lambda|X'Y) = \rho_c(\lambda|XY') = \rho_c(\lambda|X'Y') \tag{7}$$

Here ρ_c denotes a representation of probability based on purely computational estimates. Equation (7) implies that from a computational perspective, it is no more likely that the particle pair associated with a particular λ is measured with one set of X and Y values as with any other set. From this computational perspective, (5) is not violated.

How can we make physical sense of the dichotomy presented in (6) and (7)? The key is to note that in the framework developed here, undecidability is a property of the geometry of state space, and not of descriptions of physical processes occurring in space-time. A mathematical description of an actual process occurring in physical space-time should always be possible using computational/algorithmic equations, whilst a mathematical description of a proposition concerning the reality of a putative state in state space may not be so. To illustrate this dichotomy, let us return to the prototype Lorenz attractor $\mathcal{A}_L$. As discussed, the proposition $\mathbf{x} \in \mathcal{A}_L$ is undecidable. However, the space-time processes which $\mathcal{A}_L$ represents, in this case a highly truncated model of fluid convection in Newtonian space-time, are given by the computable differential equations (1), solvable by algorithm. Hence, in summary, the violation (6) of (5) only arises when comparing a real-world state $\mathbf{x}(\lambda, X, Y)$ with the hypothetical counterfactual states $\mathbf{x}(\lambda, X', Y)$ and $\mathbf{x}(\lambda, X, Y')$. By contrast, the four separate sub-ensembles of particles contributing to each of the four correlations in (4), are all associated with real-world states and not counterfactual states. As a consequence of the notion that descriptions of real-world processes in space-time are computable, (5) can be written in the form $\rho_c(\lambda|XY) = \rho_c(\lambda)$ which, from (7), is therefore not violated.

Unravelling the dichotomy between real-world processes in space-time and putative worlds in state space is central to understanding why an uncomputable theory of quantum physics can violate

Bell inequalities without violating experimenter free will or causality [6]. Free will is frequently described as an ability 'to have done otherwise', a description that is manifestly built around the notion of putative counterfactual worlds in state space (where I did do otherwise). However, one can equally well describe free will solely in terms of real-world processes, without referring to counterfactuals at all: specifically one is free when there are no constraints preventing one from doing as one wishes [8]. Similarly for causality: when Newton claps his hands and hears the sound reflected from the back wall of the college quad, he can assert that the sound was caused by the clap in one of two ways: either by claiming that if he hadn't clapped he wouldn't have heard the sound (thus invoking counterfactuals) or by linking the clap to the excitation of an acoustic wave which propagated across the quad according to the computational equations of fluid mechanics, was reflected by the quad wall, entered his ear triggering an electric signal in his neurons. The latter description only involves real-world processes occurring in space-time.

In the case of an uncomputable theory such as presented here, these two descriptions of free will and causality are profoundly inequivalent. By basing descriptions of free will and local causality strictly on computational processes occurring in space-time, and not on undecidable counterfactuals in state space, an uncomputable deterministic model need violate neither free will nor causality and still violate the Bell inequality. No conspiracies are needed to achieve this. The uncomputable model described briefly in the Appendix does this and more: using number-theoretic properties of trigonometric functions [14], it shows that whatever the choices that Alice and Bob actually make for X and Y, the correlations for the sub-ensembles are *necessarily* quantum mechanical in nature.

Above, it was mentioned that in state-space, the p-adic metric respects the primacy of a fractal invariant set better than does the familiar Euclidean metric. This has profound metaphysical implications. For example, in the theory of counterfactual causality of the renowned philosopher David Lewis [10], it is assumed that of two putative counterfactual worlds, the one which resembles reality better must be closer to reality. From this perspective, a world which differs only in some seemingly insignificant detail, e.g. in the wavelength of a single photon from a distant quasar, must be extremely close to reality. As such, it would seem grossly implausible of a theory to assert as profoundly unphysical, a world which differs from reality only in terms of something as seemingly insignificant as the wavelength of a single photon. However, relative to the p-adic metric, if this counterfactual world lies in a fractal gap of the corresponding invariant set in state space, then no matter how closely it resembles reality, and hence no matter how close it appears to reality from a Euclidean perspective, it will actually be distant from reality. In the Appendix is discussed a model where such a counterfactual world does indeed lie in a fractal gap, if the wavelength of the photon is used to determine the measurement settings in a Bell experiment.

The results above carry over *mutatis mutandis* to the finite case where propositions about states lying on periodic fractal-like invariant sets may be formally decidable but nevertheless computationally irreducible (a term defined above). In such a case, it is impossible to determine reliably the truth of such propositions $\mathbf{x} \in I_U$ using a computational subset of the universe (e.g. what we would call a computer!). The model described in the Appendix has a finite but computationally irreducible representation for finite fractal parameter p. In this model, the continuum complex Hilbert space of quantum theory arises as a singular (and not a smooth) limit [1] at $p = \infty$. This is consistent with emerging evidence from quantum complexity theory [7] that quantum theory cannot be considered the smooth limit of some corresponding finite theory.

Quantum Theory and General Relativity Theory

Could non-computability break the road-block in finding a satisfactory theory which can synthesise quantum and gravitational physics? Geroch and Hartle [5] and Penrose [20] have speculated that a quantum theory of gravity may not be computable on the basis that it is undecidable whether two simplicial 4 manifolds are topologically equivalent. The following additional reasons suggest that non-computability could lie at the heart of a quantum theory of gravity. Specifically:

- General relativity is a nonlinear theory. The structure of fractal geometry is necessarily nonlinear. As discussed, it is only when expressed in terms of evolution of probability that fractal-based dynamics, like the Schrödinger equation, appears linear.
- General relativity is a deterministic causal theory. Causal structure derives from the metric properties of space-time. As described in this essay, it is possible to violate Bell inequalities with a locally causal but uncomputable deterministic theory, providing the notion of local causality is defined purely in terms of computational processes occurring in space-time, and not in terms of undecidable counterfactual properties of state space.
- General relativity is primarily a geometric theory. Non-computability has a natural expression in terms of fractal (p-adic) geometry. Could it be that the computational pseudo-Riemannian geometry of space-time is emergent from the non-computational p-adic geometry of state space for large but finite p? (In particular, could it be that the Lorentzian signature of the space-time metric is emergent from the primitive quaternionic structure that is a feature of the specific fractal model discussed in the Appendix [18].)

Discussion

From where do new ideas come? Do they pop out of the aether as some random flashes of inspiration with no obvious precedent? Or do these ideas mostly already exist, but in a completely separate setting. As such, does the creative spark really consists of taking some pre-existing idea from its usual setting and transplanting it into an unfamiliar setting where it may provide new insights into old unsolved problems? Here, an example of the latter is presented. For many decades of his research career, the author has worked on the chaotic dynamics of climate, having first done a PhD in general relativity theory. It was this somewhat improbable combination of research topics that led to a realisation [16] that the non-computable state-space geometry of chaotic systems could provide new insights into the Bell Theorem. Perhaps this could, in turn, provide a novel path to a unification of the two most fundamental theories of 20th century physics: general relativity theory and quantum theory. If this path does turn out to be the right one, quantum theory will have to change much more than general relativity.

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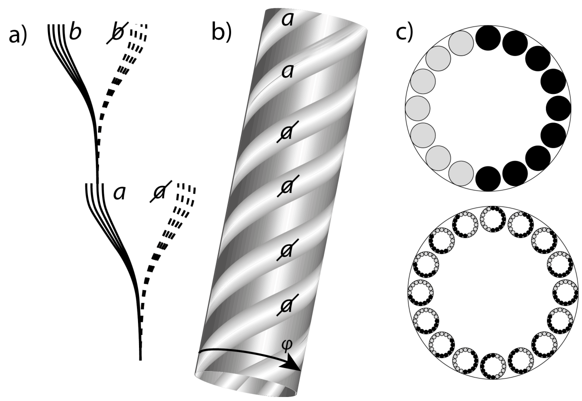

Figure 4: *A schematic of the local fractal state-space structure of the invariant set I_U in Invariant Set Theory. a) An ensemble of trajectories decoheres into two distinct clusters labelled a and $\not a$. Under a second phase of decoherence, this trajectory, itself comprising a further ensemble, decoheres into two further distinct regions labelled b and $\not b$. b) Under magnification, a trajectory segment is found to comprise a helix of p trajectories at the next fractal iterate. c) Top: a cross section through the helix of trajectories comprises p (here $p = 16$) disks coloured black or grey according to whether that trajectory evolves to the a cluster or the $\not a$ cluster. Bottom: each of these p disks itself comprises p further disks coloured black or grey according to whether each trajectory evolves to the b or $\not b$. The fractal set $\mathcal{C}_p$ of disks is homeomorphic to the set of p-adic integers.*

A Appendix

In Invariant Set Theory (IST) [17, 18], we consider a specific fractal model I_U of state-space trajectories (or histories), where a single trajectory at some $I-1$th level of fractal iterate comprises a helix of p trajectories at the Ith fractal iterate (similar to a strand of rope) - see Fig 4. Here p is a finite but arbitrarily large integer. A cross-section of such trajectories is isomorphic to $\mathcal{C}_p$ (see Fig 2b). Under interaction with the environment, these Ith trajectories diverge (with the appearance of Everettian branching at the $I-1$th iterate). In the simplest case where we partition state space into two clustering regions (corresponding to measurement eigenstates) labelled a and $\not a$, each of the Ith-iterate trajectories is labelled by the region into which the trajectory evolves. In this way, at each fractal iterate, the set of p trajectories can be represented by the complex Hilbert vector $\cos\frac{\theta}{2}|a\rangle + e^{i\phi}\sin\frac{\theta}{2}|\not a\rangle$ where $\cos^2\frac{\theta}{2}$ denotes the fraction of Ith-iterate trajectories labelled a, and ϕ denotes an angular coordinate around the helix of Ith-iterate trajectories. This correspondence with complex Hilbert states only holds when $\cos\theta$ is of the form $\frac{n_1}{p} \in \mathbb{Q}$ and $\frac{\phi}{2\pi}$ is of the form $\frac{n_2}{p} \in \mathbb{Q}$ (where $0 \le n_1, n_2 \le p$ are integers). The finite size of the helix, and its symmetry under discrete rotations, explains both the factor $\hbar$ (whose dimensions are that of phase space) and i in (3). Indeed, as discussed in [18], I_U exhibits a natural quaternionic structure (generating Pauli spin matrices). In IST the state space $\mathcal{S}_p$ associated with such qubit Hilbert vectors is a discretised form of the Bloch Sphere.

Complementarity in quantum theory (arising from the non-commutativity of observables) is a consequence of number theory in IST. In particular Niven's Theorem [14] asserts that if ϕ is a rational angle, then $\cos\phi$ is almost certainly irrational. Such a transformation arises in performing a unitary Hadamard on a complex Hilbert vector. In IST, this number theoretic result implies that if an experimenter measures on which arm a particle travels when passing through an interferometer (position measurement), the experimenter could not counterfactually have performed an interferometric experiment (momentum measurement) on that same particle. Because of Niven's Theorem, $\mathcal{S}_p$ does not map onto itself under a general rotation of the sphere. That is, $\mathcal{S}_p$ is incommensurate under general rotations. In IST, the tensor product of rational Hilbert vectors is simply the Cartesian product of $\mathcal{S}_p$ - the discretised form implying that an exponentially increasing number of degrees of freedom can be accommodated with multiple Cartesian products, unlike with the continuum state space $\mathbb{S}^2$ of a qubit in quantum theory. As a consequence, for a Bell State, IST demands that the cosine of the relative angle between Alice and Bob's measurement settings must be of the form $\frac{n_3}{p}$ and hence be rational.

Consider the first two terms in the CHSH inequality (4). They refer to correlations relative to the three measurement orientations $X = 0$, $Y = 0$ and $Y = 1$. These can be represented as vertices of a spherical triangle on the celestial sphere. Suppose in reality that a particular particle pair (with hidden variable λ) were measured relative to $X = 0$, $Y = 0$. From IST, the cosine of the angular distance between the vertices $X = 0$ and $Y = 0$ must be rational. In [18], it is shown, using the cosine rule for spherical triangles, that in such a circumstance it is impossible for the cosine of the angular distance between the vertices $X = 1$ and $Y = 1$ to also be rational. Hence if the state $\mathbf{x}(\lambda, X = 0, Y = 0) \in I_U$, the counterfactual state $\mathbf{x}(\lambda, X = 0, Y = 1) \notin I_U$ (consistent with (6)). Because the fractal geometry is described using complex Hilbert vectors, the correlations between sub-ensembles of real-world particle pairs will necessarily be quantum mechanical.

A corresponding analysis can be made of the Factorisation assumption in the Bell Theorem, expressed as $A_{XY}(\lambda) = A_X(\lambda)$, $B_{YX}(\lambda) = B_Y(\lambda)$ where A and B are Alice and Bob's deterministic spin functions which return the values ± 1. Violation of Factorisation is typically viewed as implying a violation of local causality. However, in a geometric uncomputable theory, it is again vital to make the distinction between violation of local causality based on computable space-time processes - for example when information in space-time propagates superluminally - and violation of local causality based on undecidable counterfactual reasoning. As with free choice, it is possible to violate the latter without violating the former.

IST violates Factorisation from the counterfactual perspective, but not from the space-time perspective. From the space-time perspective if λ is associated with a state on I_U, then either $\lambda \in \Lambda_1 = \{\lambda |\ X = 0,\ Y = 0 \text{ or } X = 1,\ Y = 1\}$ or $\lambda \in \Lambda_2 = \{\lambda |\ X = 0,\ Y = 1 \text{ or } X = 1,\ Y = 0\}$. If $\lambda \in \Lambda_1$, then given X (say $X = 0$) the value of Y is redundant (it is necessarily $Y = 0$), and hence Factorisation is satisfied. Similarly if $\lambda \in \Lambda_2$. By contrast, Factorisation is violated if we admit counterfactual worlds not lying on I_U. Such a counterfactual world would be associated with one where, for example, $\lambda \in \Lambda_1$ and, with $X = 0$ fixed, $Y = 0$ is mathematically perturbed to $Y = 1$. Such a counterfactual state does not lie on I_U and is therefore, by the Invariant Set Postulate, not ontic. Such a state is, moreover, p-adically distant from any state on I_U (even if the counterfactual world only differs by the wavelength of a single photon, e.g. from a distant quasar). As such, IST is locally causal from a space-time perspective, but not from a counterfactual perspective.

The uncomputable theory described here is not quantum theory. However, the closed complex Hilbert Space of quantum theory is emergent (only) in the singular limit at $p = \infty$. At this limit, all fractal gaps close, and the set of all complex Hilbert states is ontic. As the theoretical physicist Michael Berry [1] has discussed, old theories of physics are frequently the singular limit of new theories as some parameter of the new theory (here p) is set equal to either zero or infinity.